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# ON THE PRODUCT AND RATIO OF $t$ AND BESSEL RANDOM VARIABLES

BY

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**Abstract**

The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, the exact distributions of the product $\mid XY \mid$ and the ratio $\mid X/Y \mid$ are derived when $X$ and $Y$ are Student's $t$ and Bessel function random variables distributed independently of each other.

## 1. Introduction

For given random variables $X$ and $Y$, the distributions of the product $\mid XY \mid$ and the ratio $\mid X/Y \mid$ are of interest in many areas of the sciences.

In traditional portfolio selection models certain cases involve the product of random variables. The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, for example, Grubel (1968)) not only are prices of shares in local markets uncertain but also the exchange rates are uncertain so that the value of the portfolio in domestic currency is related to a product of random variables. Similarly in models of diversified production by multinationals (see, for example, Rugman (1979)) there is local production uncertainty and exchange rate uncertainty so that profits in home currency are again related to a product of random variables. An entirely different example is drawn from

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the econometric literature. In making a forecast from an estimated equation Feldstein (1971) pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered as random variables. Hence the forecast was proportional to a product of random variables.

An important example of ratios of random variables is the stress $-$ strength model in the context of reliability. It describes the life of a component which has a random strength $Y$ and is subjected to random stress $X$. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

The distributions of $\mid XY \mid$ and $\mid X/Y \mid$ have been studied by several authors especially when $X$ and $Y$ are independent random variables and come from the same family. With respect to products of random variables, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's $t$ family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, AbuSalih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). With respect to ratios of random variables, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's $t$ family, Basu and Lochner (1971) and Ali et al. (2005) for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for noncentral chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family.

However, there is relatively little work of the above kind when $X$ and $Y$ belong to different families. In the applications mentioned above, it is quite possible that $X$ and $Y$ could arise from different but similar distributions (see below for examples). In a recent paper, Nadarajah and Gupta (2005) studied the exact distributions of products of random variables when $X$ has the logistic distribution and $Y$ comes from similar but different distributions. In this paper, we study the exact distributions of $\mid XY \mid$ and $\mid X/Y \mid$ when $X$

and $Y$ are independent random variables having the Student's $t$ and Bessel function distributions with pdfs

$$f(x) = \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \tag{1}$$

and

$$f(y) = \frac{\mid y \mid^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} K_m\left(\left|\frac{y}{b}\right|\right) \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\nu > 0$, $b > 0$ and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi} x^m}{2^m \Gamma(m+1/2)} \int_1^\infty \left(t^2 - 1\right)^{m-1/2} \exp(-xt)\, dt$$

is the modified Bessel function of the second kind.

Student's $t$ and Bessel function distributions have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. Both are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data. Some examples are (see Kotz *et al* (2001) for further applications):

1. in communication theory, $X$ and $Y$ could represent the random noise corresponding to two different signals.
2. in ocean engineering, $X$ and $Y$ could represent distributions of navigation errors.
3. in finance, $X$ and $Y$ could represent distributions of log-returns of two different commodities.
4. in image and speech recognition, $X$ and $Y$ could represent "input" distributions.

In each of the examples above, it will be of interest to study the distribution of the ratio $\mid X/Y \mid$. For example, in communication theory, $\mid X/Y \mid$ could represent the relative strength of the two different signals. In ocean engineering, $\mid X/Y \mid$ could represent the relative safety of navigation. In finance, $\mid X/Y \mid$ could represent the relative popularity of the two different commodities. The distribution of the product $\mid XY \mid$ is considered here for completeness.

The exact expressions for the distributions of $\mid XY \mid$ and $\mid X/Y \mid$ are given in Sections 2 and 3. Nadarajah and Kotz (2003) have shown that the cdf corresponding to (1) can be expressed as

$$F(x)=\begin{cases} \frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x}{\sqrt{\nu}}\right)+\frac{1}{2\pi}\sum_{l=1}^{(\nu-1)/2} B\left(l,\frac{1}{2}\right)\frac{\nu^{l-1/2}x}{\left(\nu+x^2\right)^{l}}, & \text{if } \nu \text{ is odd}, \\ \frac{1}{2}+\frac{1}{2\pi}\sum_{l=1}^{\nu/2} B\left(l-\frac{1}{2},\frac{1}{2}\right)\frac{\nu^{l-1}x}{\left(\nu+x^2\right)^{l-1/2}}, & \text{if } \nu \text{ is even} \end{cases} \tag{3}$$

and this result will be crucial for the calculations. The calculations involve several special functions, including the Euler psi function defined by

$$\Psi(x) = \frac{d\log\Gamma(x)}{dx},$$

the Struve function defined by

$$H_\nu(x) = \frac{2x^{\nu+1}}{\sqrt{\pi}2^{\nu+1}\Gamma\left(\nu+3/2\right)}\sum_{k=0}^{\infty}\frac{1}{(3/2)_k(\nu+3/2)_k}\left(-\frac{x^2}{4}\right)^k,$$

the Bessel function of the first kind defined by

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma\left(\nu+1\right)}\sum_{k=0}^{\infty}\frac{1}{(\nu+1)_k k!}\left(-\frac{x^2}{4}\right)^k,$$

the hypergeometric functions defined by

$$G\left(a;x\right) = \sum_{k=0}^{\infty}\frac{1}{(a)_k}\frac{x^k}{k!}$$

and

$$H\left(a;b,c;x\right) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k(c)_k}\frac{x^k}{k!},$$

and the Lommel functions defined by

$$s_{\mu,\nu}\left(x\right) = \frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}H\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{x^2}{4}\right)$$

and

$$\begin{aligned} S_{\mu,\nu}(x) &= s_{\mu,\nu}(x) + \frac{2^{\mu+\nu-1}\Gamma(\nu)\Gamma\left((\mu+\nu+1)/2\right)}{x^{\nu}\Gamma\left((1+\nu-\mu)/2\right)} G\left(1-\nu; -\frac{x^2}{4}\right) \\ &\quad + \frac{2^{\mu-\nu-1}x^{\nu}\Gamma(-\nu)\Gamma\left((\mu-\nu+1)/2\right)}{\Gamma\left((1-\nu-\mu)/2\right)} G\left(1+\nu; -\frac{x^2}{4}\right), \end{aligned}$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

**Lemma 1.**(Equation (2.16.3.13), Prudnikov et al., 1986, volume 2) *For* $c > 0$, $z > 0$ *and* $\alpha > \nu$,

$$\begin{aligned} \int_0^{\infty} \frac{x^{\alpha-1}}{\left(x^2+z^2\right)^{\rho}} K_{\nu}(cx)dx &= 2^{\nu-2}c^{-\nu}z^{\alpha-2\rho-\nu}\Gamma(\nu)B\left(\rho+\frac{\nu-\alpha}{2}, \frac{\alpha-\nu}{2}\right) \\ &\quad \times H\left(\frac{\alpha-\nu}{2}; 1-\nu, 1-\rho+\frac{\alpha-\nu}{2}; -\frac{c^2z^2}{4}\right) \\ &\quad + 2^{-(\nu+2)}c^{\nu}z^{\alpha-2\rho+\nu}\Gamma(-\nu)B\left(\rho-\frac{\nu+\alpha}{2}, \frac{\alpha+\nu}{2}\right) \\ &\quad \times H\left(\frac{\alpha+\nu}{2}; 1+\nu, 1-\rho+\frac{\alpha+\nu}{2}; -\frac{c^2z^2}{4}\right) \\ &\quad + 2^{\alpha-2\rho-2}c^{2\rho-\alpha}\Gamma\left(\frac{\alpha+\nu}{2}-\rho\right)\Gamma\left(\frac{\alpha-\nu}{2}-\rho\right) \\ &\quad \times H\left(\rho; 1+\rho-\frac{\alpha+\nu}{2}; 1+\rho-\frac{\alpha-\nu}{2}; -\frac{c^2z^2}{4}\right). \end{aligned}$$

**Lemma 2.**(Equation (2.16.3.14), Prudnikov et al., 1986, volume 2) *For* $c > 0$, $z > 0$ *and* $\nu > -1$,

$$\int_0^{\infty} \frac{x^{\nu+1}}{\left(x^2+z^2\right)^{\rho}} K_{\nu}(cx)dx = (2z)^{\nu}\left(z/c\right)^{1-\rho}\Gamma(\nu+1)S_{-\nu-\rho,1+\nu-\rho}(cz).$$

Further properties of the above special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

## 2. Product

Theorem 1 derives an explicit expression for the cdf of $\mid XY \mid$ in terms of the hypergeometric function.

**Theorem 1.** *Suppose $X$ and $Y$ are distributed according to* (1) *and* (2), *respectively. If $\nu$ is an odd integer then the cdf of $Z =\mid XY \mid$ can be expressed as*

$$F(z) = I(\nu) + \frac{1}{\pi^{3/2}\sqrt{\nu}2^{m-1}b^{m+1}\Gamma(m+1/2)} \sum_{k=1}^{(\nu-1)/2} B\left(k, \frac{1}{2}\right) A(k), \tag{4}$$

*where $I(\cdot)$ denotes the integral*

$$I(a) = \frac{1}{\pi^{3/2}2^{m-2}b^{m+1}\Gamma(m+1/2)} \int_0^\infty \arctan\left(\frac{z}{\sqrt{a}y}\right) y^m K_m\left(\frac{y}{b}\right) dy, \tag{5}$$

$$\begin{aligned} A(k) &= 2^{-(m+2)}(\nu b)^{-m} z^{2m}\Gamma(-m)B(-m, m+k) \\ &\quad \times H\left(m+k; 1+m, 1+m; -\frac{z^2}{4\nu b^2}\right) \\ &\quad - \left\{2C + \Psi(k)\right\} 2^{m-2} b^m \Gamma(m) H\left(k; 1-m, 1; -\frac{z^2}{4\nu b^2}\right), \end{aligned} \tag{6}$$

*and $C$ denotes the Euler's constant.*

*Proof.* The cdf $F(z) = \Pr(\mid XY \mid \leq z)$ can be expressed as

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+\frac{1}{2})} \int_{-\infty}^{\infty} \left\{F\left(\frac{z}{\mid y\mid}\right) - F\left(-\frac{z}{\mid y \mid}\right)\right\} \mid y \mid^m K_m\left(\left|\frac{y}{b}\right|\right) dy \\ &= \frac{1}{\sqrt{\pi}2^{m-1}b^{m+1}\Gamma(m+\frac{1}{2})} \int_0^\infty \left\{F\left(\frac{z}{y}\right) - F\left(-\frac{z}{y}\right)\right\} y^m K_m\left(\frac{y}{b}\right) dy, \end{aligned} \tag{7}$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's $t$ random variable with degrees of freedom $\nu$. Substituting the form for $F$ given by (3) for odd degrees of freedom, (7) can be reduced to

$$F(z) = I(\nu) + \frac{1}{\pi^{3/2}\sqrt{\nu}2^{m-1}b^{m+1}\Gamma(m+1/2)} \sum_{k=1}^{(\nu-1)/2} B\left(k, \frac{1}{2}\right) J(k), \tag{8}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+2k-1} K_m(y/b)}{\left(y^2 + z^2/\nu\right)^k} dy.$$

By direct application of Lemma 1, one can easily see that $J(k) = A(k)$, where $A(k)$ is given by (6). The result of the theorem follows by substituting this form for $J(k)$ into (8). □

Theorem 2 is the analogue of Theorem 1 for the case when the degrees of freedom $\nu$ is an even integer.

**Theorem 2.** *Suppose $X$ and $Y$ are distributed according to* (1) *and* (2), *respectively. If $\nu$ is an even integer then the cdf of $Z =\mid XY \mid$ can be expressed as*

$$F(z) = \frac{1}{\pi^{3/2}\sqrt{\nu}2^{m-1}b^{m+1}\Gamma\left(m+1/2\right)} \sum_{k=1}^{\nu/2} B\left(k-\frac{1}{2}, \frac{1}{2}\right) A(k), \qquad (9)$$

*where*

$$\begin{aligned} A(k) &= 2^{-(m+2)}(\nu b)^{-m} z^{2m}\Gamma(-m) B\left(-m, m+k-\frac{1}{2}\right) \\ &\quad \times H\left(m+k-\frac{1}{2}; 1+m, 1+m; -\frac{z^2}{4\nu b^2}\right) \\ &\quad - \left\{2C+\Psi\left(k-\frac{1}{2}\right)\right\} 2^{m-2} b^m \Gamma(m) H\left(k-\frac{1}{2}; 1-m, 1; -\frac{z^2}{4\nu b^2}\right), \qquad (10) \end{aligned}$$

*and $C$ denotes the Euler's constant.*

*Proof.* Substituting the form for $F$ given by (3) for even degrees of freedom, (7) can be reduced to

$$F(z) = \frac{1}{\pi^{3/2}\sqrt{\nu}2^{m-1}b^{m+1}\Gamma\left(m+1/2\right)} \sum_{k=1}^{\nu/2} B\left(k-\frac{1}{2}, \frac{1}{2}\right) J(k), \qquad (11)$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+2k-2} K_m\left(y/b\right)}{\left(y^2+z^2/\nu\right)^{k-1/2}} dy.$$

By direct application of Lemma 1, one can easily see that $J(k) = A(k)$, where $A(k)$ is given by (10). The result of the theorem follows by substituting this form for $J(k)$ into (11). □

Figure 1 illustrates possible shapes of the pdf of $\mid XY \mid$ for $\nu = 5$ and a range of values of $m$. Note that the shapes are unimodal and that the value of $m$ largely dictates the behavior of the pdf near $z = 0$.

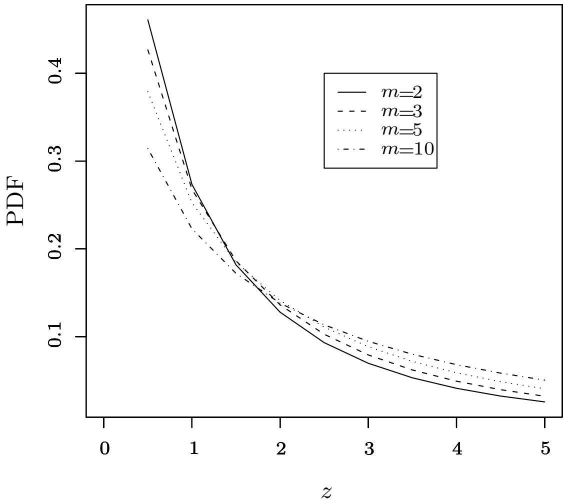


**Figure 1.** Plots of the pdf of (4) for $\nu = 5$ and $m = 2, 3, 5, 10$.

## 3. Ratoio

Theorem 3 derives an explicit expression for the cdf of $\mid X/Y \mid$ in terms of the Lommel function.

**Theorem 3.** *Suppose $X$ and $Y$ are distributed according to* (1) *and* (2), *respectively. If $\nu$ is an odd integer then the cdf of $Z = \mid X/Y \mid$ can be expressed as*

$$F(z) = I(\nu) + \frac{2\nu^{m/2}\Gamma(m+1)}{\pi^{\frac{3}{2}} b^{m+1}\Gamma(m+\frac{1}{2}) z^m} \sum_{k=1}^{(\nu-1)/2} \frac{\nu^{\frac{k}{2}} B\left(k, \frac{1}{2}\right)}{b^{1-k} z^k} S_{-(m+k),1+m-k}\left(\frac{\sqrt{\nu}}{bz}\right), \tag{12}$$

*where $I(\cdot)$ denotes the integral*

$$I(a) = \frac{1}{\pi^{\frac{3}{2}} 2^{m-2} b^{m+1} \Gamma(m+\frac{1}{2})} \int_0^\infty \arctan\left(\frac{zy}{\sqrt{a}}\right) y^m K_m\left(\frac{y}{b}\right) dy. \tag{13}$$

*Proof.* The cdf $F(z) = \Pr(\mid X/Y \mid \leq z)$ can be expressed as

$$
\begin{aligned}
F(z) &= \frac{1}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+\frac{1}{2})} \int_{-\infty}^{\infty} \{F(z \mid y \mid) - F(-z \mid y \mid)\} \mid y \mid^m K_m\left(\left|\frac{y}{b}\right|\right) dy \\
&= \frac{1}{\sqrt{\pi}2^{m-1} b^{m+1}\Gamma(m+\frac{1}{2})} \int_{0}^{\infty} \{F(zy) - F(-zy)\} y^m K_m\left(\frac{y}{b}\right) dy, \quad (14)
\end{aligned}
$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's $t$ random variable with degrees of freedom $\nu$. Substituting the form for $F$ given by (3) for odd degrees of freedom, (14) can be reduced to

$$
F(z) = I(\nu) + \frac{2z}{\pi^{3/2}\sqrt{\nu}2^m b^{m+1}\Gamma(m+\frac{1}{2})} \sum_{k=1}^{(\nu-1)/2} \nu^k z^{-2k} B\left(k, \frac{1}{2}\right) J(k), \quad (15)
$$

where $J(k)$ denotes the integral

$$
J(k) = \int_0^{\infty} \frac{y^{m+1} K_m(y/b)}{\left(y^2 + a/z^2\right)^k} dy. \quad (16)
$$

By direct application of Lemma 2, one can calculate (16) as

$$
J(k) = 2^m m! \nu^{(m-k+1)/2} b^{k-1} z^{k-m-1} S_{-(m+k),1+m-k}\left(\frac{\sqrt{\nu}}{bz}\right). \quad (17)
$$

The result of the theorem follows by substituting (17) into (15). □

Theorem 4 is the analogue of Theorem 3 for the case when the degrees of freedom $\nu$ is an even integer.

**Theorem 4.** *Suppose $X$ and $Y$ are distributed according to* (1) *and* (2), *respectively. If $\nu$ is an even integer then the cdf of $Z = \mid X/Y \mid$ can be expressed as*

$$
F(z) = \frac{2\nu^{m/2}\Gamma(m+1)}{\pi^{3/2} b^{m+1}\Gamma(m+\frac{1}{2}) z^m} \sum_{k=1}^{\nu/2} \frac{\nu^{k/2} B(k-\frac{1}{2}, \frac{1}{2})}{b^{3/2-k} z^k} S_{1/2-m-k,3/2+m-k}\left(\frac{\sqrt{\nu}}{bz}\right). \quad (18)
$$

*Proof.* Substituting the form for $F$ given by (3) for odd degrees of freedom, (14) can be reduced to

$$F(z)=\frac{2z}{\pi^{3/2}\sqrt{\nu}2^m b^{m+1}\Gamma(m+\frac{1}{2})}\sum_{k=1}^{\nu/2}\nu^{k-1/2}z^{1-2k}B\left(k-\frac{1}{2},\frac{1}{2}\right)J(k), \qquad (19)$$

where $J(k)$ denotes the integral

$$J(k) \;=\; \int_0^\infty \frac{y^{m+1}K_m\left(y/b\right)}{\left(y^2+a/z^2\right)^{k-1/2}}dy. \qquad (20)$$

By direct application of Lemma 2, one can calculate (20) as

$$J(k)=2^m m!\nu^{(m-k+3/2)/2}b^{k-3/2}z^{k-m-3/2}S_{1/2-m-k,3/2+m-k}\left(\frac{\sqrt{\nu}}{bz}\right). \qquad (21)$$

The result of the theorem follows by substituting (21) into (19). □

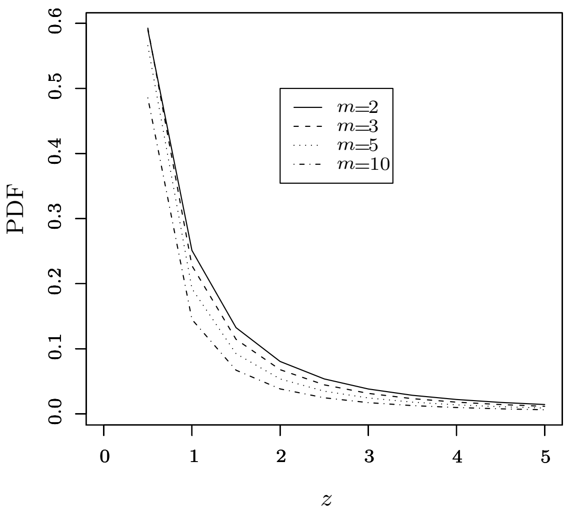


**Figure 2.** Plots of the pdf of (18) for $\nu = 10$ and $m = 2, 3, 5, 10$.

Figure 2 illustrates possible shapes of the pdf of $\mid X/Y \mid$ for $\nu = 10$ and a range of values of $m$. Note that the shapes are unimodal and that the value of $m$ largely dictates the behavior of the pdf near $z = 0$.

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## References

1. M. S. AbuSalih, Distributions of the product and the quotient of power-function random variables, *Arab J. Math.*, **4**(1983), 77-90.

2. M. M. Ali, S. Nadarajah and J. Woo, On the ratio $X/(X+Y)$ for Weibull and Levy distributions, *J. Korean Statist. Soc.*, **34**(2005), 11-20.

3. A. P. Basu and R. H. Lochner, On the distribution of the ratio of two random variables having generalized life distributions, *Technometrics*, **13**(1971), 281-287.

4. R. P. Bhargava and C. G. Khatri, The distribution of product of independent beta random variables with application to multivariate analysis, *Ann. Inst. Statist. Math.*, **33**(1981), 287-296.

5. M. S. Feldstein, The error of forecast in econometric models when the forecast-period exogenous variables are stochastic. *Econometrica*, **39**(1971), ??-??.

6. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (sixth edition), Academic Press, San Diego, 2000.

7. H. G. Grubel, Internationally diversified portfolios: welfare gains capital flows, *Amer. Econom. Rev.*, **58**(1968), ??-??.

8. H. L. Harter, On the distribution of Wald's classification statistic, *Ann. Math. Statist.*, **22**(1951), 58-67.

9. D. L. Hawkins and C. -P. Han, Bivariate distributions of some ratios of independent noncentral chi-square random variables, *Comm. Statist. Theory Methods*, **15**(1986), 261-277.

10. P. J. Korhonen and S. C. Narula, The probability distribution of the ratio of the absolute values of two normal variables, *J. Statist. Comput. Simulation*, **33**(1989), 173-182.

11. S. Kotz, T. J. and K. Podgorski, *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications*, Economics, Engineering and Finance, Birkhauser, Boston, 2001.

12. H. J. Malik and R. Trudel, Probability density function of the product and quotient of two correlated exponential random variables, *Canad. Math. Bull.*, **29**(1986), 413-418.

13. G. Marsaglia, Ratios of normal variables and ratios of sums of uniform variables, *J. Amer. Statist. Assoc.*, **60**(1965), 193-204.

14. S. Nadarajah and A. K.Gupta, Products of random variables with logistic kernel. *Bull. Inst. Math. Acad. Sinica*, **33**(2005), 381-395.

15. S. Nadarajah and S. Kotz, Skewed distributions generated by the normal kernel, *Statist. Probab. Lett.*, **65**(2003), 269-277.

16. T. Pham-Gia, Distributions of the ratios of independent beta variables and applications, *Comm. Statist. Theory Methods*, **29**(2000), 2693-2715.

17. H. Podolski, The distribution of a product of $n$ independent random variables with generalized gamma distribution, *Demonstratio Math.*, **4**(1972), 119-123.

18. S. J. Press, The $t$ ratio distribution. *J. Amer. Statist. Assoc.* , **64**(1969), 242-252.

19. S. B. Provost, On the distribution of the ratio of powers of sums of gamma random variables. *Pakistan J. Statist.*, **5**(1989), 157-174.

20. A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, *Integrals and Series* (volumes 1, 2 and 3), Gordon and Breach Science Publishers, Amsterdam, 1986.

21. P. N. Rathie and H. G. Rohrer, The exact distribution of products of independent random variables, *Metron*, **45**(1987), 235-245.

22. A. M. Rugman, *International Diversification and the Multinational Enterprise*, Lexington, 1979.

23. H. Sakamoto, On the distributions of the product and the quotient of the independent and uniformly distributed random variables, *Tohoku Math. J.*, **49**(1943), 243-260.

24. S. M. Shcolnick, On the ratio of independent stable random variables, *Stability Problems for Stochastic Models* (Uzhgorod, 1984), 349-354, Lecture Notes in Mathematics, **1155**, Springer, Berlin, 1985.

25. M. D. Springer and W. E. Thompson, The distribution of products of beta, gamma and Gaussian random variables, *SIAM J. Appl. Math.*, **18**(1970), 721-737.

26. B. M. Steece, On the exact distribution for the product of two independent beta-distributed random variables, *Metron*, **34**(1976), 187-190.

27. A. Stuart, Gamma-distributed products of independent random variables, *Biometrika*, **49**(1962), 564-565.

28. J. Tang and A. K. Gupta, On the distribution of the product of independent beta random variables, *Statist. Probab. Lett.*, **2**(1984), 165-168.

29. C. M. Wallgren, The distribution of the product of two correlated $t$ variates, *J. Amer. Statist. Assoc.*, **75**(1980), 996-1000.

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