Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **1** (2006), No. 4, pp. 485-505

# SLIGHTLY $m$-CONTINUOUS MULTIFUNCTIONS

BY

TAKASHI NOIRI AND VALERIU POPA

**Abstract**

We introduce a new class of multifunctions called a slightly $m$-continuous multifunction which is a generalization of both slightly continuous functions [10] and $m$-continuous multifunctions [33]. In this paper we obtain several properties of such a multifunction.

## 1. Introduction

Semi-open sets, preopen sets, $\alpha$-sets, and $\beta$-open sets play an important role in generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of generalizations of continuity. In 1980 Jain [10] introduced the notion of slightly continuous functions. Nour [23] defined slightly semi-continuous functions as a weak form of slight continuity and investigated the functions. Recently, Noiri and Chae [19] have further investigated slightly semi-continuous functions. On the other hand, Pal and Bhattacharyya [24] defined a function to be faintly precontinuous if the preimage of each clopen set of the codomain is preopen and obtained many properties of such functions. Slight continuity implies both slight semi-continuity and faint precontinuity but not conversely. Quite recently, the first author [18] has introduced the notion of slight $\beta$-continuity which is implied by both slight semi-continuity and faint precontinuity. A unified theory of slight continuity is presented in [35]

Received November 8, 2005 and in revised form March 14, 2006.

AMS Subject Classification: 54C08, 54C60.

Key words and phrases: $m$-continuous, multifunction, slightly continuous, slightly $m$-continuous.

where the present authors introduced and investigated the notion of slightly $m$-continuous functions.

Popa [26] and Smithson [39] independently introduced the concept of weakly continuous multifunctions. Recently, Bânzaru [5] has proved that if a multifunction is upper/lower weakly continuous then the upper/lower inverse of a clopen set is an open set.

In this paper, we introduce the notion of slightly $m$-continuous multifunctions and investigate the relationships among $m$-continuity, almost $m$-continuity, weak $m$-continuity and slight $m$-continuity for multifunctions.

## 2. Preliminaries

Throughout the present paper, $(X,\tau)$ and $(Y,\sigma)$ always represent topological spaces. Let $A$ be a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset $A$ is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

**Definition 2.1.** A subset $A$ of a topological space $(X,\tau)$ is said to be

(1) *$\alpha$-open* [16] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [11] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [13] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)))$,
(4) *$\beta$-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all clopen (resp. semi-open, preopen, $\alpha$-open, $\beta$-open) sets in $(X,\tau)$ is denoted by $\mathrm{CO}(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$).

**Definition 2.2.** The complement of a semi-open (resp. preopen, $\alpha$-open, $\beta$-open or semi-preopen) set is said to be *semi-closed* [6] (resp. *preclosed* [8], *$\alpha$-closed* [14], *$\beta$-closed* [1] or semi-preclosed [3]).

**Definition 2.3.** The intersection of all semi-closed (resp. preclosed, $\alpha$-closed, $\beta$-closed) sets of $X$ containing $A$ is called the *semi-closure* [6] (resp. *preclosure* [8], *$\alpha$-closure* [14], *$\beta$-closure* [2] or *semi-preclosure* [3]) of $A$ and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$).

**Definition 2.4.** The union of all semi-open (resp. preopen, $\alpha$-open, $\beta$-open or semi-preopen) sets of $X$ contained in $A$ is called the *semi-interior* (resp. *preinterior*, *$\alpha$-interior*, *$\beta$-interior* or *semi-preinterior*) of $A$ and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

A point $x \in X$ is called a $\theta$-cluster point of a subset $A$ of $X$ if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the *$\theta$-closure* of $A$ and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then $A$ is said to be *$\theta$-closed* [41]. The complement of a $\theta$-closed set is said to be *$\theta$-open*. The union of all $\theta$-open sets contained in $A$ is called the *$\theta$-interior* of $A$ and is denoted by $\mathrm{Int}_\theta(A)$. It is shown in [41] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set $V$ of $X$ and $\mathrm{Cl}_\theta(S)$ is closed in $X$ for every subset $S$ of $X$.

A subset $A$ of $(X, \tau)$ is said to be *$\delta$-open* [41] if for each $x \in A$ there exists a regular open set $G$ of $X$ such that $x \in G \subset A$. The union of all $\delta$-open sets contained in $A$ is called *$\delta$-interior* of $A$ and is denoted by $\mathrm{Int}_\delta(A)$. A point $x \in X$ is called a $\delta$-cluster point of a subset $A$ of $X$ if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the *$\delta$-closure* of $A$ and is denoted by $\mathrm{Cl}_\delta(A)$. If $A = \mathrm{Cl}_\delta(A)$, then $A$ is said to be *$\delta$-closed* [41]. For a topological space $(X, \tau)$, the family of all $\delta$-open sets of $(X, \tau)$ forms a topology for $X$ which is weaker than $\tau$. This topology has a base consisting of all regular open sets in $(X, \tau)$. It is called the semi-regularization of $\tau$ and is usually denoted by $\tau_s$ or $\tau_\delta$.

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ represents a multivalued function. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we shall denote the upper and lower inverse of a set $B$ of a space $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \quad \text{and} \quad F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

**Definition 2.5.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *slightly continuous* [10] (resp. *slightly semi-continuous* [23], *faintly precontinuous* [24], *slightly $\beta$-continuous* [18]) if for each point $x \in X$ and each clopen set $V$ containing $f(x)$ there exists an open (resp. semi-open, preopen, $\beta$-open) set $U$ containing $x$ such that $f(U) \subset V$.

**Definition 2.6.** A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(a) *upper slightly continuous* (resp. *upper slightly semi-continuous*, *upper slightly precontinuous* or *upper faintly precontinuous*, *upper slightly $\beta$-continuous*) if for each point $x \in X$ and each clopen set $V$ of $Y$ containing $F(x)$, there exists an open (resp. semi-open, preopen, $\beta$-open ) set $U$ of $X$ containing $x$ such that $F(U) \subset V$,

(b) *lower slightly continuous* (resp. *lower slightly semi-continuous*, *lower slightly precontinuous* or *lower faintly precontinuous*, *lower slightly $\beta$-continuous*) if for each point $x \in X$ and each clopen set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists an open (resp. semi-open, preopen, $\beta$-open ) set $U$ of $X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

## 3. Slightly $m$-Continuous Multifunctions

**Definition 3.1.** A subfamily $m_X$ of the power set $\mathcal{P}(X)$ of a nonempty set $X$ is called a *minimal structure* (briefly *$m$-structure*) on $X$ if $\emptyset \in m_X$ and $X \in m_X$. By $(X, m_X)$, we denote a nonempty subset $X$ with a minimal structure $m_X$ on $X$. Each member of $m_X$ is said to be *$m_X$-open* and the complement of an $m_X$-open set is said to be *$m_X$-closed*

**Remark 3.1.** Let $(X, \tau)$ be a topological space. Then the families $\tau$, SO$(X)$, PO$(X)$, $\alpha(X)$ and $\beta(X)$ are all $m$-structures on $X$.

**Definition 3.2.** Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the *$m_X$-closure* of $A$ and the *$m_X$-interior* of $A$ are defined in [12] as follows:

(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

**Remark 3.2.** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_X = \tau$ (resp. SO$(X)$, PO$(X)$, $\alpha(X)$, $\beta(X)$), then we have

(1) $m_X\text{-Cl}(A) = \text{Cl}(A)$ (resp. $\text{sCl}(A)$, $\text{pCl}(A)$, $\alpha\text{Cl}(A)$, ${}_\beta\text{Cl}(A)$),
(2) $m_X\text{-Int}(A) = \text{Int}(A)$ (resp. $\text{sInt}(A)$, $\text{pInt}(A)$, $\alpha\text{Int}(A)$, ${}_\beta\text{Int}(A)$) .

**Lemma 3.1.**(Maki [12]) *Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following hold:*

(1) $m_X\text{-Cl}(X - A) = X - (m_X\text{-Int}(A))$ *and* $m_X\text{-Int}(X - A) = X - (m_X\text{-Cl}(A))$,

(2) *If* $(X - A) \in m_X$*, then* $m_X$-Cl$(A) = A$ *and if* $A \in m_X$*, then* $m_X$-Int$(A) = A$*,*

(3) $m_X$-Cl$(\emptyset) = \emptyset$*,* $m_X$-Cl$(X) = X$*,* $m_X$-Int$(\emptyset) = \emptyset$ *and* $m_X$-Int$(X) = X$*,*

(4) *If* $A \subset B$*, then* $m_X$-Cl$(A) \subset m_X$-Cl$(B)$ *and* $m_X$-Int$(A) \subset m_X$-Int$(B)$*,*

(5) $A \subset m_X$-Cl$(A)$ *and* $m_X$-Int$(A) \subset A$*,*

(6) $m_X$-Cl$(m_X$-Cl$(A))=m_X$-Cl$(A)$ *and* $m_X$-Int$(m_X$-Int$(A))=m_X$-Int$(A)$*.*

**Lemma 3.2.** (Popa and Noiri [32]) *Let* $X$ *be a nonempty set with a minimal structure* $m_X$ *and* $A$ *a subset of* $X$*. Then* $x \in m_X$-Cl$(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing* $x$*.*

**Definition 3.3.** A function $f : (X, m_X) \to (Y, \sigma)$, where $(X, m_X)$ is a nonempty set $X$ with an minimal structure $m_X$ and $(Y, \sigma)$ is a topological space, is said to be *slightly* $m$*-continuous* [35] if for each $x \in X$ and each clopen set $V$ of $Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$.

**Definition 3.4.** A multifunction $F : (X, m_X) \to (Y, \sigma)$, where $(X, m_X)$ is a nonempty set $X$ with an minimal structure $m_X$ and $(Y, \sigma)$ is a topological space, is said to be

(a) *upper slightly* $m$*-continuous* if for each point $x \in X$ and each clopen set $V$ of $Y$ containing $F(x)$, there exists $U \in m_X$ containing $x$ such that $F(U) \subset V$,

(b) *lower slightly* $m$*-continuous* if for each point $x \in X$ and each clopen set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

**Remark 3.3.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. If $m_X = \tau$ (resp. SO$(X)$, PO$(X)$, $\beta(X)$). Then an upper/lower slightly $m$-continuous multifunction is upper/lower slightly continuous (resp. upper/lower slightly semi-continuous, upper/lower slightly precontinuous, upper/lower slightly $\beta$-continuous).

**Definition 3.5.**(Popa and Noiri [36]) A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(a) *upper m-continuous* (resp. *upper almost m-continuous*, *upper weakly m-continuous*) if for each point $x \in X$ and each open set $V$ of $Y$ containing $F(x)$, there exists $U \in m_X$ containing $x$ such that $F(U) \subset V$ (resp. $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$, $F(U) \subset \mathrm{Cl}(V)$),

(b) *lower m-continuous* (resp. *lower almost m-continuous*, *lower weakly m-continuous*) if for each point $x \in X$ and each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ (resp. $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$, $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$) for each $u \in U$.

**Theorem 3.1.** *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) *$F$ is upper slightly m-continuous;*
(2) $F^+(V) = m_X\text{-}\mathrm{Int}(F^+(V))$ *for each* $V \in \mathrm{CO}(Y)$;
(3) $F^-(V) = m_X\text{-}\mathrm{Cl}(F^-(V))$ *for each* $V \in \mathrm{CO}(Y)$.

**Proof.** (1) $\Rightarrow$ (2): Let $V$ be any clopen set of $Y$ and $x \in F^+(V)$. Then $F(x) \in V$. There exists $U \in m_X$ containing $x$ such that $F(U) \subset V$. Thus $x \in U \subset F^+(V)$ and hence $x \in m_X\text{-}\mathrm{Int}(F^+(V))$. Therefore, we have $F^+(V) \subset m_X\text{-}\mathrm{Int}(F^+(V))$. By Lemma 3.1, we obtain $F^+(V) = m_X\text{-}\mathrm{Int}(F^+(V))$.

(2) $\Rightarrow$ (3): Let $K$ be any clopen set of $Y$. Then $Y - K$ is clopen in $Y$. By (2) and Lemma 3.1, we have $X - F^-(K) = F^+(Y - K) = m_X\text{-}\mathrm{Int}(F^+(Y - K)) = X - [m_X\text{-}\mathrm{Cl}(F^-(K))]$. Therefore, we obtain $F^-(K) = m_X\text{-}\mathrm{Cl}(F^-(K))$.

(3) $\Rightarrow$ (2): This follows from the fact that $F^-(Y - B) = X - F^+(B)$ for every subset $B$ of $Y$.

(2) $\Rightarrow$ (1): Let $x \in X$ and $V$ be any clopen set of $Y$ containing $F(x)$. Then $x \in F^+(V) = m_X\text{-}\mathrm{Int}(F^+(V))$. There exists $U \in m_X$ containing $x$ such that $x \in U \subset F^+(V)$. Therefore, we have $x \in U, U \in m_X$ and $f(U) \subset V$. Hence $F$ is upper slightly $m$-continuous.

**Theorem 3.2.** *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) *$F$ is lower slightly m-continuous;*
(2) $F^-(V) = m_X\text{-}\mathrm{Int}(F^-(V))$ *for each* $V \in \mathrm{CO}(Y)$;
(3) $F^+(V) = m_X\text{-}\mathrm{Cl}(F^+(V))$ *for each* $V \in \mathrm{CO}(Y)$.

*Proof.* (1) $\Rightarrow$ (2): Let $V \in \mathrm{CO}(Y)$ and $x \in F^-(V)$. Then $F(x) \cap V \neq \emptyset$ and by (1) there exists $U \in m_X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$. Therefore, we have $U \subset F^-(V)$ and hence $x \in U \subset m_X$-$\mathrm{Int}(F^-(V))$. Thus, we obtain $F^-(V) \subset m_X$-$\mathrm{Int}(F^-(V))$ and by Lemma 3.1 $F^-(V) = m_X$-$\mathrm{Int}(F^-(V))$.

(2) $\Rightarrow$ (3): Let $V \in \mathrm{CO}(Y)$. Then $Y - V \in \mathrm{CO}(Y)$ and by (2) we have $X - F^+(V) = F^-(Y - V) = m_X$-$\mathrm{Int}(F^-(Y - V)) = X - m_X$-$\mathrm{Cl}(F^+(V))$. Hence we obtain $F^+(V) = m_X$-$\mathrm{Cl}(F^+(V))$.

(3) $\Rightarrow$ (1): Let $x$ be any point of $X$ and $V$ any clopen set of $Y$ such that $F(x) \cap V \neq \emptyset$. Then $x \in F^-(V)$ and $x \notin X - F^-(V) = F^+(Y - V)$. By (3), we have $x \notin m_X$-$\mathrm{Cl}(F^+(Y - V))$. By Lemma 3.2, there exists $U \in m_X$ containing $x$ such that $U \cap F^+(Y - V) = \emptyset$; hence $U \subset F^-(V)$. Therefore, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and $F$ is lower slightly $m$-continuous. □

**Definition 3.6.** A minimal structure $m_X$ on a nonempty set $X$ is said to have the *property* ($\mathcal{B}$) [12] if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

**Lemma 3.3.** (Popa and Noiri [33]) *For an minimal structure $m_X$ on a nonempty set $X$, the following properties are equivalent:*

(1) *$m_X$ has the property ($\mathcal{B}$);*
(2) *If $m_X$-$\mathrm{Int}(V) = V$, then $V \in m_X$;*
(3) *If $m_X$-$\mathrm{Cl}(F) = F$, then $X - F \in m_X$.*

**Corollary 3.1.** *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where $m_X$ has the property ($\mathcal{B}$), the following are equivalent:*

(1) *$F$ is upper slightly $m$-continuous;*
(2) *$F^+(V)$ is $m_X$-open in $(X, m_X)$ for each $V \in \mathrm{CO}(Y)$;*
(3) *$F^-(V)$ is $m_X$-closed in $(X, m_X)$ for each $V \in \mathrm{CO}(Y)$.*

**Corollary 3.2.** *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, where $m_X$ has the property ($\mathcal{B}$), the following are equivalent:*

(1) *$F$ is lower slightly $m$-continuous;*
(2) *$F^-(V)$ is $m_X$-open in $(X, m_X)$ for each $V \in \mathrm{CO}(Y)$;*
(3) *$F^+(V)$ is $m_X$-closed in $(X, m_X)$ for each $V \in \mathrm{CO}(Y)$.*

**Definition 3.7.** A topological space $(X, \tau)$ is said to be *extremally disconnected* (briefly E.D.) if the closure of each open set of $X$ is open in $X$.

**Theorem 3.3.** *Let $(Y, \sigma)$ be E.D. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) *$F$ is upper slightly $m$-continuous;*
(2) *$m_X$-$\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ for every open set $V$ of $(Y, \sigma)$;*
(3) *$F^+(\mathrm{Int}(C)) \subset m_X$-$\mathrm{Int}(F^+(C))$ for every closed set $C$ of $(Y, \sigma)$.*

*Proof.* (1) $\Rightarrow$ (2): Let $V$ be any open set of $Y$. Then $\mathrm{Cl}(V) \in \mathrm{CO}(Y)$. By Theorem 3.1, $F^-(\mathrm{Cl}(V)) = m_X$-$\mathrm{Cl}(F^-(\mathrm{Cl}(V)))$ and $F^-(V) \subset F^-(\mathrm{Cl}(V))$. Therefore, by Lemma 3.1 we have $m_X$-$\mathrm{Cl}(F^-(V)) \subset m_X$-$\mathrm{Cl}(F^-(\mathrm{Cl}(V))) = F^-(\mathrm{Cl}(V))$. This implies that $m_X$-$\mathrm{Cl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): Let $C$ be any closed set of $(Y, \sigma)$. Set $V = Y - C$. Then $V$ is open in $(Y, \sigma)$. By Lemma 3.1, we have $X - [m_X$-$\mathrm{Int}(F^+(C))] = m_X$-$\mathrm{Cl}(X - F^+(C)) = m_X$-$\mathrm{Cl}(F^-(Y - C)) \subset F^-(\mathrm{Cl}(Y - C)) = F^-(Y - \mathrm{Int}(C)) = X - F^+(\mathrm{Int}(C))$. Therefore, we have $F^+(\mathrm{Int}(C)) \subset m_X$-$\mathrm{Int}(F^+(C))$.

(3) $\Rightarrow$ (1): Let $x \in X$ and $V \in \mathrm{CO}(Y)$ containing $F(x)$. Then by (3) we have $x \in F^+(V) = F^+(\mathrm{Int}(V)) \subset m_X$-$\mathrm{Int}(F^+(V))$. Therefore, there exists $U \in m_X$ such that $x \in U \subset F^+(V)$. Thus $x \in U, U \in m_X$ and $F(U) \subset V$. Hence $F$ is upper slightly $m$-continuous. □

**Theorem 3.4.** *Let $(Y, \sigma)$ be E.D. For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) *$F$ is lower slightly $m$-continuous;*
(2) *$m_X$-$\mathrm{Cl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ for every open set $V$ of $(Y, \sigma)$;*
(3) *$F^-(\mathrm{Int}(C)) \subset m_X$-$\mathrm{Int}(F^-(C))$ for every closed set $C$ of $(Y, \sigma)$.*

*Proof.* The proof is similar to that of Theorem 3.3. □

**Lemma 3.4.**(Noiri [17] and Sivaraj [38]) *For a topological space $(Y, \sigma)$, the following are equivalent:*

(1) *$(Y, \sigma)$ is extremally disconnected;*
(2) *The closure of every semi-open set of $(Y, \sigma)$ is open;*
(3) *The closure of every preopen set of $(Y, \sigma)$ is open;*
(4) *The closure of every $\beta$-open set of $(Y, \sigma)$ is open.*

**Theorem 3.5.** *Let $(Y,\sigma)$ be E.D. For a multifunction $F:(X,m_X)\to(Y,\sigma)$, the following are equivalent:*

(1) *$F$ is upper slightly $m$-continuous;*
(2) *$m_X$-Cl$(F^-(V))\subset F^-(\mathrm{Cl}(V))$ for every semi-open (resp. preopen, $\beta$-open) set $V$ of $(Y,\sigma)$;*
(3) *$F^+(\mathrm{Int}(C))\subset m_X$-Int$(F^+(C))$ for every semi-closed (resp. preclosed, $\beta$-closed) set $C$ of $(Y,\sigma)$.*

*Proof.* The proof is similar to that of Theorem 3.3 and it follows from Theorem 3.1 and Lemma 3.4. □

**Theorem 3.6.** *Let $(Y,\sigma)$ be E.D. For a multifunction $F:(X,m_X)\to(Y,\sigma)$, the following are equivalent:*

(1) *$F$ is lower slightly $m$-continuous;*
(2) *$m_X$-Cl$(F^+(V))\subset F^+(\mathrm{Cl}(V))$ for every semi-open (resp. preopen, $\beta$-open) set $V$ of $(Y,\sigma)$;*
(3) *$F^-(\mathrm{Int}(C))\subset m_X$-Int$(F^-(C))$ for every semi-closed (resp. preclosed, $\beta$-closed) set $C$ of $(Y,\sigma)$.*

*Proof.* The proof is similar to that of Theorem 3.4 and it follows from Theorem 3.2 and Lemma 3.4. □

**Remark 3.4.** Let $(X,\tau)$ and $(Y,\sigma)$ be two topological spaces. We put $m_X=\tau$ (resp. SO$(X)$, PO$(X)$, $\beta(X)$). Then Theorems 3.1–3.6 establish characterizations for upper/lower slightly continuous (resp. upper/lower slightly semi-continuous, upper/lower slightly precontinuous, upper/lower slightly $\beta$-continuous) multifunctions.

## 4. Slight $m$-continuity and Other Forms of $m$-continuity

In this section, we investigate the relationships between upper/lower slightly $m$-continuous multifunctions and other related multifunctions.

**Theorem 4.1.** *If a multifunction $F:(X,m_X)\to(Y,\sigma)$ is upper weakly $m$-continuous, then it is upper slightly $m$-continuous.*

*Proof.* Let $x \in X$ and $V \in \mathrm{CO}(Y)$ containing $F(x)$. Since $F$ is upper weakly $m$-continuous, there exists $U \in m_X$ containing $x$ such that $F(U) \subset \mathrm{Cl}(V) = V$. Therefore, $F$ is upper slightly $m$-continuous. □

**Theorem 4.2.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower weakly $m$-continuous, then it is lower slightly $m$-continuous.*

*Proof.* The proof is similar to that of Theorem 4.1. □

**Remark 4.1.** The converse to Theorem 4.1 is not true in general as shown in Example 1 of [23] and Example 2.7 of [24].

**Corollary 4.1.**(Bânzaru [5]) *If a multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper weakly continuous, then $F^+(V)$ is open in $X$ for every clopen set $V$ of $Y$.*

*Proof.* The proof follows from Corollary 3.1 and Theorem 4.1. □

**Corollary 4.2.**(Bânzaru [5]) *If a multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower weakly continuous, then $F^-(V)$ is open in $X$ for every clopen set $V$ of $Y$.*

*Proof.* The proof follows from Corollary 3.2 and Theorem 4.2. □

**Lemma 4.1.** *A multifunction $F : (X, m_X) \to (Y, \sigma)$ is upper almost $m$-continuous (resp. lower almost $m$-continuous) if and only if for each regular open set $V$ containing F(x) (resp. meeting $F(x)$) there exists $U \in m_X$ containing $x$ such that $F(U) \subset V$ (resp. $F(u) \cap V \neq \emptyset$ for every $u \in U$).*

**Theorem 4.3.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is upper slightly $m$-continuous and $(Y, \sigma)$ is E.D., then $F$ is upper almost $m$-continuous.*

*Proof.* Let $x \in X$ and $V$ be any regular open set of $(Y, \sigma)$ containing $F(x)$. Then by Lemma 5.6 of [24] we have $V \in \mathrm{CO}(X)$ since $(Y, \sigma)$ is E.D. Since $F$ is upper slightly $m$-continuous, there exists $U \in m_X$ containing $x$ such that $F(U) \subset V$. By Lemma 4.1 $F$ is upper almost $m$-continuous. □

**Theorem 4.4.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower slightly $m$-continuous and $(Y, \sigma)$ is E.D., then $F$ is lower almost $m$-continuous.*

*Proof.* The proof is similar to that of Theorem 4.3. □

**Definition 4.1.** A topological space $(Y, \sigma)$ is said to be

(a) *0-dimensional* [43] if each point of $Y$ has a neighborhood base consisting of clopen sets, equivalently if for each point $y$ in $Y$ and each closed set $B$ not containing $y$, there exists a clopen set containing $y$ and not meeting $B$,

(b) *mildly compact* [40] or *slightly compact* [24] if every clopen cover of $Y$ admits a finite subcover.

A subset $A$ of a topological space $(Y, \sigma)$ is said to be *mildly compact relative to* $Y$ if every cover of $A$ by clopen sets of $Y$ has a finite subcover.

**Theorem 4.5.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is upper slightly $m$-continuous, $(Y, \sigma)$ is 0-dimensional and F(x) is mildly compact relative to $Y$ for each $x \in X$, then $F$ is upper $m$-continuous.*

*Proof.* Let $x \in X$ and $V$ be any open set of $(Y, \sigma)$ containing $F(x)$. Then by the 0-dimensionality of $(Y, \sigma)$, for each $y \in F(x)$ there exists $W_y \in \mathrm{CO}(Y)$ such that $y \in W_y \subset V$. Since $F(x)$ is mildly compact relative to $Y$, there exist a finite number of points, say, $y_1, y_2, \ldots, y_n \in F(x)$ such that $W_{y_i} \in \mathrm{CO}(Y)$ for each $i$ and $F(x) \subset \bigcup_{i=1}^{n} W_{y_i} \subset V$. Now put $W = \bigcup_{i=1}^{n} W_{y_i}$. Then we have $W \in \mathrm{CO}(Y)$ and $F(x) \subset W \subset V$. Since $F$ is upper slightly $m$-continuous, there exists $U \in m_X$ containing $x$ such that $F(U) \subset W \subset V$. Thus $F$ is upper $m$-continuous. □

**Lemma 4.2.** *Let $(Y, \sigma)$ be a 0-dimensional topological space. If $F$ is closed in $Y$ and $y \in Y - F$, then there exist two disjoint clopen sets containing $y$ and $F$, respectively.*

*Proof.* Let $y \notin F$ and $F$ be closed in $Y$. Then $Y - F$ is an open set. Since $(Y, \sigma)$ is 0-dimensional, there exists a clopen set $W$ such that $x \in W \subset X - F$. Put $D = X - W$, then $D$ is clopen, $F \subset D$ and $D \cap W = \emptyset$. □

**Lemma 4.3.** *Let $(Y, \sigma)$ be a 0-dimensional topological space and $A$ a subset of $Y$. Then for every open set $D$ which intersects $A$, there exists a clopen set $D_A$ such that $A \cap D_A \neq \emptyset$ and $D_A \subset D$.*

*Proof.* Let $y \in A \cap D$, then $y \notin (X - D)$. Since $X - D$ is closed in $Y$, by Lemma 4.2 there exist two disjoint clopen sets $U$ and $V$ containing $y$ and

$X-D$, respectively. Thus $y \in U, X-D \subset V, U \cap V = \emptyset$ and $U, V$ are clopen sets. Put $D_A = X - V$, then $y \in D_A, A \cap D_A \neq \emptyset$ and $D_A \subset D$. □

**Theorem 4.6.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower slightly $m$-continuous, $(Y, \sigma)$ is 0-dimensional, then $F$ is lower $m$-continuous.*

*Proof.* Let $x \in X$ and $V$ be any open set of $(Y, \sigma)$ such that $F(x) \cap V \neq \emptyset$. By Lemma 4.3, there exists a clopen set $V_x$ such that $F(x) \cap V_x \neq \emptyset$ and $V_x \subset V$. Since $F$ is lower slightly $m$-continuous and $F(x) \cap V_x \neq \emptyset$, there exists $U \in m_X$ containing $x$ such that $F(u) \cap V_x \neq \emptyset$ for every $u \in U$. Since $V_x \subset V$, it follows that $F(u) \cap V \neq \emptyset$ for every $u \in U$. Therefore, $F$ is lower $m$-continuous. □

Let $(X, \tau)$ be a topological space. Since the intersection of two clopen sets of $(X, \tau)$ is clopen, the clopen subsets of $(X, \tau)$ may be used as a base for a topology on $X$. The topology is called the *ultra-regularization* [18] of $\tau$ and is denoted by $\tau_u$. A topological space $(X, \tau)$ is said to be *ultra-regular* [9] if $\tau = \tau_u$.

**Theorem 4.7.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is upper slightly $m$-continuous, $(Y, \sigma)$ is ultra-regular and F(x) is mildly compact relative to $Y$ for each $x \in X$, then $F$ is upper $m$-continuous.*

*Proof.* The proof is similar to that of Theorem 4.5. □

**Theorem 4.8.** *If a multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower slightly $m$-continuous, $(Y, \sigma)$ is ultra-regular, then $F$ is lower $m$-continuous.*

*Proof.* The proof is similar to that of Theorem 4.6. □

## 5. Some Properties of Slight $m$-continuity

**Definition 5.1.** A nonempty set $X$ with a minimal structure $m_X$, $(X, m_X)$, is said to be *m-connected* [25] if $X$ cannot be written as the union of two nonempty sets of $m_X$.

**Theorem 5.1.** *Let $F : (X, m_X) \to (Y, \sigma)$ be a upper/lower slightly $m$-continuous surjection. If $(X, m_X)$ is $m$-connected, $m_X$ has the property ($\mathcal{B}$) and F(x) is connected for each $x \in X$, then $(Y, \sigma)$ is connected.*

**Proof.** Assume that $(Y, \sigma)$ is not connected. Then there exist nonempty open sets $U$ and $V$ such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \in U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since $F$ is surjective, there exist $x$ and $y$ in $X$ such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;
(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;
(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are $m_X$-open.

(i) Let $F$ be upper slightly $m$-continuous. By Corollary 3.1 $F^+(U)$ is $m_X$-open since $U$ is clopen in $(Y, \sigma)$. Similarly, $F^+(V)$ is $m_X$-open. Consequently, $(X, m_X)$ is not $m$-connected.

(ii) Let $F$ be lower slightly $m$-continuous. By Corollary 3.2, $F^+(V)$ is $m_X$-closed since $V$ is clopen in $(Y, \sigma)$. By (1) and (2), $F^+(U)$ is $m_X$-open. Similarly, $F^+(V)$ is $m_X$-open. Consequently, $(X, m_X)$ is not $m$-connected. □

**Remark 5.1.** By Theorems 4.1 and 4.2, every upper/lower weakly $m$-continuous multifunction is upper/lower slightly $m$-continuous. Therefore, if $m_X = \tau$ (resp. SO$(X)$, PO$(X)$, $\beta(X)$) and $F : (X, m_X) \to (Y, \sigma)$ is upper/lower weakly $m$-continuous, the results established in [27] and [39] (resp. [20], [21], [34]) are obtained.

**Definition 3.2.** A nonempty set $X$ with a minimal structure $m_X$, $(X, m_X)$, is said to be *$m$-compact* [33] if every cover of $X$ by sets of $m_X$ has a finite subcover.

**Theorem 5.2.** *Let $F : (X, m_X) \to (Y, \sigma)$ be an upper slightly $m$-continuous surjection such that $F(x)$ is mildly compact realtive to $(Y, \sigma)$ for each $x \in X$. If $(X, m_X)$ is $m$-compact, then $(Y, \sigma)$ is mildly compact.*

*Proof.* Let $\{V_i : i \in I\}$ be any clopen cover of $Y$. For each $x \in X$, $F(x)$ is mildly compact realtive to $(Y, \sigma)$ and there exists a finite subset $I(x)$ of $I$ such that $F(x) \subset \bigcup_{i \in I(x)} V_i$. Set $V(x) = \bigcup_{i \in I(x)} V_i$, then $V(x)$ is clopen and $F(x) \subset V(x)$. Since $F$ is upper slightly $m$-continuous, there exists $U(x) \in m_X$ containing $x$ such that $F(U(x)) \subset V(x)$. The family

$\{U(x) : x \in X\}$ is a cover of $X$ by sets of $m_X$. Since $(X, m_X)$ is $m$-compact, there exist a finite number of points, say, $x_1, x_2, \ldots, x_n$ in $X$ such that $X \subset \cup\{U(x_k) : x_k \in X, 1 \leq k \leq n\}$. Therefore, we obtain

$$Y \subset \cup\{F(U(x_k)) : x_k \in X, 1 \leq k \leq n\} \subset \cup\{V_i : i \in I(x_k), 1 \leq k \leq n\}.$$

This shows that $(Y, \sigma)$ is mildly compact. □

**Definition 5.3.** A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to have an *$m$-clopen graph* if for each $(x, y) \in (X \times Y) - \mathrm{G}(F)$, there exist $U \in m_X$ containing $x$ and a clopen set $V$ of $Y$ containing $y$ such that $(U \times V) \cap \mathrm{G}(F) = \emptyset$.

**Lemma 5.1.** *A multifunction $F : (X, m_X) \to (Y, \sigma)$ has an $m$-clopen graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exist $U \in m_X$ containing $x$ and a clopen set $V$ of $Y$ containing $y$ such that $F(U) \cap V = \emptyset$.*

**Definition 5.4.** A topological space $(X, \tau)$ is said to be *ultra-Hausdorff* [40] if for each distinct points $x, y \in X$, there exist $U, V \in \mathrm{CO}(X)$ containing $x$ and $y$, respectively, such that $U \cap V = \emptyset$.

**Theorem 5.3.** *If $F : (X, m_X) \to (Y, \sigma)$ is an upper slightly $m$-continuous multifunction such that $F(x)$ is mildly compact relative to $(Y, \sigma)$ and $(Y, \sigma)$ is ultra-Hausdorff, then $\mathrm{G}(F)$ is $m$-clopen.*

*Proof.* Suppose that $(x_0, y_0) \in (X \times Y) - \mathrm{G}(F)$. Then $y_0 \notin F(x_0)$. Since $Y$ is ultra-Hausdorff, for each $y \in F(x_0)$ there exist clopen sets $V(y)$ and $W(y)$ in $Y$ containing $y$ and $y_0$, respectively, such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y) : y \in F(x_0)\}$ is a clopen cover of $F(x_0)$. Since $F(x_0)$ is mildly compact relative to $(Y, \sigma)$, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in $F(x_0)$ such that $F(x_0) \subset \bigcup_{i=1}^{n} V(y_i)$. Put $V = \bigcup_{i=1}^{n} V(y_i)$ and $W = \bigcap_{i=1}^{n} W(y_i)$. Then $V$ and $W$ are clopen sets, $F(x_0) \subset V, y_0 \in W$ and $V \cap W = \emptyset$. Since $F$ is upper slightly $m$-continuous, there exists $U \in m_X$ containing $x_0$ such that $F(U) \subset V$. This implies that $F(U) \cap W = \emptyset$ and by Lemma 5.1 $\mathrm{G}(F)$ is $m$-clopen. □

**Definition 5.5.** Let $X$ be a nonempty set which has a minimal structure $m_X$ and $A$ a subset of $X$. The *$m_X$-frontier* of $A$ [35], denoted by $m_X$-Fr$(A)$, is defined by $m_X$-Fr$(A) = m_X$-Cl$(A) \cap$ $m_X$-Cl$(X - A) = m_X$-Cl$(A)$ - [$m_X$-Int$(A)$].

**Theorem 5.4.** *Let $X$ be a nonempty set with a minimal structure $m_X$ and $(Y, \sigma)$ a topological space. The set of all points $x \in X$ at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not upper slightly $m$-continuous (resp. lower slightly $m$-continuous) is identical with the union of $m_X$-frontiers of the upper (resp. lower) inverse images of clopen sets containing (resp. meeting) $F(x)$.*

*Proof.* Suppose that $F$ is not upper slightly $m$-continuous at $x \in X$. Then there exists a clopen sets $V$ of $Y$ containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in m_X$ containing $x$. By Lemma 3.2 $x \in m_X$-Cl$(X - F^+(V))$. On the other hand, we have $x \in F^+(V) \subset m_X$-Cl$(F^+(V))$ and hence $x \in m_X$-Fr$(F^+(V))$.

Conversely, suppose that $F$ is upper slightly $m$-continuous at $x \in X$ and let $V$ be a clopen set of $Y$ containing $F(x)$. Then there exists $U \in m_X$ containing $x$ such that $U \subset F^+(V)$; hence $x \in m_X$-Int$(F^+(V))$. Therefore, $x \notin m_X$-Fr$(F^+(V))$ for each clopen set $V$ of $Y$ containing $f(x)$. The case of lower slightly $m$-continuity is similarly shown. □

**Remark 5.2.** By Theorems 4.1 and 4.2, every upper/lower weakly $m$-continuous multifunction is upper/lower slightly $m$-continuous. Therefore, if $m_X = \tau$ (resp. SO$(X)$, PO$(X)$, $\alpha(X)$, $\beta(X)$) and $F : (X, m_X) \to (Y, \sigma)$ is upper/lower weakly $m$-continuous, the results established in [29] (resp. [28], [30], [34], [31]) are obtained.

## 6. New Varieties of Slight-continuity

Let $A$ be a subset of a topological space $(X, \tau)$. A point $x$ of $X$ is called a *semi-$\theta$-cluster point* of $A$ if sCl$(U) \cap A \neq \emptyset$ for every $U \in$ SO$(X)$ containing $x$. The set of all semi-$\theta$-cluster points of $A$ is called the *semi-$\theta$-closure* [7] of $A$ and is denoted by sCl$_\theta(A)$. A subset $A$ is said to be *semi-$\theta$-closed* if $A =$ sCl$_\theta(A)$. The complement of a semi-$\theta$-closed set is said to be *semi-$\theta$-open.* A subset $A$ is said to be *semi-regular* [7] if it is semi-open and semi-closed.

**Definition 6.1.** Let $(X,\tau)$ be a topological space. A subset $A$ of $X$ is said to be

(1) *b-open* [4] if $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$,
(2) *$\delta$-preopen* [37] if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$,
(3) *$\delta$-semi-open* [25] if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A)))$.

The family of all $b$-open (resp. $\delta$-preopen, $\delta$-semi-open, semi-$\theta$-open) sets in $(X,\tau)$ is denoted by $\mathrm{BO}(X)$ (resp. $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$, $\mathrm{S}\theta\mathrm{O}(X)$).

**Lemma 6.1.** (Noiri and Popa [22]) *For a subset $A$ of a topological space $(X,\tau)$, the following properties hold:*

(1) *If $A$ is a semi-open set, then* sCl*$(A)$ is semi-regular,*
(2) *If $A$ is a semi-regular set, then it is semi-$\theta$-open,*
(3) *If $A$ is a semi-$\theta$-open set, then it is $\delta$-semi-open,*
(4) *If $A$ is a $\delta$-semi-open set, then it is semi-open.*

In [25], it is shown that openness and $\delta$-semi-opennes are independent. And it is also shown in [22] that the concepts of $\delta$-preopen sets and semi-preopen sets are independent of each other. We have the following diagram.

**DIAGRAM I**

$$\begin{array}{ccccccccc}
\delta\text{-open} & \rightarrow & \text{open} & \rightarrow & \alpha\text{-open} & \rightarrow & \text{preopen} & \rightarrow & \delta\text{-preopen} \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\text{semi-}\theta\text{-open} & \rightarrow & \delta\text{-semi-open} & \rightarrow & \text{semi-open} & \rightarrow & b\text{-open} & \rightarrow & \text{semi-preopen}
\end{array}$$

**Lemma 6.2.** (Noiri and Popa [22]) *For a subset $A$ of a topological space $(X,\tau)$, the following properties hold:*

(1) *$A$ is $\delta$-semi-open in $(X,\tau)$ if and only if $A$ is semi-open in $(X,\tau_s)$,*
(2) *$A$ is $\delta$-preopen in $(X,\tau)$ if and only if $A$ is preopen in $(X,\tau_s)$.*

**Definition 6.2.** A multifunction $F:(X,\tau)\rightarrow(Y,\sigma)$ is said to be

(1) *upper slightly super-continuous* (resp. *upper slightly semi-$\theta$-continuous*, *upper slightly b-continuous*, *upper slightly $\delta$-precontinuous*, *upper slightly $\delta$-semi-continuous*, *upper slightly $\alpha$-continuous*) if for each $x \in X$ and every clopen set $V$ of $(Y,\sigma)$ such that $F(x) \subset V$, there exists a $\delta$-open

(resp. semi-$\theta$-open, $b$-open, $\delta$-preopen, $\delta$-semi-open, $\alpha$-open) set $U$ of $(X,\tau)$ containing $x$ such that $F(U) \subset V$,

(2) *lower slightly super-continuous* (resp. *lower slightly semi-$\theta$-continuous*, *lower slightly b-continuous*, *lower slightly $\delta$-precontinuous*, *lower slightly $\delta$-semi-continuous*, *lower slightly $\alpha$-continuous*) if for each $x \in X$ and every clopen set $V$ of $(Y,\sigma)$ such that $F(x) \cap V \neq \emptyset$, there exists a $\delta$-open (resp. semi-$\theta$-open, $b$-open, $\delta$-preopen, $\delta$-semi-open, $\alpha$-open) set $U$ of $(X,\tau)$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

**Remark 6.1.** Let $F : (X,\tau) \to (Y,\sigma)$ be a multifunction and $m_X = \tau_s$ (resp. S$\theta$O$(X)$, BO$(X)$, $\delta$SO$(X)$, $\delta$PO$(X)$, $\alpha(X)$). Then an upper/lower slightly $m$-continuous multifunction $F : (X, m_X) \to (Y,\sigma)$ is upper/lower slightly super-continuous (resp. upper/lower slightly semi-$\theta$-continuous, upper/lower slightly $b$-continuous, upper/lower slightly $\delta$-semi-continuous, upper/lower slightly $\delta$-precontinuous, upper/lower slightly $\alpha$-continuous).

The families $\tau_s$, S$\theta$O$(X)$, BO$(X)$, $\delta$SO$(X)$, $\delta$PO$(X)$ and $\alpha(X)$ have the property $(\mathcal{B})$. Especially, $\tau_s$ and $\alpha(X)$ are topologies for $X$. Therefore, we can apply all results obtained in Section 3−5 to these new multifunctions. The following theorem is the typical characterizations.

**Theorem 6.1.** *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

(1) *$F$ is upper slightly super-continuous* (*resp. upper slightly semi-$\theta$-continuous, upper slightly b-continuous, upper slightly $\delta$-semi-continuous, upper slightly $\delta$-precontinuous, upper slightly $\alpha$-continuous*) *if and only if for every clopen set $V$ of $Y$, $F^+(V)$ is $\delta$-open* (*resp. semi-$\theta$-open, b-open, $\delta$-semi-open, $\delta$-preopen, $\alpha$-open*) *in $X$,*

(2) *$F$ is lower slightly super-continuous (resp. lower slightly semi-$\theta$-continuous, lower slightly b-continuous, lower slightly $\delta$-semi-continuous, lower slightly $\delta$-precontinuous, lower slightly $\alpha$-continuous*) *if and only if for every clopen set $V$ of $Y$, $F^-(V)$ is $\delta$-open* (*resp. semi-$\theta$-open, b-open, $\delta$-semi-open, $\delta$-preopen, $\alpha$-open*) *in $X$.*

*Proof.* The proof is obvious from the definition. □

By Theorem 6.1 and DIAGRAM I, we obtain the following diagram:

**DIAGRAM II**

$$\begin{array}{ccccccccc} \text{u/l s. super-C} & \to & \text{u/l s. C} & \to & \text{u/l s. } \alpha\text{-C} & \to & \text{u/l s. } p\text{-C} & \to & \text{u/l s. } \delta\text{-}p\text{-C} \\ \downarrow & & & & \downarrow & & \downarrow & & \\ \text{u/l s. } s\text{-}\theta\text{-C} & \to & \text{u/l s. } \delta\text{-}s\text{-C} & \to & \text{u/l s. } s\text{-C} & \to & \text{u/l s. } b\text{-C} & \to & \text{u/l s. } \beta\text{-C} \end{array}$$

In the diagram above, we abbreviate as follows: $u/l$ = upper/lower, $s.$ = slightly, $C$ = continuous, $p$ = pre and $s$ = semi.

**Theorem 6.2.** *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

(1) *$F$ is upper slightly super-continuous if and only if $F : (X, \tau_s) \to (Y, \sigma)$ is upper slightly continuous,*

(2) *$F$ is lower slightly super-continuous if and only if $F : (X, \tau_s) \to (Y, \sigma)$ is lower slightly continuous.*

*Proof.* This is obvious by Theorem 6.1. □

**Theorem 6.3.** *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

(1) *$F$ is upper slightly $\delta$-semi-continuous (resp. upper slightly $\delta$-precontinuous) if and only if $F : (X, \tau_s) \to (Y, \sigma)$ is upper slightly semi-continuous (resp. upper slightly precontinuous),*

(2) *$F$ is lower slightly $\delta$-semi-continuous (resp. lower slightly $\delta$-precontinuous) if and only if $F : (X, \tau_s) \to (Y, \sigma)$ is lower slightly semi-continuous (resp. lower slightly precontinuous).*

*Proof.* This is an immediate consequence of Lemma 6.2. □

**Theorem 6.4.** *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *$F$ is upper slightly semi-$\theta$-continuous;*

(2) *For each point $x \in X$ and each clopen set $V$ of $Y$ containing $F(x)$, there exists a semi-open set $U$ of $X$ containing $x$ such that $F(\mathrm{sCl}(U)) \subset V$;*

(3) *For each point $x \in X$ and each clopen set $V$ of $Y$ containing $F(x)$, there exists a semi-regular set $U$ of $X$ containing $x$ such that $F(U) \subset V$.*

*Proof.* (1) $\Rightarrow$ (2): This is obvious from the fact that for any semi-$\theta$-open set $G$ and each $x \in G$, there exists a semi-open set $H$ such that $x \in H \subset \mathrm{sCl}(H) \subset G$.

(2) $\Rightarrow$ (3): By Lemma 6.1, $\mathrm{sCl}(H)$ is a semi-regular set for each semi-open set $H$.

(3) $\Rightarrow$ (1): Let $V$ be any clopen set of $Y$ and $x \in F^+(V)$. Then there exists a semi-regular set $U$ of $X$ containing $x$ such that $F(U) \subset V$. By Lemma 6.1, every semi-regular set is semi-$\theta$-open. Therefore, $F$ is upper slightly semi-$\theta$-continuous. □

**Theorem 6.5.** *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *$F$ is lower slightly semi-$\theta$-continuous;*
(2) *For each point $x \in X$ and each clopen set $V$ of $Y$ such that $V \cap F(x) \neq \emptyset$, there exists a semi-open set $U$ of $X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in \mathrm{sCl}(U)$;*
(3) *For each point $x \in X$ and each clopen set $V$ of $Y$ such that $V \cap F(x) \neq \emptyset$, there exists a semi-regular set $U$ of $X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

*Proof.* The proof is similar to that of Theorem 6.4. □

## References

1. M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, $\beta$-open sets and $\beta$-continuous mappings, *Bull. Fac. Sci. Assiut Univ.*, **12** (1983), 77-90.

2. M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, $\beta$-closure and $\beta$-interior, *J. Fac. Ed. Ain Shams Univ.*, **10** (1986), 235-245.

3. D. Andrijević, Semipreopen sets, *Mat. Vesnik*, **38** (1986), 24-32.

4. D. Andrijević, On $b$-open sets, *Mat. Vesnik*, **48** (1996), 59-64.

5. T. Bânzaru, More remarks on some forms of continuity for multifunctions, *Bul. Ştiinţ. Univ. Politeh. Timiş Ser. Mat. Fiz.*, **44(58)** (1999), 34-39.

6. S. G. Crossley and S. K. Hildbrand, Semi-closure, *Texas J. Sci.*, **22** (1971), 99-112.

7. G. Di Maio and T. Noiri, On $s$-closed spaces, *Indian J. Pure Appl. Math.*, **18** (1987), 226-233.

8. S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On $p$-regular spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **27(75)** (1983), 311-315.

9. R. L. Ellis, A nonarchimedian analog of the Tietze-Urysohn extension theorem, *Indag. Math.*, **70** (1967), 332-333.

10. R. C. Jain, The role of regularly open sets in general topology, Ph.D. Thesis, Meerut Univ., Meerut, 1980.

11. N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36-41.

12. H. Maki, On generalizing semi-open and preopen sets, *Meeting on Topological Spaces Theory and its Applications*, 24-25 August, 1996, Yatsushiro Coll. Tech., 13-18.

13. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt.*, **53** (1982), 47-53.

14. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, $\alpha$-continuous and $\alpha$-open mappings, *Acta Math. Hungar.*, **14** (1983), 213-218.

15. M. N. Mukherjee and C. K. Basu, On semi-$\theta$-closed sets, semi-$\theta$-connectedness and some associated mappings, *Bull. Calcutta Math. Soc.*, **83** (1991), 227-238.

16. O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.*, **15** (1965), 961-970.

17. T. Noiri, Characterizations of extremally disconnected spaces, *Indian J. Pure Appl. Math.*, **19** (1988), 325-329.

18. T. Noiri, On slightly $\beta$-continuous functions, *Internat. J. Math. Math. Sci.*, **28** (2001), 469-478..

19. T. Noiri and G. I. Chae, A note on slightly semi-continuous functions, *Bull. Calcutta Math. Soc.*, **92** (2000), 87-92.

20. T. Noiri and V. Popa, On upper and lower weakly quasicontinuous multifunctions, *Rev. Roumaine Math. Pures Appl.*, **37** (1992), 499-508.

21. T. Noiri and V. Popa, Almost weakly continuous multifunctions, *Demonstratio Math.*, **26** (1993), 363-368.

22. T. Noiri and V. Popa, *A unified theory of contra-continuity for functions*, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **45** (2002), 115-137 (2003).

23. T. M. Nour, Slightly semi-continuous functions, *Bull. Calcutta Math. Soc.*, **87** (1995), 187-190.

24. M. C. Pal and P. Bhattacharyya, Faint precontinuous functions, *Soochow J. Math.*, **21** (1995), 273-289.

25. J. H. Park, Y. Lee and M. J. Son, On $\delta$-semiopen sets in topological spaces, *J. Indian Acad. Math.*, **19** (1997), 59-67.

26. V. Popa, Weakly continuous multifunctions, *Boll. Un. Mat. Ital.* (5), **15** (1978), 379-388.

27. V. Popa, On some properties of weakly continuous multifunctions, *Univ. Bacău, Stud. Cerc. Mat.*, 1980, 121-127.

28. V. Popa and T. Noiri, Properties of upper and lower weakly quasicontinuous multifunctions, *Univ. Bacău, Stud. Cerc. St. Ser. Mat.*, **8**(1998), 97-112.

29. V. Popa and T. Noiri, Some properties of upper and lower weakly continuous multifunctions, *An. Univ. Craiova, Ser. Mat. Inform.*, **25** (1998), 24-30.

30. V. Popa and T. Noiri, Some properties of almost weakly continuous multifunctions, *Demonstratio Math.*, **3** (1999), 605-614.

31. V. Popa and T. Noiri, On upper and lower $\beta$-continuous multifunctions, *Anal. Univ. Sci. Budapest*, **43** (2000), 25-48.

32. V. Popa and T. Noiri, On $M$-continuous functions, *Anal. Univ. Dunărea de Jos-Galaţi, Ser. Mat. Fiz. Mec. Teor. Fasc. II*, **18**(2000), (23), 31-41.

33. V. Popa and T. Noiri, On the definitions of some generalized forms of continuity under minimal conditions, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, **22** (2001), 9-18.

34. V. Popa and T. Noiri, On upper and lower weakly $\alpha$-continuous multifunctions, *Novi Sad J. Math.*, **32** (2002), 7-24.

35. V. Popa and T. Noiri, Slightly $m$-continuous functions, *Radovi Mat.*, **10** (2001), 1-14.

36. V. Popa and T. Noiri, A unified theory of weak continuity for multifunctions, (submitted).

37. S. Raychaudhuri and M. N. Mukherjee, On $\delta$-almost continuity and $\delta$-preopen sets, *Bull. Inst. Math. Acad. Sinica*, **21** (1993), 375-366.

38. D. Sivaraj, A note on extremally disconnected spaces, *Indian J. Pure Appl. Math.*, **17** (1986), 1373-1375.

39. R. E. Smithson, Almost and weak continuity for multifunctions, *Bull. Calcutta Math. Soc.*, **70** (1978), 383-390.

40. R. Staum, The algebra of bounded continuous functions into a nonarchimedean field, *Pacific J. Math.*, **50** (1974), 169-185.

41. N. V. Veličko, $H$-closed topological spaces, *Amer. Math. Soc. Transl.* (2), **78** (1968), 103-118.

42. T. H. Yalvaç, Semi-regular sets, semi-$\theta$-open sets and semi-$\theta$-connectedness, *Demonstratio Math.*, **29** (1996), 1-6.

43. S. Willard, *General Topology*, Addison-Wesley, 1970.

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142 JAPAN.

E-mail: t.noiri@nifty.com

Department of Mathematics, University of Bacău, 600114 Bacău, Rumania.

E-mail: vpopa@ub.ro