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ON NEIGHBORHOODS OF STRONGLY STARLIKE FUNCTIONS OF ORDER α AND TYPE β WITH RESPECT TO SYMMETRIC POINTS

BY

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Abstract

In this paper we introduce a class of strongly starlike functions of order α and type β with respect to symmetric points and investigate the neighborhoods and coefficients bounds of such functions.

1. Introduction

Let $H(D)$ denote the class of all functions f holomorphic in the open unit disc D in C and A be the class of all functions $f \in H(D)$ with the normalizations $f(0) = 0$ and $f'(0) = 1$. Any $f \in A$ is said to be strongly starlike of order α, $0 < \alpha \leq 1$ if it satisfies

$$\left|\text{Arg}\ \frac{zf'(z)}{f(z)}\right| < \frac{\pi\alpha}{2}\,, \quad z \in D.$$

This class $S^*(\alpha)$ of strongly starlike functions of order α was introduced by D.A. Brannan and W. E. Kirwan [2] and in dependently by J. Stankiewicz [8].

In this paper we introduce a class of strongly starlike functions of order α and type β with respect to symmetric points. We investigate some interesting

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properties of this class, e.g., the neighborhoods of such functions, coefficients bounds, and the criteria for a function to be in this class.

Definition 1.1. Let $f \in A$. For $0 < \alpha \leq 1$ and $0 \leq \beta < 1$ in D, f is said to be in the class $\overline{S}^*_{\alpha,\beta}(\alpha)$ of strongly starlike functions of order α and type β with respect to symmetric points if

$$\left|\arg\left(\frac{2zf'(z)}{f(z)-f(-z)}-\beta\right)\right| < \frac{\alpha\pi}{2} \text{ in } D.$$

Clearly, $\overline{S}^*_{s,0}(\alpha) = \overline{S}^*_{s}(\alpha)$ the class of all strongly starlike functions of order α with respect to symmetric points which was introduced by R. Parvatham and M. Premabai [4].

Also, $\overline{S}^*_{s,0}(1) = S^*_s$ the class of all starlike functions with respect to symmetric points which was introduced by K. Sakaguchi [7].

$f \in \overline{S}^*_{s,\beta}(\alpha)$ means that the image of D under $\left(\frac{2zf'(z)}{f(z)-f(-z)}-\beta\right)$ lies in the region Ω defined by

$$\Omega = \left\{z \in C : |\arg z| < \frac{\alpha\pi}{2}\,, \quad 0 < \alpha \leq 1\right\}.$$

Equivalently, $f \in \overline{S}^*_{s,\beta}(\alpha)$ if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} \neq \beta + te^{i\alpha\pi/2}\,, \quad t \in R^+.$$

Or $f \in \overline{S}^*_{s,\beta}(\alpha)$ if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} = (1-\beta)p(z)^{\alpha} + \beta,$$

where $p \in P$, the class of all functions $p(z)$ analytic in D for which $\text{Re}\{p(z)\} > 0$ and $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, $0 < \alpha \leq 1$, $0 \leq \beta < 1$.

Definition 1.2. Any $f \in A$ is said to be strongly convex of order α and type β with respect to symmetric points in D if $\forall\ z \in D$, $\left|\arg\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}-\beta\right)\right| < \frac{\alpha\pi}{2}$, $0 < \alpha \leq 1$, $0 \leq \beta < 1$. Let $\overline{K}_{s,\beta}(\alpha)$ be the class of all strongly convex functions of order α and type β with respect to symmetric points.

Note. $\overline{K}_{s,0}(\alpha) = K_s(\alpha)-$the class of all strongly convex functions of order α with respect to symmetric points which was introduced by R. Parvatham and M. Premabai [4].

Also, $\overline{K}_{s,0}(1) = K_s-$the class of convex functions with respect to symmetric points which was introduced by R.N. Das and P. Singh [3].

The relation between the classes $\overline{S}_{s,\beta}(\alpha)$ and $\overline{K}_{s,\beta}(\alpha)$ is given as

$$f \in \overline{K}_{s,\beta}(\alpha) \Leftrightarrow zf'(z) \in \overline{S}^*_{s,\beta}(\alpha).$$

Any $f \in A$ has the Taylor's expansion $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in D.

2. Main Results

In order to derive our first result for the coefficients bounds of the class $\overline{S}^*_{s,\beta}(\alpha)$, we need the following lemma due to Pommerenke [1].

Lemma 2.1. *If* $p(z) = 1 + c_1z + c_2z^2 + \cdots \in P$, *then*

$$|c_k| \leq 2. \tag{1}$$

Theorem 2.1. *Let* $f(z) = z + \sum_{k=2}^{\infty} a_kz^k$ *belong to* $\overline{S}^*_{s,\beta}(\alpha)$ $(0 < \alpha \leq 1, 0 \leq \beta < 1)$. *Then*

$$\begin{aligned} |a_2| &\leq \alpha(1-\beta) \\ |a_3| &\leq \alpha^2(1-\beta). \end{aligned}$$

The result is sharp.

Proof. For $f(z) \in \overline{S}^*_{s,\beta}(\alpha)$, there is $p(z) \in P$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = (1-\beta)p(z)^{\alpha} + \beta.$$

Assume that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then direct calculation gives us

$$2a_2 = \alpha(1-\beta)p_1$$

and

$$2a_3 = \alpha(1-\beta)p_2 + \frac{\alpha(\alpha-1)}{2}(1-\beta)p_1^2. \tag{2}$$

By using Lemma 2.1, we get that

$$\begin{aligned} |a_2| &\leq \alpha(1-\beta) \\ |a_3| &\leq \alpha^2(1-\beta)\ ; \end{aligned}$$

For a_2, equality holds if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} = (1-\beta)\left(\frac{1+\epsilon z}{1-\epsilon z}\right)^{\alpha} + \beta, \quad |\epsilon| = 1,$$

and for a_3, equality holds if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} = (1-\beta)\left(\frac{1+\epsilon z^2}{1-\epsilon z^2}\right)^{\alpha} + \beta, \quad |\epsilon| = 1.$$

This completes our proof. □

Now, let us see a characterization formula for f to be in $\overline{S}^*_{s,\beta}(\alpha)$ by means of convolution. For this, we need to define convolution of $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ as $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$.

Definition 2.1. Let $S^{*'}_{s,\beta}(\alpha)$ be the class of all functions $h(z)$ such that

$$h(z) = \frac{f_2(z) - (te^{\pm i\alpha\frac{\pi}{2}} + \beta)f_3(z)}{1 - (te^{\pm i\alpha\frac{\pi}{2}} + \beta)}\ ; \quad t \in R^+,$$

where $f_2(z) = \dfrac{z}{(1-z)^2}$ and $f_3(z) = \dfrac{z}{1-z^2}$.

Clearly, $f(z) = f(z) * f_1(z)$, $zf'(z) = f(z) * f_2(z)$ and $\dfrac{f(z)-f(-z)}{2} = f(z) * f_3(z)$, where $f_1(z) = \dfrac{z}{1-z}$, $f_2(z) = \dfrac{z}{(1-z)^2}$ and $f_3(z) = \dfrac{z}{1-z^2}$.

Note that $S^{*'}_{s,0}(\alpha) = S^{*'}_{s}(\alpha)$ – the class which was introduced by R. Parvatham and M. Premabai [4].

The characterization formula for f to be in $\overline{S}^{*}_{s,\beta}(\alpha)$ is given in the following theorem

Theorem 2.2. *$f \in \overline{S}^{*}_{s,\beta}(\alpha)$ if and only if $\forall\ H \in S^{*'}_{s,\beta}(\alpha)$ and $\forall\ z \in D$, $\dfrac{(f*H)(z)}{z} \neq 0$.*

Proof. Let us first assume that for $f \in A$, $\dfrac{(f*H)(z)}{z} \neq 0\ \ \forall\, H \in S^{*'}_{s,\beta}(\alpha)$ and $\forall\ z \in D$.

From the definition of $H(z)$, it follows that

$$\begin{aligned}\frac{(f*H)(z)}{z} &= \frac{(f*f_2)(z) - (te^{\pm i\alpha\frac{\pi}{2}}+\beta)(f*f_3)(z)}{[1-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)]z}\\ &= \frac{zf'(z) - (te^{\pm i\alpha\frac{\pi}{2}}+\beta)\left(\frac{f(z)-f(-z)}{2}\right)}{\left[1-(te^{\pm\alpha\frac{\pi}{2}}+\beta)\right]z} \neq 0, \quad t \in R^{+}.\end{aligned}$$

Equivalently, $\dfrac{2zf'(z)}{f(z)-f(-z)} \neq te^{\pm i\alpha\frac{\pi}{2}}+\beta, \quad t \in R^{+}$; or $\dfrac{2zf'(z)}{f(z)-f(-z)} - \beta \neq te^{\pm i\alpha\frac{\pi}{2}}$, $t \in R^{+}$. As $t \in R^{+}$, $te^{\pm i\alpha\frac{\pi}{2}}$ covers the half lines $|\arg\omega| = \dfrac{\alpha\pi}{2}$ and $\dfrac{1}{1-\beta}\left(\dfrac{2zf'(z)}{f(z)-f(-z)} - \beta\right) = 1$ at $z = 0$. Hence

$$\frac{2zf'(z)}{f(z)-f(-z)} - \beta \in \Omega = \left\{z \in C : |argz| < \frac{\alpha\pi}{2}\right\};$$

or $f \in \overline{S}^{*}_{s,\beta}(\alpha)$.

Conversely, let $f \in \overline{S}^{*}_{s,\beta(\alpha)}$. Then

$$\frac{2zf'(z)}{f(z)-f(-z)} - \beta \neq te^{\pm i\alpha\frac{\pi}{2}}. \tag{3}$$

Now,

$$\begin{aligned}\frac{(f*H)(z)}{z} &= \frac{(f*f_2)(z)-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)(f*f_3)(z)}{[1-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)]z}\\ &= \left(\frac{2zf'(z)}{f(z)-f(-z)}-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)\right)\cdot\frac{f(z)-f(-z)}{2z[1-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)]}\,.\end{aligned}$$

(3) gives $\dfrac{(f*H)(z)}{z}\neq 0$ in D which completes the proof of the theorem. □

The notion of δ-neighborhood was first introduced by St. Ruscheweyh [5].

Definition 2.2. For $\delta\geq 0$, the δ-neighborhood of $f(z)=z+\sum_{k=2}^{\infty}a_kz^k\in A$ is defined by

$$N_\delta(f)=\left\{g(z)=z+\sum_{k=2}^{\infty}b_kz^k;\sum_{k=2}^{\infty}k|a_k-b_k|\leq\delta\right\}.$$

To investigate the δ-neighborhoods of functions belonging to the class $\overline{S}^*_{s,\beta}(\alpha)$, we need the following lemmas:

Lemma 2.2. *Let* $H(z)=z+\sum_{n=2}^{\infty}h_nz^n\in S^{*'}_{s,\beta}(\alpha)$. *Then* $|h_n|<\dfrac{\sqrt{(n-\beta)^2+2\beta}}{(1-\beta)\sin\frac{\alpha\pi}{2}}$ *for* $n=2,3,\ldots$.

Proof. Since $H(z)\in S^{*'}_{s,\beta}(\alpha)$, we have

$$\begin{aligned}H(z) &= \frac{1}{1-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)}\left[\frac{z}{(1-z)^2}-\left(te^{\pm i\alpha\frac{\pi}{2}}+\beta\right)\frac{z}{1-z^2}\right]\\ &= \frac{1}{1-(te^{\pm i\alpha\frac{\pi}{2}}\beta)}[(z+2z^2+3z^3+\cdots+nz^n+\cdots)\\ &\quad -(te^{\pm i\alpha\frac{\pi}{2}}+\beta)(z+z^3+z^5+\cdots+z^{2n+1}+\cdots)]\\ &= z+\sum_{n=2}^{\infty}h_nz^n.\end{aligned}$$

Then comparing the coefficients on either side, we get

$$h_n = \begin{cases} \dfrac{n}{1-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)} & \text{when } n \text{ is even,} \\ \dfrac{n-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)}{1-(te^{\pm i\alpha\frac{\pi}{2}}+\beta)} & \text{when } n \text{ is odd.} \end{cases}$$

Hence when n is odd,

$$\begin{aligned} |h_n|^2 &= \frac{\left[n-\left(\beta+t\cos\alpha\frac{\pi}{2}\right)\right]^2+t^2\sin^2\frac{\alpha\pi}{2}}{\left(1-\left(\beta+t\cos\alpha\frac{\pi}{2}\right)\right)^2+t^2\sin^2\frac{\alpha\pi}{2}} \\ &= \frac{n^2-2n\beta+\beta^2+2(\beta-n)t\cos\frac{\alpha\pi}{2}+t^2}{1-2\beta+\beta^2+2(\beta-1)t\cos\frac{\alpha\pi}{2}+t^2} \\ &= 1+\frac{(n^2-1)-2(n-1)\beta-2(n-1)t\cos\frac{\alpha\pi}{2}}{1-2\beta+\beta^2+2(\beta-1)t\cos\frac{\alpha\pi}{2}+t^2} \\ &= 1+\frac{(n-1)\left\{(n+1-2\beta)-2t\cos\frac{\alpha\pi}{2}\right\}}{(1-\beta)^2+2(\beta-1)t\cos\frac{\alpha\pi}{2}+t^2} \\ &\le 1+\frac{(n-1)(n+1-2\beta)}{(1-\beta)^2+2(\beta-1)t\cos\frac{\alpha\pi}{2}+t^2} \\ &\le \max_t\left\{1+\frac{(n-1)(n+1-2\beta)}{(1-\beta)^2\left[1-\frac{2t}{1-\beta}\cos\frac{\alpha\pi}{2}+\left(\frac{t}{1-\beta}\right)^2\right]}\right\} \\ &= 1+\frac{(n-1)(n+1-2\beta)}{(1-\beta)^2\sin^2\frac{\alpha\pi}{2}} \quad \text{since } t\ge 0. \end{aligned}$$

Therefore

$$\begin{aligned} |h_n| &\le \frac{\sqrt{n^2-\cos^2\frac{\alpha\pi}{2}+\beta^2\sin^2\frac{\alpha\pi}{2}-2\beta\left(n-\cos^2\frac{\alpha\pi}{2}\right)}}{(1-\beta)\sin\frac{\alpha\pi}{2}} \\ &< \frac{\sqrt{(n-\beta)^2+2\beta}}{(1-\beta)\sin\frac{\alpha\pi}{2}}. \end{aligned}$$

□

Lemma 2.3. *For $f\in A$ and for every $\epsilon\in C$ such that $|\epsilon|<\delta$, if $F_\epsilon(z)=\dfrac{f(z)+\epsilon z}{1+\epsilon}\in\overline{S}^*_{s,\beta}(\alpha)$, then for every $H\in S^{*'}_{s,\beta}(\alpha)$, $\left|\dfrac{(f*H)(z)}{z}\right|\ge\delta$, $z\in D$.*

Proof. Let $F_\epsilon \in \overline{S}^*_{s,\beta}(\alpha)$. Then by Theorem 2.2, $\dfrac{(F_\epsilon * H)(z)}{z} \neq 0$, $\forall$ $H \in S^{*'}_{s,\beta}(\alpha)$, $z \in D$. Equivalently, $\dfrac{(f * H)(z) + \epsilon z}{(1+\epsilon)z} \neq 0$ in D or $\dfrac{(f * H)(z)}{z} \neq -\epsilon$ which shows that $\left|\dfrac{(f * H)(z)}{z}\right| \geq \delta$. □

Theorem 2.3. *For $f \in A$ and $\epsilon \in C$, $|\epsilon| < \delta < 1$ assume $F_\epsilon(z) \in \overline{S}^*_{s,\beta}(\alpha)$. Then*

$$N_{\delta(1-\beta)\sin\frac{\alpha\pi}{2}}(f) \subset \overline{S}^*_{s,\beta}(\alpha).$$

Proof. Let $H(z) \in S^{*'}_{s,\beta}(\alpha)$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is in $N_\delta(f)$. Then

$$\begin{aligned}\left|\frac{(g * H)(z)}{z}\right| &= \left|\frac{(f * H)(z)}{z} + \frac{((g-f) * H)(z)}{z}\right| \\ &\geq \left|\frac{(f * H)(z)}{z}\right| - \left|\frac{((g-f) * H)(z)}{z}\right| \\ &\geq \delta - \left|\sum_{k=2}^{\infty} \frac{(b_k - a_k)h_k z^k}{z}\right| \qquad \text{by Lemma 2.3.}\end{aligned}$$

Thus

$$\begin{aligned}\left|\frac{(g * H)(z)}{z}\right| &\geq \delta - |z| \sum_{k=2}^{\infty} |h_k|\,|b_k - a_k| \\ &> \delta - \frac{1}{(1-\beta)\sin\frac{\alpha\pi}{2}} \sum_{k=2}^{\infty} \sqrt{(k-\beta)^2 + 2\beta}\,|b_k - a_k|.\end{aligned}$$

Since $g(z) \in N_\delta(f)$, therefore $g(z) \in N_{\delta'}(f)$ for all $\delta' > \delta$. Hence, we get

$$\left|\frac{(g * H)(z)}{z}\right| > \delta - \frac{\delta'}{(1-\beta)\sin\frac{\alpha\pi}{2}} = 0, \qquad \text{for } \delta' = \delta(1-\beta)\sin\frac{\alpha\pi}{2}.$$

Thus $\dfrac{(g * H)(z)}{z} \neq 0$ in D for all $H \in S^{*'}_{s,\beta}(\alpha)$ which means by Theorem 2.2, $g \in \overline{S}^*_{s,\beta}(\alpha)$; in other words, $N_{\delta(1-\beta)\sin\frac{\alpha\pi}{2}}(f) \subset \overline{S}^*_{s,\beta}(\alpha)$. □

Next, we will show that the class $\overline{S}^*_{s,\beta}(\alpha)$ is closed under convolution with functions f which are convex univalent in D, that is, $(f * g)(z) \in \overline{S}^*_{s,\beta}(\alpha)$

whenever $f \in K$ and $g \in \overline{S}^*_{s,\beta}(\alpha)$. For this we shall need the following lemmas:

Lemma 2.4. *If* $g \in \overline{S}^*_{s,\beta}(\alpha)$, *then* $G(z) = \dfrac{g(z)-g(-z)}{z} \in S^*$.

Proof. Since $g \in \overline{S}^*_{s,\beta}(\alpha)$, therefore

$$\left|\arg \frac{2zg'(z)}{g(z)-g(-z)} - \beta\right| < \frac{\alpha\pi}{2} \text{ in } D$$

or $\left(\dfrac{2zg'(z)}{g(z)-g(-z)} - \beta\right)$ lies in the convex region

$$\Omega = \left\{z \in C : |argz| < \frac{\alpha\pi}{2}\,, \quad 0 < \alpha \le 1\right\}.$$

Hence

$$\frac{zG'(z)}{G(z)} = \frac{zg'(z)}{2G(z)} + \frac{(-z)g'(-z)}{2G(-z)}.$$

There exists ξ_0, ξ_1 in Ω such that

$$\frac{zG'(z)}{G(z)} = \xi_0 + \xi_1 = \xi_2$$

for some $\xi_2 \in \Omega$ since Ω is the convex sector. Thus $G \in \overline{S}^*(\alpha) \subset S^*$. □

Lemma 2.5.([6]) *If* ϕ *is a convex univalent function with* $\phi(0) = 0 = \phi'(0) - 1$ *in* D *and* g *is starlike univalent in* D, *then for each analytic function* F *in* D, *the image of* D *under* $\dfrac{(\phi * Fg)(z)}{(\phi * g)(z)}$ *is a subset of the convex hull of* $F(D)$.

Theorem 2.4. *Let* $\phi(z) \in K$, $f(z) \in \overline{S}^*_{s,\beta}(\alpha)$. *Then* $(\phi * f)(z) \in \overline{S}^*_{s,\beta}(\alpha)$.

Proof. Assume $G(z) = \dfrac{1}{1-\beta}\left\{\dfrac{2zf'(z)}{f(z)-f(-z)} - \beta\right\}$. Then for $F(z) = (\phi * f)(z)$, we have

$$2zF'(z) = \phi * 2zf'(z).$$

Hence

$$\frac{1}{1-\beta}\left[\frac{2zF'(z)}{F(z)-F(-z)}-\beta\right]=\frac{\phi*G(f(z)-f(-z))}{\phi*(f(z)-f(-z))}.$$

By Lemma 2.5, the image of D under $\dfrac{\phi*G(f(z)-f(-z))}{\phi*(f(z)-f(-z))}$ is a subset of the convex hull of $G(D)$. Then $G(D)\subset\Omega=\left\{\omega:|arg\omega|<\dfrac{\alpha\pi}{2}\right\}$ and hence $\dfrac{1}{1-\beta}\left[\dfrac{2zF'(z)}{F(z)-F(-z)}-\beta\right]$ lies in Ω which means $(\phi*f)(z)\in\overline{S}^*_{s,\beta}(\alpha)$. □

Theorem 2.5. *If* $f\in\overline{K}_{s,\beta}(\alpha)$, *then* $\dfrac{f(z)+\epsilon z}{1+\epsilon}\in\overline{S}^*_{s,\beta}(\alpha)$ *for* $|\epsilon|<\dfrac{1}{4}$.

Proof. Let $f(z)=z+\sum\limits_{k=2}^{\infty}a_kz^k$. Then

$$\begin{aligned}\frac{f(z)+\epsilon z}{1+\epsilon}&=\frac{z(1+\epsilon)+\sum_{k=2}^{\infty}a_kz^k}{1+\epsilon}\\&=\frac{f(z)*\left\{z(1+\epsilon)+\sum_{k=2}^{\infty}z^k\right\}}{1+\epsilon}\\&=f(z)*\frac{\left(z-\frac{\epsilon}{1+\epsilon}z^2\right)}{1-z}=f(z)*h(z),\end{aligned}$$

where $h(z)=\dfrac{z-\frac{\epsilon}{1+\epsilon}z^2}{1-z}$. Now,

$$\begin{aligned}\frac{zh'(z)}{h(z)}&=\frac{z-\frac{2\epsilon}{1+\epsilon}z^2}{z-\frac{\epsilon}{1+\epsilon}z^2}+\frac{z}{1-z}\\&=\frac{-\rho z}{1-\rho z}+\frac{1}{1-z}\ ,\ \text{where}\ \ \rho=\frac{\epsilon}{1+\epsilon}\ .\end{aligned}$$

Hence $|\rho|<\dfrac{|\epsilon|}{1-|\epsilon|}<\dfrac{1}{3}$ gives $|\epsilon|<\dfrac{1}{4}$. Thus

$$\text{Re}\left(\frac{zh'(z)}{h(z)}\right)\geq\frac{1-2|\rho|\ |z|-|\rho|\ |z|^2}{(1-|\rho|\ |z|)(1+|z|)}>0$$

if $|\rho|\ \ (|z|^2+2|z|)-1<0$. This inequality holds for all $\rho<\dfrac{1}{3}$ and $|z|<1$, which is true for $|\epsilon| < \dfrac{1}{4}$. Therefore h is starlike in D and so

$$\int_0^z \frac{h(t)}{t} dt = z + \sum_{k=2}^{\infty} \frac{h_k z^k}{k} = h(z) * \log\left(\frac{1}{1-z}\right)$$

is convex for $|\epsilon| < \dfrac{1}{4}$

$$\begin{aligned}(f * h)(z) &= (h * f)(z) = \left[h(z) * \left(zf'(z) * \log\left(\frac{1}{1-z}\right)\right)\right] \\ &= zf'(z) * \left[h(z) * \log\left(\frac{1}{1-z}\right)\right]\end{aligned}$$

$$f(z) \in \overline{K}_{s,\beta}(\alpha) \Rightarrow zf'(z) \in \overline{S}^*_{s,\beta}(\alpha) \text{ and } h(z) * \log\left(\frac{1}{1-z}\right) \in K.$$

Now, by Theorem 2.4, we have

$$zf'(z) * \left[h(z) * \log\left(\frac{1}{1-z}\right)\right] \in \overline{S}^*_{s,\beta}(\alpha).$$

Thus

$$(f * h)(z) = \frac{f(z) + \epsilon z}{1+\epsilon} \in \overline{S}^*_{s,\beta}(\alpha). \qquad \square$$

Theorem 2.6. Let $f \in \overline{K}_{s,\beta}(\alpha)$. Then

$$N_{\frac{1}{4}(1-\beta)\sin\frac{\alpha\pi}{2}}(f) \subset \overline{S}^*_{s,\beta}(\alpha).$$

Proof. Let $f \in \overline{K}_{s,\beta}(\alpha)$. Then from Theorem 2.5, we have $\dfrac{f(z)+\epsilon}{1+\epsilon} \in \overline{S}^*_{s,\beta}(\alpha)$ for $|\epsilon| < \dfrac{1}{4}$. Then an application of Theorem 2.3 gives

$$N_{\frac{1}{4}(1-\beta)\sin\frac{\alpha\pi}{2}}(f) \subset \overline{S}^*_{s,\beta}(\alpha). \qquad \square$$

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