A CLASS OF ANALYTIC FUNCTIONS DEFINED
BY THE CARLSON-SHAFFER OPERATOR

BY

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Abstract
The Carlson-Shaffer operator $L(a, c)f = \phi(a, c) * f$, where $f(z) = z + a_2 z^2 + \cdots$ is analytic in the unit disk $E = \{z : |z| < 1\}$ and $\phi(a, c; z)$ is an incomplete beta function, is used to define the class $T(a, c)$. An analytic function $f$ belongs to $T(a, c)$ if $L(a, c)f$ is starlike in $E$. The object of the present paper is to derive some properties of functions $f$ in the class $T(a, c)$.

1. Introduction

Let $A$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f \in A$ is said to be starlike of order $\alpha$ in $E$ if

$$\text{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in E)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote this class as $S^*(\alpha)$. Also we denote by
A function \( f \in A \) is said to be convex (univalent) in \( E \) if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in E).
\]
We denote this class as \( K \). Clearly \( f \in K \) if and only if \( zf' \in S^* \).

The class \( A \) is closed under the Hadamard product or convolution
\[
(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,
\]
where
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.
\]

Let \( \phi(a,c) \) be defined by
\[
\phi(a,c; z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in E; \ c \neq 0, -1, -2, \ldots),
\]
where \( (\lambda)_n = \lambda(\lambda + 1) \cdots (\lambda + n - 1)(n \in \mathbb{N} = \{1, 2, 3, \ldots\}) \). The function \( \phi(a,c) \) is an incomplete beta function. Carlson and Shaffer [1] defined a linear operator on \( A \) by the convolution as follows:
\[
L(a,c)f = \phi(a,c) * f \quad (f \in A; \ c \neq 0, -1, -2, \ldots).
\]

\( L(a,c) \) maps \( A \) into itself. \( L(c,c) \) is the identity and if \( a \neq 0, -1, -2, \ldots \),
then \( L(a,c) \) has a continuous inverse \( L(c,a) \) and is an one-to-one mapping of \( A \) onto itself. \( L(a,c) \) provides a convenient representation of differentiation and integration. If \( g(z) = zf'(z) \), then \( g = L(2,1)f \) and \( f = L(1,2)g \).

By using \( L(a,c) \) we now introduce the subclass of \( A \) as follows.

**Definition.** A function \( f \in A \) is said to be in the class \( T(a,c) \) if
\[
L(a,c)f \in S^* \quad (c \neq 0, -1, -2, \ldots).
\]

Miller and Mocanu [4, Theorem 2] have proved that if \( c(c \neq 0) \) and \( a \)
are real and satisfy
\[
a > N(c) = \begin{cases} 
|c| + \frac{1}{2} & (|c| \geq \frac{1}{3}) \\
\frac{3}{2}c^2 + \frac{2}{3} & (|c| \leq \frac{1}{3}),
\end{cases}
\]
then the function
\[
\Phi(c, a; z) = 1 + \sum_{n=1}^{\infty} \frac{(c)_n}{(a)_n} \frac{z^n}{n!}
\]
is convex in \( E \).

In [5] Noor gave the following.

**Lemma A.** ([5, Lemma 2.1]) If \( c(c \neq 0) \) and \( a \) are real and satisfy (1.5), then \( \phi(c, a; z) \) is convex in \( E \).

**Theorem A.** ([5, Theorem 3.2]) Let \( f \in T(a, c) \), where \( a \) and \( c \) satisfy the conditions of Lemma A. Then \( f \in S^* \) and hence \( f \) is univalent in \( E \).

**Theorem B.** ([5, Theorem 3.3]) Let \( f \in T(a, c) \) with \( a \) and \( c \) satisfying (1.5). Then the disk \( E \) is mapped onto a domain that contains the disk
\[
D = \left\{ w : |w| < \frac{2(c + a)}{a} \right\}.
\]

**Theorem C.** ([5, Theorem 3.4]) Let \( a(a \neq 0), c \) and \( d \) be real and \( c > N(d) \), where \( N(d) \) is defined as in (1.5). Then
\[
T(a, d) \subset T(a, c).
\]

**Theorem D.** ([5, Theorem 3.5]) Let \( a(a \neq 0) \) and \( c \) be real and satisfy \( c > N(a) \), where \( N(a) \) is defined in the similar way of (1.5). Let \( \psi \) be a convex function in \( E \). If \( f \in T(a, c) \) then \( \psi \ast f \in T(a, c) \).

**Theorem E.** ([5, Theorem 3.7]) Let \( f \in T(a, c) \) and let \( F \) be defined by
\[
F(z) = \frac{\beta + 1}{z^\beta} \int_0^z t^{\beta-1} f(t) dt \quad (\beta \in N).
\]
Then
\[ \text{Re} \left\{ \frac{z(L(a, c)F(z))'}{L(a, c)F(z)} \right\} > \alpha \quad (z \in E), \] (1.10)
where
\[ \alpha = \frac{- (2\beta + 1) + \sqrt{4\beta^2 + 4\beta + 9}}{4}. \] (1.11)

However, we find that Lemma A is not always true for \( c(c \neq 0 \text{ real}) \) and \( a \) satisfying (1.5).

Counterexample. Let \( a = 1 \) and \( \frac{1}{3} \leq c < \frac{1}{2} \). Then \( a > N(c) = c + \frac{1}{2} \) and
\[ \phi(c, 1; z) = z + \sum_{n=1}^{\infty} \frac{(c)_n}{n!} z^{n+1} = \frac{z}{(1-z)^c}. \] (1.12)

For \( z = \rho e^{i\theta} (0 < \rho < 1) \) and \( 1 - \frac{c}{2} < \cos \theta < 1 (0 < \theta < \frac{\pi}{2}) \), we have
\[ 1 + \frac{z\phi''(c, 1; z)}{\phi'(c, 1; z)} = 1 + \frac{(c+1)\rho e^{i\theta}}{1 - \rho e^{i\theta}} + \frac{(c-1)\rho e^{i\theta}}{1 + (c-1)\rho e^{i\theta}}. \]

Hence
\[
\lim_{\rho \to 1} \text{Re} \left\{ 1 + \frac{\rho e^{i\theta} \phi''(c, 1; \rho e^{i\theta})}{\phi'(c, 1; \rho e^{i\theta})} \right\} = \frac{1-c}{2} + (c-1) \text{Re} \left\{ \frac{e^{i\theta}}{1 + (c-1)e^{i\theta}} \right\} \\
= \frac{1-c}{2} + (c-1) \frac{c-1 + \cos \theta}{|1 + (c-1)e^{i\theta}|^2} \\
< 0,
\] (1.13)
which implies that the function \( \phi(c, 1; z)(\frac{1}{3} \leq c < \frac{1}{2}) \) is not convex in \( E \).

In view of \( \frac{z}{(1-z)^2} \in S^\star \), we see that
\[ f_c(z) = \phi(c, 1; z) * \frac{z}{(1-z)^2} \in T(1, c). \]

But \( f_c(z) = z\phi'(c, 1; z)(\frac{1}{3} \leq c < \frac{1}{2}) \) is not starlike in \( E \). Thus the counterexample shows that Theorem A is not true when \( a = 1 \) and \( \frac{1}{3} \leq c < \frac{1}{2} \). In [5], the proof of Theorem B used Lemma A, and so its validity is not justified.
Similarly the proof of Theorem C in [5] is not valid. Also the result from Theorem E is not sharp.

In this paper we discuss similar problems and obtain useful results for the class $T(a, c)$.

2. Preliminary Results

To prove our results, we need the following lemmas.

**Lemma 2.1.** ([4, Corollary 4.1]) If $a$, $b$ and $c$ are real and satisfy $-1 \leq a \leq 1$, $b \geq 0$ and $c > 1 + \max\{2 + |a + b - 2|, 1 - (a - 1)(b - 1)\}$, then

$$zF(a, b; c; z) \in S^*,$$

where

$$F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!} \quad (2.2)$$

is the Gaussian hypergeometric function.

Applying Lemma 2.1, we derive the following result.

**Lemma 2.2.** If $a$ and $c$ are real and satisfy $-1 \leq a \leq 1$ and $c > 3 + |a|$, then $\phi(a, c; z)$ defined by (1.2) is convex in $E$.

**Proof.** From (1.2) we have

$$z\phi'(a, c; z) = z + \sum_{n=1}^{\infty} \frac{(n + 1)(a)_n z^{n+1}}{(c)_n}$$

$$= z + \sum_{n=1}^{\infty} \frac{(a)_n(2)_n z^{n+1}}{(c)_n n!}$$

$$= zF(a, 2; c; z). \quad (2.3)$$

Since $-1 \leq a \leq 1$ and $c > 3 + |a|$, it follows from (2.3) and Lemma 2.1 (with $b = 2$) that $z\phi'(a, c; z)$ is starlike in $E$, which leads to $\phi(a, c) \in K$. \hfill $\Box$
Lemma 2.3. ([6]) If \( f \in K \) and \( g \in S^* \), then \( f \ast g \in S^* \).

Let \( f \) and \( g \) be analytic in \( E \). The function \( f \) is subordinate to \( g \), written \( f < g \) or \( f(z) < g(z) \), if \( g \) is univalent in \( E \), \( f(0) = g(0) \) and \( f(E) \subset g(E) \).

Lemma 2.4. ([2]) Let \( \alpha (\alpha \neq 0) \) and \( \beta \) be complex numbers and let \( p \) and \( h \) be analytic in \( E \) with \( p(0) = h(0) \). If \( Q(z) = \alpha h(z) + \beta \) is convex and \( \text{Re} \ Q(z) > 0 \) in \( E \), then

\[
p(z) + \frac{zp'(z)}{\alpha p(z) + \beta} < h(z)
\]

implies that \( p(z) < h(z) \).

Lemma 2.5. ([3]) Let \( \alpha (\alpha \neq 0) \) and \( \beta \) be complex numbers and let \( h \) be analytic and univalent in \( E \) and \( Q(z) = \alpha h(z) + \beta \). Let \( p \) be analytic in \( E \) and satisfy

\[
p(z) + \frac{zp'(z)}{\alpha p(z) + \beta} < h(z) \quad (p(0) = h(0)). \tag{2.4}
\]

If

(i) \( \text{Re} \ Q(z) > 0 \) for \( z \in E \), and
(ii) \( Q \) and \( \frac{1}{Q} \) are convex in \( E \),

then the solution of the differential equation

\[
q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \quad (q(0) = h(0)) \tag{2.5}
\]
is univalent in \( E \) and is the best dominant of (2.4).

Lemma 2.6. ([7]) Let \( \mu \) be a positive measure on the unit interval \([0,1]\). Let \( g(t,z) \) be a function analytic in \( E \) for each \( t \in [0,1] \) and integrable in \( t \) for each \( z \in E \) and for almost all \( t \in [0,1] \), and suppose that \( \text{Re} \ g(t,z) > 0 \) in \( E \), \( g(t,-\rho) \) is real and

\[
\text{Re} \ \frac{1}{g(t,z)} \geq \frac{1}{g(t,-\rho)} \quad (|z| \leq \rho < 1; \ t \in [0,1]).
\]
If \( g(z) = \int_{0}^{1} g(t, z) d\mu(t) \), then
\[
\text{Re} \frac{1}{g(z)} \geq \frac{1}{g(-\rho)} \quad (|z| \leq \rho).
\] (2.6)

3. The Class \( T(a, c) \)

**Theorem 3.1.** Let \( a \) and \( c \) be real and satisfy
\[
c \neq 0, -1 < c \leq 1 \quad \text{and} \quad a > 3 + |c|.
\] (3.1)
Then \( T(a, c) \subset S^* \).

**Proof.** If \( f \in T(a, c) \), then \( L(a, c)f = \phi(a, c) * f \in S^* \). Since \( a \) and \( c \) satisfy (3.1), we have from Lemma 2.2 that \( \phi(c, a) \in K \). Therefore an application of Lemma 2.3 leads to
\[ f = \phi(c, a) * (\phi(a, c) * f) \in S^*. \]
This completes the proof of the theorem. \( \square \)

**Theorem 3.2.** Let \( a \) and \( c \) satisfy (3.1). If \( f \in T(a, c) \), then \( f(E) \) contains the disk
\[
D = \left\{ w : |w| < \frac{a}{2(|c| + a)} \right\}.
\] (3.2)

**Proof.** Let
\[ f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \in T(a, c), \]
where \( a \) and \( c \) satisfy (3.1), and \( w_0 (w_0 \neq 0) \) be any complex number such that \( f(z) \neq w_0 \) for \( z \in E \). Then the function
\[ g(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left( a_2 + \frac{1}{w_0} \right) z^2 + \cdots \]
is analytic and univalent in \( E \) by Theorem 3.1, and hence
\[
\frac{1}{|w_0|} - |a_2| \leq \left| a_2 + \frac{1}{w_0} \right| \leq 2.
\] (3.3)
Since

\[ L(a, c)f(z) = z + \sum_{n=1}^{\infty} (a)_n (c)_n a_{n+1} z^{n+1} \in S^*, \]

we have \( \frac{a_2}{\alpha} \leq 2 \), and it follows from (3.3) that

\[ |w_0| \geq \frac{1}{2 + |a_2|} \geq \frac{a}{2(|c| + a)}. \quad (3.4) \]

This gives the desired result. \( \square \)

**Theorem 3.3.** Let \( a, c \) and \( d \) be real. If

\[ d \neq 0, \ -1 < d \leq 1 \quad \text{and} \quad c > 3 + |d|, \quad (3.5) \]

then \( T(a, d) \subset T(a, c) \).

**Proof.** If \( f \in T(a, d) \), then \( L(a, d)f = \phi(a, d) \ast f \in S^* \). Since \( c \) and \( d \) satisfy (3.5), \( \phi(d, c) \in K \) by Lemma 2.2. Hence it follows from Lemma 2.3 that

\[
L(a, c)f = \phi(a, c) \ast f = (\phi(a, d) \ast \phi(d, c)) \ast f
\]

\[ = \phi(d, c) \ast (\phi(a, d) \ast f) \in S^*, \]

that is, \( f \in T(a, c) \). The proof is complete. \( \square \)

**Theorem 3.4.** Let \( f \in T(a, c) \) and \( \psi \in K \). Then \( \psi \ast f \in T(a, c) \).

**Proof.** Since \( L(a, c)f \in S^* \) and \( \psi \in K \), it follows from Lemma 2.3 that

\[ L(a, c)(\psi \ast f) = \psi \ast L(a, c)f \in S^*. \]

Hence \( \psi \ast f \in T(a, c) \). \( \square \)

In view of Theorem 3.4, we see that the assumption “\( a(a \neq 0) \) and \( c \) are real and satisfy \( c > N(a) \), where \( N(a) \) is defined in the similar way of (1.5)” in Theorem D is redundant.

**Theorem 3.5.** If \( a \geq 1 \), then

\[ T(a + 1, c) \subset T(a, c). \quad (3.6) \]
Proof. It is known that for \( f \in A \),
\[
z(L(a, c)f(z))' = aL(a + 1, c)f(z) - (a - 1)L(a, c)f(z). \tag{3.7}
\]

Let us put
\[
g(z) = \frac{z(L(a, c)f(z))'}{L(a, c)f(z)}. \tag{3.8}
\]
Then \( g(0) = 1 \) and from (3.7) and (3.8) we get
\[
\frac{aL(a + 1, c)f(z)}{L(a, c)f(z)} = g(z) + a - 1. \tag{3.9}
\]
Differentiating both sides of (3.9) logarithmically and using (3.8) we have
\[
\frac{z(L(a + 1, c)f(z))'}{L(a + 1, c)f(z)} = g(z) + \frac{zg'(z)}{g(z) + a - 1}. \tag{3.10}
\]
If \( f \in T(a + 1, c) \), then (3.10) leads to
\[
g(z) + \frac{zg'(z)}{g(z) + a - 1} < \frac{1 + z}{1 - z}. \tag{3.11}
\]
Since \( Q(z) = \frac{1 + z}{1 - z} + a - 1 \) is convex in \( E \) and \( \text{Re} \ Q(z) > a - 1 \geq 0(z \in E) \), it follows from (3.11) and Lemma 2.4 that \( g(z) < \frac{1 + z}{1 - z} \), which is equivalent to \( f \in T(a, c) \). This proves (3.6). \( \square \)

**Theorem 3.6.** Let \( f \in T(a, c) \) and
\[
F(z) = \frac{\beta + 1}{z^\beta} \int_0^z t^{\beta-1} f(t)dt \quad (\beta > 0). \tag{3.12}
\]
Then \( L(a, c)F \in S^*(\sigma(\beta)) \), where
\[
\sigma(\beta) = \left(4 \int_0^1 \frac{t^\beta}{(1 + t)^2}dt \right)^{-1} - \beta. \tag{3.13}
\]
The result is sharp, that is, the order \( \sigma(\beta) \) cannot be increased.

Proof. From (3.12) we have \( F \in A \) and
\[
\beta L(a, c)F(z) + z(L(a, c)F(z))' = (\beta + 1)L(a, c)f(z)
\]
or

\[
\frac{z(L(a,c)F(z))'}{L(a,c)F(z)} + \beta = (\beta + 1)\frac{L(a,c)f(z)}{L(a,c)F(z)}. \tag{3.14}
\]

Differentiating both sides of (3.14) logarithmically we deduce that

\[
p(z) + \frac{zp'(z)}{p(z) + \beta} = \frac{z(L(a,c)f(z))'}{L(a,c)f(z)}, \tag{3.15}
\]

where

\[
p(z) = \frac{z(L(a,c)F(z))'}{L(a,c)F(z)}. \tag{3.16}
\]

Since \( f \in T(a,c) \), it follows from (3.15) that

\[
p(z) + \frac{zp'(z)}{p(z) + \beta} < \frac{1 + z}{1 - z} \quad (p(0) = 1). \tag{3.17}
\]

Taking \( \alpha = 1, \beta > 0, h(z) = \frac{1 + z}{1 - z} \) and \( Q(z) = \frac{1 + z}{1 - z} + \beta \), it is clear that the conditions (i) and (ii) in Lemma 2.5 are satisfied. Thus, by Lemma 2.5, the differential equation

\[
q(z) + \frac{zq'(z)}{q(z) + \beta} = \frac{1 + z}{1 - z} \quad (q(0) = 1) \tag{3.18}
\]

has a univalent solution \( q(z) \),

\[
p(z) < q(z) < \frac{1 + z}{1 - z}, \tag{3.19}
\]

and \( q(z) \) is the best dominant of (3.17). It is easy to verify that the solution \( q(z) \) of (3.18) is

\[
q(z) = \frac{z^{\beta + 1}}{(1 - z)^2 \int_0^z \frac{u^\beta}{(1 - u)^2} du} - \beta
\]

\[
= \left( (1 - z)^2 \int_0^1 \frac{t^\beta}{(1 - tz)^2} dt \right)^{-1} - \beta. \tag{3.20}
\]

It is well known that for \( c_1 > b_1 > 0 \) and \( z \in E \), the Gaussian hyperge-
ometric function defined by (2.2) satisfies

\[ F(a_1, b_1; c_1; z) = \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1 - b_1)} \int_0^1 t^{b_1-1}(1 - t)^{c_1-b_1-1} \frac{(1 - tz)^{a_1}}{(1 - t)^{a_1}} dt \]  \hspace{1cm} (3.21)

and

\[ F(a_1, b_1; c_1; z) = F(b_1, a_1; c_1; z) = (1 - z)^{c_1-a_1-b_1} F(c_1 - b_1, c_1 - a_1; c_1; z). \]  \hspace{1cm} (3.22)

By using (3.21) and (3.22), \( q(z) \) given by (3.20) can be expressed as

\[ q(z) = \frac{\beta + 1}{(1 - z)^2 F(2, \beta + 1; \beta + 2; z)} - \beta \]

\[ = \frac{\beta + 1}{(1 - z) F(1, \beta; \beta + 2; z)} - \beta. \]  \hspace{1cm} (3.23)

From (3.21) and (3.23) we have

\[ q(z) = \frac{1}{g(z)} - \beta, \]  \hspace{1cm} (3.24)

where

\[ g(z) = \int_0^1 g(t, z) d\mu(t), \]

\[ g(t, z) = \frac{1}{\beta + 1} \left( \frac{1 - z}{1 - tz} \right), \quad d\mu(t) = \beta(\beta + 1)t^{\beta-1}(1 - t) dt \quad (\beta > 0). \]  \hspace{1cm} (3.25)

Note that for \( |z| \leq \rho < 1 \) and \( 0 \leq t \leq 1 \),

\[ \text{Re} \frac{1}{g(t, z)} \geq (\beta + 1) \left( \frac{1 + t\rho}{1 + \rho} \right) = \frac{1}{g(t, -\rho)} > 0. \]

Now applying Lemma 2.6, it follows from (3.24) and (3.25) that

\[ \text{Re} q(z) \geq \frac{1}{g(-\rho)} - \beta = \frac{\beta + 1}{\int_0^1 \frac{1 + t\rho}{1 + t\rho} d\mu(t)} - \beta \quad (|z| \leq \rho), \]

which leads to

\[ \inf_{z \in E} \text{Re} q(z) = \frac{1}{g(-1)} - \beta = q(-1). \]  \hspace{1cm} (3.26)
Since $q(z)$ is the best dominant of (3.17), from (3.16)-(3.20) and (3.26) we conclude that

$$\text{Re} \frac{z(L(a, c)F(z))'}{L(a, c)F(z)} > q(-1) = \left(4 \int_0^1 \frac{t^\beta}{(1+t)^2} dt \right)^{-1} - \beta = \sigma(\beta)$$

and the bound $\sigma(\beta)$ cannot be increased. The proof is now complete. \qed

We note that Theorem 3.6 is better than Theorem E by Noor.

**References**


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