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CLASSIFICATION OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

BY

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1. **Introduction.** Staikos and Sficas [10] and Philos [6] have recently divided the solutions of n-th order functional differential equations into disjoint classes according to their oscillatory character and their behavior as $t \to \infty$. By imposing certain integral conditions on the functions in these equations, they were able to determine which of these classes make up the set of all solutions. An earlier study of this type for second order equations was carried out by Ladas *et al.* [5].

Here we study the decomposition of the solutions of the equation

$$(*) \qquad [r(t)x^{(n-\nu)}(t)]^{(\nu)} + \delta f(t, x(g(t))) = 0$$

under less restrictive integral conditions than those imposed in [6] and [10]. We also obtain (see Theorem 1 below) results regarding the oscillation and convergence to zero of the bounded solutions of (*).

For an extensive bibliography on nonlinear oscillation problems for functional differential equations, the reader is referred to the survey paper of Kartsatos [4]. Other recent results related to the theorems in this paper can be found in the papers of Grammatikopoulos [1], Philos [8], and the references contained therein.

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2. **An oscillation theorem.** Consider the n-th order ($n \geq 2$) nonlinear differential equation

$$[r(t)x^{(n-\nu)}(t)]^{(\nu)} + \delta f(t, x(g(t))) = 0 \tag{1}$$

where $\delta = \pm 1$, $1 \leq \nu \leq n-1$, and r, g: $[t_0, \infty) \to R$ and f: $[t_0, \infty) \times R \to R$ are continuous. Throughout this paper we assume that $r(t) > 0$ on $[t_0, \infty)$, $\int^{\infty} [1/r(s)]\, ds = \infty$, $g(t) \to \infty$ as $t \to \infty$, and that $f(t, y)$ is nondecreasing in y with $yf(t, y) > 0$ for $y \neq 0$.

Furthermore, it is understood that the results here pertain only to the nontrivial continuable solutions $x(t)$ of (1), i.e., $x(t)$ is defined on an interval of the form $[t_x, \infty)$ and for every $T \geq t_x$ $\sup\{|x(t)|: t \geq T\} > 0$. Such a solution is said to be oscillatory if its set of zeros is unbounded and nonoscillatory otherwise.

Parts of the proofs given here make use of the following two lemmas due to Grammatikopoulos [2,3]. (See also Sficas and Stavroulakis [11]).

LEMMA 1. *Let u be a positive $(n-\nu)$-times continuously differentiable function on the interval $[a, \infty)$ and let μ be a positive continuous function on $[a, \infty)$ such that*

$$\int^{\infty} [1/\mu(t)]\, dt = \infty$$

and the function $w \equiv \mu u^{(n-\nu)}$ is ν-times continuously differentiable on $[a, \infty)$. Moreover, let

$$\omega_k = \begin{cases} u^{(k)}, & \text{if } 0 \leq k \leq n-\nu-1 \\ w^{(k-n+\nu)}, & \text{if } n-\nu \leq k \leq n. \end{cases}$$

If $\omega_n(t) \equiv w^{(\nu)}(t)$ is of constant sign and not identically zero for all large t, then there exist $t_u \geq a$ and an integer l, $0 \leq l \leq n$, with $n+l$ even for ω_n nonnegative or $n+l$ odd for ω_n nonpositive, and such that for every $t \geq t_u$

$$l > 0 \quad \textit{implies} \quad \omega_k(t) > 0 \quad (k = 0, 1, \cdots, l-1)$$

and

$$l \leq n-1 \quad \textit{implies} \quad (-1)^{l+k}\omega_k(t) > 0 \quad (k = l, l+1, \cdots, n-1).$$

LEMMA 2. *If the functions u, μ, w and ω_k are as in Lemma 1 and for some $k = 0, 1, \cdots, n-2$*

$$\lim_{t\to\infty} \omega_k(t) = c, \qquad c \in R$$

then

$$\lim_{t\to\infty} \omega_{k+1}(t) = 0 .$$

We will also make use of the following notation in the remainder of this paper. For any $T \geq t_0$ and all $t \geq T$ we let

$$z(t) = r(t)x^{(n-\nu)}(t),$$

$$J(T, t) = \int_T^t [(t-s)^{\nu-1} s^{n-\nu-1}/r(s)]\, ds/(\nu-1)!\,(n-\nu-1)!,$$

and

$$\omega_k(t) = \begin{cases} x^{(k)}(t), & 0 \leq k \leq n-\nu-1 \\ z^{(k-n+\nu)}(t), & n-\nu \leq k \leq n . \end{cases}$$

With this notation, we have the following result.

THEOREM 1. *The condition that*

$$\int^{\infty} J(T, s)|f(s, A)|\, ds = \infty \tag{2}$$

for every constant $A \neq 0$ and all large T is necessary and sufficient for every bounded solution $x(t)$ of (1) to be:

(I) *oscillatory if n is even and $\delta = 1$, or if n is odd and $\delta = -1$;*

and

(II) *either oscillatory or satisfy $\omega_k(t) \to 0$ monotonically as $t \to \infty$ for $k = 0, 1, \cdots, n-1$ if n is odd and $\delta = 1$ or if n is even and $\delta = -1$.*

Proof. To prove sufficiency, assume that $x(t)$ is a bounded nonoscillatory solution of (1). We will give the details of the proof only for the case when $x(t)$ is eventually positive since the proof for the case $x(t)$ eventually negative is similar. To this end let $t_1 > \max\{t_0, 0\}$ be such that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$. Then from (1) we have

$$\delta z^{(\nu)}(t) \leq 0, \tag{3}$$

for $t \geq t_1$. Notice that (3) and Lemma 1 together imply that the integer l assigned to $x(t)$ by Lemma 1 is such that $n+l$ is odd for $\delta=1$ and is even for $\delta=-1$. It follows from Lemma 1 that if $\delta=1$, then $l=1$ for n even and $l=0$ for n odd for otherwise successive integrations would show that $x(t)$ is unbouned. By similar reasoning, if $\delta=-1$ we obtain $l=0$ for n even and $l=1$ for n odd. Furthermore, Lemma 1 implies that for the cases where $l=1$

$$(-1)^{i+1}\omega_i(t)>0 \quad \text{for} \quad i=1,2,\cdots,n-1 \tag{4}$$

holds, and for $l=0$

$$(-1)^{i}\omega_i(t)>0 \quad \text{for} \quad i=1,2,\cdots,n-1 \tag{5}$$

holds.

Next, multiply (1) by $J(T,t)$, $T\geq t_1$, and integrate over $[T,t]$ to obtain

$$\int_T^t J(T,s)f(s,x(g(s)))\,ds=-\delta\int_T^t J(T,s)\,z^{(\nu)}(s)\,ds. \tag{6}$$

But

$$\begin{aligned}\int_T^t J(T,s)z^{(\nu)}(s)\,ds &=J(T,t)z^{(\nu-1)}(t)-J'(T,t)z^{(\nu-2)}(t)+\cdots+(-1)^{\nu-1}J^{(\nu-1)}(T,t)z(t)\\ &\quad+(-1)^{\nu}t^{n-\nu-1}x^{(n-\nu-1)}(t)/(n-\nu-1)!+\cdots+(-1)^{n-1}x(t)+L\end{aligned}$$

where L is a constant. It then follows from (4) – (6) that

$$\int_T^t J(T,s)f(s,x(g(s)))\,ds<\infty. \tag{7}$$

Now if $l=1$, then $x'(t)>0$ and this together with (7) contradicts (2). Moreover, $l=1$ implies that either $\delta=1$ with n even or $\delta=-1$ with n odd, and thus (1) has no bounded positive nonoscillatory solutions in this case. For the case $l=0$ notice that (5) implies that $x(t)$ is decreasing. Therefore, if (II) does not hold. then there exists a constant $A_1>0$ such that $A_1\leq x(g(t))$ for $t\geq T$. Again, (7) contradicts (2) and the sufficiency part of the proof is complete.

To prove necessity, assume that every bounded solution of (1) satisfies either (I) or (II), but that there exists a constant $c\neq 0$ such that

$$\int^{\infty} J(T, s)|f(s, 3c)|\,ds < \infty .$$

Choose $T_0 > \max\{t_0, 1\}$ so that

$$\int_{T_0}^{\infty} J(T_0, s)|f(s, 3c)|\,ds < |c| \tag{8}$$

and consider the integral equation

$$x(t) = 2c + \delta(-1)^{n-1}\int_t^{\infty}\left(\int_t^s [(s-u)^{\nu-1}(u-t)^{n-\nu-1}/r(u)]\,du\right) \cdot f(s, x(g(s)))\,ds/(\nu-1)!\,(n-\nu-1)! . \tag{9}$$

It is not difficult to verify by differentiation that a solution of (9) is also a solution of (1). We will show that (1) has a bounded nonoscillatory solution by using the following special case of Tychonov's fixed point theorem:

THEOREM. *Let F be a Fréchet space and X be a closed convex subset of F. If G: $X \to X$ is continuous and the closure $\overline{G(X)}$ is a compact subset of X, then there exists at least one fixed point x in X.*

In order to utilize this theorem, let $u_0 = \min\{T_0, \min_{t\geq t_0} g(t)\}$ and let F be the Fréchet space of all continuous functions x: $[u_0, \infty) \to R$ with the topology of uniform convergence on compact subintervals of $[u_0, \infty)$.

If $c > 0$, let the closed convex subset X of F be defined by

$$X = \{x \in F : c \leq x(g(t)) \leq 3c,\ t \geq u_0\},$$

and define the operator G on X by

$$(Gx)(t) = \begin{cases} 2c + (-1)^{n-1}\,\delta Q(t), & \text{if } t \geq T_0 \\ 2c + (-1)^{n-1}\,\delta Q(T_0), & \text{if } u_0 \leq t \leq T_0 \end{cases}$$

where

$$Q(m) = \int_m^{\infty}\left(\int_m^s [(s-u)^{\nu-1}(u-m)^{n-\nu-1}/r(u)]\,du\right) \cdot f(s, x(g(s)))\,ds/(n-\nu-1)!\,(\nu-1)! .$$

To complete the proof for the case $c > 0$, we show that G

satisfies all the hypotheses of the fixed point theorem stated above. First observe that for any $x \in X$

$$|(Gx)(t) - 2c| \leq Q(T_0)$$

for $t \geq u_0$, and that

$$Q(T_0) \leq \int_{T_0}^{\infty} J(T_0, s) f(s, 3c)\, ds < c.$$

Thus we see that G maps X into X.

To shown that G is continuous let $\{x_\lambda\}$, $\lambda = 1, 2, \cdots$ be any sequence of functions in X converging uniformly to $x \in X$ on every compact subinterval of $[u_0, \infty)$. Let $t \geq u_0$ and $T_2 > \max\{t, T_0\}$; then $f(t, x_\lambda(g(t))) \to f(t, x(g(t)))$ uniformly on $[u_0, T_2]$. But $|(Gx_\lambda)(t) - (Gx)(t)| \leq \int_{T_0}^{T_2} J(T_0, s) |f(s, x_\lambda(g(s))) - f(s, x(g(s)))|\, ds$, and we see from (8) that Gx_λ converges uniformly to Gx on any compact subinterval of $[u_0, \infty)$. Hence we conclude that G is continuous.

Finally, in order to show that $\overline{GX}$ is a compact subset of X it is sufficient to show that GX is relatively compact since $GX \subset X$ and X is closed. Furthermore, since X is bounded, it suffices to show that GX is equicontinuous. For this purpose we distinguish two cases. If $n - \nu \neq 1$, then from the definitions of Gx, J, and X we have that there exists a constant L_1 such that

$$\begin{aligned} |(Gx)'(t)| &\leq \int_t^{\infty} \left(\int_t^s [(s-u)^{\nu-1}(u-t)^{n-\nu-2}/r(u)]\, du \right) \\ &\qquad \cdot f(s, x(g(s))) ds/(\nu-1)!(n-\nu-2)! \\ &\leq L_1 \int_T^{\infty} J(T_0, s) f(s, 3c)\, ds \end{aligned}$$

for $t \geq u_0$. Hence it follows from (8) that there is a constant L_2 such that $|(Gx)'(t)| \leq L_2$ where L_2 is independent of both x and t. It then follows that GX is equicontinuous on $[u_0, \infty)$.

If $n - \nu = 1$, we have for each $t \geq T_0$ that

$$|(Gx)'(t)| \leq L_3 \int_t^{\infty} (s-t)^{\nu-1} f(s, x(g(s)))\, ds/r(t)$$

for some positive constant L_3. Since Gx is constant on $[u_0, T_0]$, then

$$|(Gx)'(t)| \le L_3 \int_{T_0}^{\infty} (s - T_0)^{\nu-1} f(s, 3c)\, ds / r(t)$$

for all $t \ge u_0$. Noticing that $[(s - T_0)^{\nu-1}/J(T_0, s)] \to 0$ as $s \to \infty$, we see that

$$|(Gx)'(t)| \le [L_4/r(t)] \int_{T_0}^{\infty} J(T_0, s) f(s, 3c)\, ds$$

for some constant L_4. Therefore for any given closed subinterval $[u_0, T_1]$, of $[u_0, \infty)$, with $T_1 > T_0$, there exists a constant $L(T_1)$ such that

$$|(Gx)'(t)| \le L(T_1)$$

where $L(T_1)$ is independent of both $x \in X$ and t in $[u_0, T_1]$. Also, if $t_2 > t_1 \ge T_1$, then

$$\begin{aligned} &|(Gx)(t_2) - (Gx)(t_1)| \\ &\qquad \le \int_{t_2}^{\infty} J(T_1, s) f(s, 3c)\, ds + \int_{t_1}^{\infty} J(T_1, s) f(s, 3c)\, ds \\ &\qquad \le 2 \int_{T_1}^{\infty} J(T_1, s) f(s, 3c)\, ds \end{aligned}$$

where the last integral tends to zero as $T_1 \to \infty$ independent of $x \in X$ and t_1, t_2 in $[T_1, \infty)$. It is now easy to see that GX is equicontinuous on $[u_0, \infty)$ for $n - \nu = 1$.

We now have all the hypotheses of the fixed point theorem satisfied for the case $c > 0$. The argument for the case $c < 0$ is similar and will be omitted. From the fixed point theorem we have the existence of $x \in X$ such that $Gx = x$. But then x is a bounded solution of (1) that satisfies neither (I) nor (II). This contradiction completes the proof of the theorem.

REMARK. By differentiating both sides of equation (9) we see that $x'(t)$ has fixed sign. Therefore we have actually proved that if there exists a constant $c \neq 0$ such that $\int^{\infty} J(T, s) |f(s, 3c)|\, ds < \infty$, then (1) has a solution that converges to a nonzero constant as $t \to \infty$.

It is also interesting to observe that equation (1) may have unbounded nonoscillatory solutions with condition (2) holding. This is illustrated by the simple examples

$$[tx'(t)]'' + x(t)/8t^2 = 0, \qquad t > 0$$

and

$$[tx'(t)]''' + 9x(t)/16t^3 = 0, \qquad t > 0$$

which have the solutions $x_1(t) = t^{1/2}$ and $x_2(t) = t^{3/2}$ respectively.

REMARK. Results similar to Theorem 1 have been obtained by other authors, for example see Philos [7] and Sficas and Stavroulakis [11]. However, Theorem 1 above differs from their results in the form of the integral conditions imposed. To see that Theorem 1 applies to equations not covered by the results in [7] or [11], consider the equation

$$[t^2 x'(t)/e^t]' + (\operatorname{sech} t)(\tanh t)\, x(t) = 0, \qquad t > 3.$$

Condition (2) and the other hypotheses of Theorem 1 hold but none of the results in [7] or [11] apply to this example.

Other theorems giving sufficient conditions for solutions to behave as in Theorem 1 can be found in [1] and [8].

3. **Classification results.** In this section we classify all solutions of (1) according to their oscillatory character and behavior as $t \to \infty$. For this purpose we shall use the same classification used by Philos [6] and Staikos and Sficas [10].

Let $S(\delta)$ denote the set of all solutions of (1) and let the subsets $S^{\sim}(\delta)$, $S^{0}(\delta)$, $S_1^{+\infty}(\delta)$, $S_2^{+\infty}(\delta)$, $S_1^{-\infty}(\delta)$, $S_2^{-\infty}(\delta)$, $S^{+\infty}(\delta)$, and $S^{-\infty}(\delta)$ be subsets of $S(\delta)$ defined as follows.

(a) $S^{\sim}(\delta)$ is the set of all oscillatory $x \in S(\delta)$.

(b) $S^{0}(\delta)$ is the set of all nonoscillatory $x \in S(\delta)$ with $\omega_k(t) \to 0$ monotonically as $t \to \infty$ for $k = 0, 1, \cdots, n-1$.

(c) $S_1^{+\infty}(\delta)$ is the set of all $x \in S(\delta)$ for which there exists an integer N, $0 \leq N \leq n-1$, with $n+N$ odd and such that:

(P$_1$) $\lim_{t\to\infty} \omega_k(t) = \infty$ for every $k = 0, 1, \cdots, N$;

(P$_2$) If $N \leq n-2$, then $\lim_{t\to\infty} \omega_{N+1}(t)$ exists in R;

(P$_3$) If $N \leq n-3$, then for every $k = N+2, \cdots, n-1$ $\lim_{t\to\infty} \omega_k(t) = 0$, $\omega_k(t) \neq 0$ for all large t, and $\omega_k(t)\, \omega_{k+1}(t) \leq 0$ for all large t.

(d) $S_2^{+\infty}(\delta)$ is the set of all $x \in S(\delta)$ which possess properties

(P_1)-(P_3) for some integer N, $0 \leq N \leq n-1$, with $n+N$ even.

(e) $S_1^{-\infty}(\delta)$ is the set of all $x \in S(\delta)$ for which the function $-x$ possesses properties (P_1)-(P_3) for some integer N, $0 \leq N \leq n-1$, with $n+N$ odd.

(f) $S_2^{-\infty}(\delta)$ is the set of all $x \in S(\delta)$ for which the function $-x$ possesses properties (P_1)-(P_3) for some integer N, $0 \leq N \leq n-1$, with $n+N$ even.

(g) $S^{+\infty}(\delta) = S_1^{+\infty}(\delta) \cup S_2^{+\infty}(\delta)$.

(h) $S^{-\infty}(\delta) = S_1^{-\infty}(\delta) \cup S_2^{-\infty}(\delta)$.

The following lemma will be used in the proof of our next theorem; a proof of this lemma can be found in [9].

LEMMA 3. *Consider the linear differential equation*

$$y' - P'(t)[y - H(t)]/P(t) = 0 \tag{10}$$

where $H, P' : [t_0, \infty) \to R$ *are continuous,* $|P'(t)| > 0$, *and* $\lim_{t\to\infty} P(t)$ *belongs to the set* $\{0, -\infty, \infty\}$. *If* $\lim_{t\to\infty} H(t)$ *exists in the extended real line, then so does* $\lim_{t\to\infty} y(t)$ *for any solution* $y(t)$ *of* (10) *which satisfies* $y(t_0) = 0$. *Moreover,* $\lim_{t\to\infty} |H(t)| = \infty$ *implies that* $\lim_{t\to\infty} |y(t)| = \infty$.

THEOREM 2. *If* (2) *is satisfied, then the solutions of* (1) *have the following decomposition.*

(III) *For n even*

$$S(+1) = S^{\sim}(+1) \cup S^{+\infty}(+1) \cup S^{-\infty}(+1)$$

and

$$S(-1) = S^{\sim}(-1) \cup S^{0}(-1) \cup S^{+\infty}(-1) \cup S^{-\infty}(-1).$$

(IV) *For n odd*

$$S(+1) = S^{\sim}(+1) \cup S^{0}(+1) \cup S^{+\infty}(+1) \cup S^{-\infty}(+1)$$

and

$$S(-1) = S^{\sim}(-1) \cup S^{+\infty}(-1) \cup S^{-\infty}(-1).$$

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and let $t_1 > \max\{t_0, 0\}$ be such that $|x(g(t))| > 0$ and $|x(t)| > 0$ for $t \geq t_1$. Then, since $\omega_n(t) = z^{(\nu)}(t)$ does not change sign on $[t_1, \infty)$, each

$\omega_k(t)$ eventually has fixed sign for $k=0, 1,\cdots, n-1$. Therefore, for each k satisfying $1 \le k \le n-1$, $\lim_{t\to\infty} \omega_k(t)$ exists in the extended real number system.

Now suppose that $x(t)>0$ on $[T, \infty)$, $T \ge t_1$, and that $\lim_{t\to\infty} x(t) \ne 0$. Then there exists a constant $M>0$ so that $x(g(t)) \ge M$ for $t \ge T$. Define the functions h_i by

$$h_i(t) = \int_T^t J^{(\nu-i)}(T, s) z^{(i)}(s)\, ds$$

and integrate by parts to obtain

$$\begin{aligned} h_i(t) &= J^{(\nu-i)}(T, t) z^{(i-1)}(t) - \int_T^t J^{(\nu-i+1)}(T, s) z^{(i-1)}(s)\, ds \\ &= J^{(\nu-i)}(T, t) J^{(\nu-i+1)}(T, t) z^{(i-1)}(t) / J^{(\nu-i+1)}(T, t) \\ &\quad - \int_T^t J^{(\nu-i+1)}(T, s) z^{(i-1)}(s)\, ds \\ &= J^{(\nu-i)}(T, t) h'_{i-1}(t) / J^{(\nu-i+1)}(t) - h_{i-1}(t)\,. \end{aligned}$$

Hence we see that, for $t \ge T$, $h_{i-1}(t)$ is a solution of the equation

$$(L_i) \qquad y' - J^{(\nu-i+1)}(T, t)[y - H_i(t)]/J^{(\nu-i)}(t) = 0$$

for $i = 1, 2,\cdots, \nu$ where $H_i(t) = -h_i(t)$. Moreover, $h_{i-1}(T) = 0$. Noticing also that

$$\begin{aligned} h_\nu(t) &= \int_T^t J(T, s) z^{(\nu)}(s)\, ds \\ &= -\delta \int_T^t J(T, s) f(s, x(g(s)))\, ds \end{aligned}$$

we have

$$|h_\nu(t)| \ge \int_T^t J(T, s) f(s, M)\, ds\,.$$

Thus (2) implies that $|h_\nu(t)| = |H_\nu(t)| \to \infty$ as $t \to \infty$ and, applying Lemma 3 to equation (L_ν), we conclude that $|h_{\nu-1}(t)| \to \infty$ as $t \to \infty$. Continuing in this fashion we obtain $|H_0(t)| = |h_0(t)| \to \infty$ as $t \to \infty$.

Next, let

$$q_j(t) = \int_T^t s^j x^{(j+1)}(s)\, ds$$

for $j = 1, 2,\cdots, n-\nu$. Integrating by parts yields

$$q_j(t) = t^j x^{(j)}(t) - j \int_T^t s^{j-1} x^{(j)}(s)\, ds - T^j x^{(j)}(T)$$

$$= t(t^{j-1})\, x^{(j)}(t) - j \int_T^t s^{j-1} x^{(j)}(s)\, ds - T^j x^{(j)}(T),$$

or

$$q_j(t) = tq'_{j-1}(t) - jq_{j-1}(t) - T^j x^{(j)}(T)\,.$$

Thus $q_{j-1}(t)$ is a solution of the equation

$$(L_{1,j}) \qquad y' - j[y - Q_j(t)]/t = 0$$

where $Q_j(t) = -(T^j x^{(j)}(T) + q_j(t))/j$ and $q_{j-1}(T) = 0$. Furthermore, notice that

$$q_{n-\nu-1}(t) = h_0(t) = \int_T^t s^{n-\nu-1} x^{(n-\nu)}(s)\, ds$$

and hence $|q_{n-\nu-1}(t)| \to \infty$ and $|Q_{n-\nu-1}(t)| \to \infty$ as $t \to \infty$. Applying Lemma 3 to the equation $(L_{1,n-\nu-1})$ we obtain $|q_{n-\nu-2}(t)| \to \infty$ and $|Q_{n-\nu-2}(t)| \to \infty$ as $t \to \infty$. Continuing in this way we obtain $|q_0(t)| \to \infty$ as $t \to \infty$. But $q_0(t) = \int_T^t x'(s)\, ds = x(t) - x(T)$ so $x(t) \to \infty$ as $t \to \infty$.

It follows from the above argument that if $x(t)$ is a solution of (1) that is eventually positive, then as $t \to \infty$ either $x(t) \to 0$ or $x(t) \to \infty$. If $x(t) \to 0$, we conclude from Lemma 2 that $\omega_k(t) \to 0$ for $k = 0, 1, \cdots, n-1$ and, since each $\omega_k(t)$ is eventually monotonic, we have $x(t) \in S^0(\delta)$. For the case $x(t) \to \infty$, let $N \leq n-1$ be the greatest nonnegative integer such that $\omega_k(t) \to \infty$ as $t \to \infty$ for $k = 0, 1, \cdots, N$. If $N \leq n-2$, then $\lim_{t\to\infty} \omega_{N+1}(t) = C \in R$ since each $\omega_k(t)$ is monotonic. Furthermore, if $N \leq n-3$, then Lemma 2 implies that $\omega_k(t) \to 0$ as $t \to \infty$ for $k = N+2, N+3, \cdots, n-1$. In addition, it is easy to see that if $\omega_k(t) \to 0$ as $t \to \infty$, then $\omega_k(t)\omega_{k+1}(t) \leq 0$ for $k = N+2, N+3, \cdots, n-1$. Thus $x(t) \in S^{+\infty}(\delta)$.

For the case when $x(t)$ is eventually negative, a similar argument leads to the conclusion that $x(t) \in S^{-\infty}(\delta)$.

To complete the proof we need only observe that by Theorem 1 $S^0(1) \neq \emptyset$ implies n is odd and $S^0(-1) \neq \emptyset$ implies n is even.

REMARK. Philos [6; Th. 1] and Staikos and Sficas [10: Th. 1] also obtained results of this type. But as was pointed out in the remark following Theorem 1, our results apply to equations not covered by results in [6] or [10], and in this sense is a generalization of some special cases of their work.

The next theorem provides additional information concerning the structure of the set $S(+1)$.

THEOREM 3. *Suppose that condition* (2) *holds and that for every* $c>0$ *we have*

$$\int^{\infty}\left(\int_{T}^{s}[1/r(u)]\,du\right)|f(s,cg(s))|\,ds=\infty,\qquad \text{if}\quad \nu=1, \tag{11}$$

$$\int^{\infty}s|f(s,cg(s))|\,ds=\infty,\qquad \text{if}\quad 1<\nu<n-1, \tag{12}$$

and

$$\int^{\infty}s\left|f\left(s,c\int_{T}^{g(s)}[1/r(u)]\,du\right)\right|ds=\infty,\qquad \text{if}\quad \nu=n-1. \tag{13}$$

Then for n even

$$S(+1)=S^{\sim}(+1)\cup S_2^{+\infty}(+1)\cup S_2^{-\infty}(+1)$$

while for n odd

$$S(+1)=S^{\sim}(+1)\cup S^{0}(+1)\cup S_2^{+\infty}(+1)\cup S_2^{-\infty}(+1).$$

Proof. Let $x(t)$ be a solution of (1) with $\delta=1$, i.e. $x(t)\in S(+1)$. We already know from Theorem 2 that for n even

$$S(+1)=S^{\sim}(+1)\cup S^{+\infty}(+1)\cup S^{-\infty}(+1)$$

and for n odd

$$S(+1)=S^{\sim}(+1)\cup S^{0}(+1)\cup S^{+\infty}(+1)\cup S^{-\infty}(+1).$$

If $x(t)\in S^{+\infty}(+1)=S_1^{+\infty}(+1)\cup S_2^{+\infty}(+1)$, then (P_1) implies that

$$x(t)\longrightarrow\infty\quad\text{as}\quad t\to\infty. \tag{14}$$

First assume that the integer N associated with $x(t)$ by the definitions of $S_1^{+\infty}(\delta)$ and $S_2^{+\infty}(\delta)$ satisfies $N\geq 1$, then (P_1) also implies that

(15) $$x'(t) \longrightarrow \infty \quad \text{as} \quad t \to \infty \quad \text{for} \quad \nu < n-1$$

and

(16) $$r(t)x'(t) \longrightarrow \infty \quad \text{as} \quad t \to \infty \quad \nu = n-1 .$$

It is easy to see from (15) that if $\nu < n-1$, then there exist $t_1 > \max\{t_0, 0\}$ and a constant $c_1 > 0$ such that for $t \geq t_1$ both $x(t)$ and $x(g(t))$ are positive and

(17) $$x(g(t)) \geq c_1\, g(t) .$$

In like manner, if $\nu = n-1$ it follows from (16) that there exist $t_2 > \max\{t_0, 0\}$ and a constant $c_2 > 0$ so that for $t \geq t_2$ we have $x(t) > 0$, $x(g(t)) > 0$ and

(18) $$x(g(t)) \geq c_2 \int_{t_2}^{g(t)} [1/r(u)]\, du .$$

For notational convenience in applying (17) and (18) we will let $T = t_1$ and $c = c_1$ when $\nu < n-1$ and $T = t_2$ and $c = c_2$ in case $\nu = n-1$. Since $\delta = 1$ we also have from (1) that

(19) $$f(t, x(g(t))) = -\,\omega_n(t) = -\,z^{(\nu)}(t) > 0$$

for $t \geq T$ which, together with (14), implies that $\omega_{n-1}(t) = z^{(\nu-1)}(t)$ is positive and bounded above. Notice also that if $n = 2$, then $\nu = 1 = n-1$ and it follows that $\omega_{n-1}(t) = \omega_1(t) = r(t)x'(t)$ is bounded above contradicting (15). Hence for the case $N \geq 1$ we have $n > 2$. If $\nu = 1 < n-1$ we multiply (19) by $\int_T^t [1/r(u)]\, du$ obtaining

$$\left(\int_T^t [1/r(u)]\, du\right) f(t, x(g(t))) = -\left(\int_T^t [1/r(u)]\, du\right) z'(t) .$$

Integrating the right member of the last equation by parts and using (17) leads to

$$\int_T^t \left(\int_T^s [1/r(u)]\, du\right) f(s, cg(s))\, ds \leq L_1 + \omega_{n-2}(t)$$

for some constant L_1. Then from (11) we have $\omega_{n-2}(t) \to \infty$ as $t \to \infty$ which, in view of Lemmas 1 and 2, implies that $N \geq n-2$. But $N \leq n-2$ since $\omega_{n-1}(t)$ is bounded above, so we have $N = n-2$. Therefore $n + N$ is even and $x(t) \in S_2^{+\infty}(+1)$ provided $\nu = 1$ and $N \geq 1$. Furthermore, if $1 < \nu < n-1$, multiplying (19) by t and integrating the right member by parts yields

$$\int_T^t sf(s,\ x(g(s)))\,ds = -tz^{(\nu-1)}(t) + L_2 + \omega_{n-2}(t)$$

where L_2 is a constant. Applying (12) and (17) if $1<\nu<n-1$ and (13) and (18) if $\nu=n-1$ we again obtain $N=n-2$, i.e. $x(t)\in S_2^{+\infty}(+1)$. Hence $S_1^{+\infty}(+1)=\varnothing$ in case $N\geq 1$.

Now suppose that $N=0$ and that $x(t)\in S_1^{+\infty}(+1)$, then n is odd so we have $n\geq 3$. Also (14) implies (19) for all sufficiently large t, say $t\geq t_3>\max\{t_0,\ 0\}$. Moreover, since n is odd it follows from (P_3) that $\omega_2(t)>0$ for $t\geq t_3$. Notice that if $\nu<n-1$, then either $\omega_2(t)=x''(t)$ or $\omega_2(t)=r(t)\,x''(t)$ so that $x''(t)>0$ and $\omega_1(t)=x'(t)$ is increasing. Clearly, if $\nu=n-1$, then $\omega_1(t)=r(t)x'(t)$ is increasing. Also, observe that (14) implies that $x'(t)$ is eventually positive. Therefore for the case $\nu<n-1$ there exists $t_4>t_3$ such that $x'(t)>x'(t_4)$ for $t\geq t_4$, and for $\nu=n-1$ there exists $t_5\geq t_3$ so that $r(t)\,x'(t)>r(t_5)\,x'(t_5)>0$ for $t\geq t_5$. Thus we have that either

$$x(t)\geq x(t_4)+x'(t_4)(t-t_4)$$

or

$$x(t)\geq x(t_5)+r(t_5)\,x'(t_5)\int_{t_5}^t [1/r(u)]\,du,$$

and we see that either (17) or (18) holds with suitable choices for c_1 and t_1 or c_2 and t_2. It then follows, by arguments similar to the ones used above for the case $N\geq 1$, that $\omega_{n-2}(t)\to\infty$ as $t\to\infty$. But if $n\geq 5$, then (P_3) implies that $\omega_{n-2}(t)<0$ and we have a contradiction. On the other hand if $n=3$, then we have $\omega_{n-2}(t)=\omega_1(t)\to\infty$ and $\omega_0(t)\to\infty$ as $t\to\infty$ which contradicts the assumption that $N=0$. Thus $S_1^{+\infty}(+1)=\varnothing$.

If $x(t)\in S^{-\infty}(+1)=S_1^{-\infty}(+1)\cup S_2^{-\infty}(+1)$, then arguments similar to those above show that $S_1^{-\infty}(+1)=\varnothing$ and the theorem is proved.

The three equations

(20) $$(tx'(t))''+x(t)/8t^2=0,\qquad t>0$$

(21) $$(tx''(t))''+3x(t)/16t^3=0,\qquad t>0$$

and

$$(22) \qquad (tx''(t))''' + 45x^{1/5}(t)/32t^2 = 0, \quad t > 0$$

all satisfy the hypotheses of Theorem 3 and have the solutions $x_1(t) = -t^{1/2}$, $x_2(t) = t^{3/2}$ and $x_3(t) = t^{5/2}$ respectively. Notice that each $x_1(t)$, $x_2(t)$ and $x_3(t)$ belongs to $S_2^{+\infty}(+1) \cup S_2^{-\infty}(+1)$ with $N = n - 2$.

It is interesting to compare Theorem 3 with similar results obtained by Philos [6; Th. 2] and Staikos and Sficas [10; Th. 3]. In [6] and [10] the authors used more restrictive integral conditions than (2) and (11) — (13). In so doing they were able to show that when n is odd $S_2^{+\infty}(+1) = S_2^{-\infty}(+1) = \emptyset$. Notice that examples (20) and (22) above have solutions of this type.

Furthermore, under their more restrictive integral conditions, Philos [6; Th. 3] and Staikos and Sficas [10; Th. 2] were able to show that

$$S(-1) = S^{\sim}(-1) \cup S^0(-1) \cup S_1^{+\infty}(-1) \cup S_1^{-\infty}(-1)$$

for n even, and for n odd that

$$S(-1) = S^{\sim}(-1) \cup S_1^{+\infty}(-1) \cup S_1^{-\infty}(-1).$$

To see that conditions (2) and (11) — (13) above do not imply this decomposition of solutions of (1) when $\delta = -1$, consider the equation

$$[x^{(4)}(t)/(t+1)]' - 120x^{1/5}(t)/t(t+1)^2 = 0, \qquad t > 0.$$

Here $\nu = 1$ and (2) and (11) are satisfied. Moreover, this example has the solution $x(t) = x^5/120$ which belongs to the set $S_2^{+\infty}(-1)$. Thus under our integral conditions, when $\delta = -1$, Theorem 2 already gives us the best possible decomposition.

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