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ON THE TOTAL CURVATURE OF IMMERSED MANIFOLDS, V: *C*-SURFACES IN EUCLIDEAN *m*-SPACE

BY

BANG-YEN CHEN

Abstract. A C-surface is a surface in an Euclidean m-space E^m which is the image of a flat surface under a conformal mapping on E^m. In this paper, we prove that the total mean curvature of any compact C-surface is $\geq 2\pi^2$. Some related problems are also studied.

1. Introduction. In the classical theory of surfaces in an Euclidean m-space E^m, the two most important curvatures are the so-called Gauss curvature G and the mean curvature α. It is well-known that the Gauss curvature is intrinsic and its integral gives the beautiful Gauss-Bonnet formula. For the mean curvature of a compact surface M in E^m, the total mean curvature satisfies [2I, 9]

$$\int_M \alpha^2\, dV \geq 4\pi\,. \tag{1.1}$$

The equality holds when and only when M is an ordinary 2-sphere in an affine 3-space. It is known that the total mean curvature is invariant under conformal mappings on E^m [3, 8]. Moreover, it is closely related to the eigenvalues of Laplacian [1, 2IV, 7] and the self-intersection number of M [10]. In [2IV], it is shown that if M is a compact flat surface in E^4, then the total mean curvature is $\geq 2\pi^2$. It is interesting to know the answer to the following.

Problem. Is it true that every compact flat surface in E^m, $m \geq 4$, has total mean curvature $\geq 2\pi^2$?

It has been proved that the answer to this problem is affirmative if one of the following conditions hold; (a) M is pseudoumbilical [4], (b) M has flat normal connection, or (c) M is isometric to the product of two plane circles of the same radius.

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In the following, we call a surface M in E^m to be a *C-surface* if it is the image of a compact flat surface in E^m under a conformal mapping on E^m. By a *conformal Clifford torus* we mean a compact surface in E^m which is the image of the Clifford torus $T^2 = S^1(a) \times S^1(a) \subset E^4 \subset E^m$ under a conformal mapping on E^m [5] ($S^1(a)$ is a plane circle of radius a).

The main purpose of this paper is to prove the following theorem which gives a complete answer to the proposed problem.

THEOREM 1. *If M is a compact C-surface in E^m, then we have*

$$\int_M \alpha^2 \, dV \geq 2\pi^2 . \tag{1.2}$$

The quality sign of (1.2) *holds when and only when M is a conformal Clifford torus.*

In view of Theorm 1 it is interesting to know when two compact flat surfaces are related by a conformal mapping on E^m. In this respect, we prove the following.

THEOREM 2. *If there is a conformal mapping on E^m which carries a compact flat surface M into another compact flat surface $\bar{M}$, then there exists a similarity transformation on E^m which carries M into $\bar{M}$.*

By similarity transformations on E^m we mean motions and homothetics on E^m.

2. **Preliminaries.** Let M be a surface in an Euclidean m-space E^m. We choose a local field of orthonormal frames $e_1, e_2, e_3, \cdots, e_m$ in E^m such that, restricted to M, the vectors e_1, e_2 are tangent to M and $e_3, \cdots, e_m$ are normal to M. Let $\omega^1, \omega^2, \omega^3, \cdots, \omega^m$ be the field of dual frames. Then the structure equations of E^m are given by

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \qquad \omega^A_B + \omega^B_A = 0 , \tag{2.1}$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B, \qquad A, B, C = 1, 2, 3, \cdots, m . \tag{2.2}$$

We restrict these forms to M. Then $\omega^3 = \cdots = \omega^m = 0$. Since

$$0 = d\omega^r = -\sum \omega^r_i \wedge \omega^i, \qquad r, s, t = 3, \cdots, m; \; i, j, k = 1, 2, \tag{2.3}$$

by Cartan's lemma, we may write

$$\omega_i^r = \sum h_{ij}^r \omega^j, \qquad h_{ij}^r = h_{ji}^r . \tag{2.4}$$

We call $h = \sum h_{ij}^r \omega^i \omega^j e_r$ the second fundamental form M. The mean curvature vector H is given by

$$\frac{1}{2} \sum (h_{11}^r + h_{22}^r) e_r .$$

The Gauss curvature G and the mean curvature α are given respectively by

$$G = \sum_{r=3}^{m} (h_{11}^r h_{22}^r - h_{12}^r h_{12}^r), \tag{2.5}$$

$$\alpha = \frac{1}{2} \left[\sum_{r=3}^{m} (h_{11}^r + h_{22}^r)^2 \right]^{1/2} . \tag{2.6}$$

For a normal vector $e = \sum_r a_r e_r$ at $x \in M$, the second fundamental tensor $A(x, e)$ at (x, e) is given by $(\sum_{r=3}^{m} a_r h_{ij}^r)$. The Lipschitz-Killing curvature $K(x, e)$ is defined by

$$K(x, e) = \det (A(x, e)) . \tag{2.7}$$

3. **Proof of Theorem 1.** Since every compact C-surface in E^m is the image of a compact flat surface in E^m under a conformal mapping on E^m and the total mean curvature of a compact surface in E^m is invariant under the conformal mappings on E^m, we may assume that M itself is a compact flat surface in E^m.

For each $x \in M$, we denote by $T_x^{\perp}$ the normal space of M at x. Let $A(e)$ denote the second fundamental tensor at (x, e) (for any normal vector e at x). We define a linear mapping γ from $T_x^{\perp}$ into the space of all symmetric matrices of order 2 by

$$\gamma \left(\sum_r a_r e_r \right) = \sum_r a_r A(e_r) . \tag{3.1}$$

Let O_x denote the kernel of γ. Then we have $\dim O_x \geq m - 5$. We define N_x to be the subspace of the normal space given by

$$T_x^{\perp} = N_x \oplus O_x, \qquad N_x \perp O_x .$$

Then we have $A(\bar{e}) = 0$ for any $\bar{e} \in O_x$. We choose $e_3, \cdots, e_m$ at x in such a way that $e_6, \cdots, e_m \in O_x$. Then for any unit normal vector e at x, (3.2) $$e = \sum \cos \theta_r e_r,$$

the Lipschitz-Killing curvature $K(x, e)$ at (x, e) is given by

(3.3) $$K(x, e) = \left(\sum_{r=3}^{5} \cos \theta_r h_{11}^r\right)\left(\sum_{s=3}^{5} \cos \theta_s h_{22}^s\right) - \left(\sum_{t=3}^{5} \cos \theta_t h_{12}^t\right)^2.$$

The right-hand side of (3.3) is a quadratic form on $\cos \theta_r$. Hence, by choosing a suitable unit orthogonal normal vectors e_3, e_4, e_5 at x, we may write

(3.4) $$K(x, e) = \lambda_1(x) \cos^2 \theta_3 + \lambda_2(x) \cos^2 \theta_4 + \lambda_3(x) \cos^2 \theta_5, \qquad \lambda_1 \geq \lambda_2 \geq \lambda_3.$$

Moreover, since M is a flat surface, we have

(3.5) $$\lambda_1 + \lambda_2 + \lambda_3 = 0, \qquad \lambda_A = h_{11}^{2+A} h_{22}^{2+A} - (h_{12}^{2+A})^2, \qquad A = 1, 2, 3.$$

In particular, we have $\lambda_1 \geq 0$ and $\lambda_3 \leq 0$. Now, we consider the cases $\lambda_2 \geq 0$ and $\lambda_2 < 0$ separately.

Case 1: $\lambda_1, \lambda_2 \geq 0$. From (3.5), we have

(3.6) $$K(x, e) = \lambda_1(\cos^2 \theta_3 - \cos^2 \theta_5) + \lambda_2(\cos^2 \theta_4 - \cos^2 \theta_5).$$

Hence

(3.7) $$\begin{aligned} \int_{S_x} |K(x, e)| d\sigma &= \int_{S_x} |\lambda_1(\cos^2 \theta_3 - \cos^2 \theta_5) \\ &\qquad + \lambda_2(\cos^2 \theta_4 - \cos^2 \theta_5)| d\sigma \\ &\leq \lambda_1(x) \int_{S_x} |\cos^2 \theta_3 - \cos^2 \theta_5| d\sigma \\ &\qquad + \lambda_2(x) \int_{S_x} |\cos^2 \theta_4 - \cos^2 \theta_5| d\sigma, \end{aligned}$$

where S_x is the unit hypersphere of $T_x^\perp$ and $d\sigma$ is the volume element of S_x. By a formula on spherical integration, we have

(3.8) $$\int_{S_x} |\cos^2 \theta_r - \cos^2 \theta_s| d\sigma = 2c_{m-1}/\pi^2, \qquad r \neq s,$$

where c_{m-1} denotes the area of the unit $(m-1)$-sphere. Hence if we denote the left hand side of (3.7) by $K^*(p)$, then we have

(3.9) $$\lambda_1(x) + \lambda_2(x) \geq K^*(x)\pi^2/2c_{m-1}.$$

On the other hand, (2.6) gives

$$
\begin{aligned}
4\alpha^2 &= (h_{11}^3 + h_{22}^3)^2 + (h_{11}^4 + h_{22}^4)^2 + (h_{11}^5 + h_{22}^5)^2 \\
&\geq (h_{11}^3)^2 + (h_{22}^3)^2 + 2\lambda_1 + 2(h_{12}^3)^2 + (h_{11}^4)^2 + (h_{22}^4)^2 \\
&\qquad + 2\lambda_2 + 2(h_{12}^4)^2 \\
&\geq 4(\lambda_1 + \lambda_2) + 4(h_{12}^3)^2 + 4(h_{12}^4)^2 \\
&\geq 4(\lambda_1 + \lambda_2)\,.
\end{aligned} \tag{3.10}
$$

Combining (3.9) and (3.10) we obtain

$$\alpha^2 \geq K^*(x)\pi^2/2c_{m-1}\,. \tag{3.11}$$

Case 2: $\lambda_2, \lambda_3 < 0$. From (3.5) we have

$$K(x, e) = \lambda_2(\cos^2\theta_4 - \cos^2\theta_3) + \lambda_3(\cos^2\theta_5 - \cos^2\theta_3)$$

Hence (3.5) and (3.8) we have

$$
\begin{aligned}
\int_{S_x} |K(x, e)|\,d\sigma &\leq -\lambda_2 \int_{S_x} |\cos^2\theta_4 - \cos^2\theta_3|\,d\sigma \\
&\qquad - \lambda_3 \int_{S_x} |\cos^2\theta_5 - \cos^2\theta_3|\,d\sigma \\
&= 2\lambda_1 c_{m-1}/\pi^2\,.
\end{aligned} \tag{3.12}
$$

From which we find

$$\lambda_1 \geq K^*(x)\pi^2/2c_{m-1}\,. \tag{3.13}$$

On the other hand, we have

$$
\begin{aligned}
4\alpha^2 &= (h_{11}^3 + h_{22}^3)^2 + (h_{11}^4 + h_{22}^4)^2 + (h_{11}^5 + h_{22}^5)^2 \\
&\geq (h_{11}^3)^2 + (h_{22}^3)^2 + 2\lambda_1 + 2(h_{12}^3)^2 \\
&\geq 4\lambda_1 + 4(h_{12}^3)^2 \geq 4\lambda_1\,.
\end{aligned} \tag{3.14}
$$

Hence, by (3.13), we get

$$\alpha^2 \geq K^*(x)\pi^2/2c_{m-1}\,.$$

Consequently, we always have the following inequality:

$$\int_M \alpha^2\,dV \geq \pi^2 \int_M K^*\,dV/2c_{m-1}\,. \tag{3.15}$$

On the other hand, since M is compact and flat, a well-known inequality of Chern-Lashof gives

$$\int_M K^*\,dV \geq 4c_{m-1}\,. \tag{3.16}$$

Therefore, from (3.15) and (3.16), we obtain (1.2).

If the equality sign of (1.2) holds, the inequalities in (3.7) and

(3.12) become equalities. Hence at least one of λ_1 and λ_2 is zero for case 1) and at least one of λ_2 and λ_3 is zero for case 2). The second case is impossible by our assumption. Thus we have $\lambda_2 \equiv 0$ on M. Furthermore, since the inequalities of (3.10) are actually equalities in this case, we have $h^3_{11} = h^3_{22}$, $h^3_{12} = h^4_{12} = 0$, $h^4_{11} = h^4_{22}$ and $h^5_{11} + h^5_{22} = 0$. Consequently, by $\lambda_2 \equiv 0$ we obtain

$$h^3_{11} = h^3_{22}, \quad h^3_{12} = 0, \quad h^4_{ij} = 0, \quad h^5_{11} + h^5_{22} = 0\,. \tag{3.17}$$

This shows that M is a compact, pseudo-umbilical, flat surface in E^m with flat normal connection and whose total mean curvature is $2\pi^2$. From these we may conclude that M is a Clifford torus in an affine 4-space E^4 (cf. Theorem 2 of [4]). This proves Theorem 1.

4. **Proof of Theorem 2.** A conformal mapping on an Euclidean m-space E^m can be decomposed into a product of similarity transformations and inversions. It is obvious that similarity transformations always carry a compact flat surface into a compact flat surface. Thus it suffices to study the problem for inversions. Let φ be an inversion on E^m such that the center of φ does not lie on the surface M. We choose the origin to be the center of the inversion φ. Let x and $\bar{x}$ be the position vectors of the original surface M and the inverse surface $\bar{M}$, respectively and c the radius of the inversion φ. Then we have

$$\bar{x} = (c^2/r^2)x, \qquad r^2 = x\cdot x\,,$$

where "$\cdot$" denotes the inner product on E^m. From this we obtain

$$d\bar{x} = (c^2/r^2)dx - (2c^2/r^3)(dr)x, \tag{4.1}$$

$$d\bar{x}\cdot d\bar{x} = (c^4/r^4)(dx\cdot dx)\,. \tag{4.2}$$

Let $e_3,\cdots, e_m$ be any orthonormal normal frame on M. Then

$$\bar{e}_t = \frac{2(x\cdot e_t)}{r^2}\,x - e_t, \qquad t = 3,\cdots, m \tag{4.3}$$

form an orthonormal normal frame on $\bar{M}$. From (4.1) and (4.3) we obtain

$$d\bar{x}\cdot d\bar{e}_t = \left(\frac{2c^2(x\cdot e_t)}{r^4}\right)(dx\cdot dx) - \left(\frac{c^2}{r^2}\right)(dx\cdot de_t)\,. \tag{4.4}$$

Thus, by (4.2) and (4.4), we find that the principal curvatures $k_i(e_t)$ of M in the direction e_t satisfy

$$\bar{k}_i(\bar{e}_t) = -\left(\frac{r^2}{c^2}\right) k_i(e_t) - \frac{2}{c^2}(x \cdot e_t), \qquad i = 1, 2, \tag{4.5}$$

where $\bar{k}_i(\bar{e}_t)$ denote the corresponding principal curvatures of $\bar{M}$. Therefore, the Gauss curvatures G and $\bar{G}$ of M and $\bar{M}$ satisfy

$$G(\bar{x}) = \frac{r^4}{c^4} G(x) + \frac{4r^2}{c^4}(x \cdot H) + \frac{4}{c^4}(x_n \cdot x_n), \tag{4.6}$$

where H and x_n denote the mean curvature vector and the normal component of x on M, respectively.

If M and $\bar{M}$ are both flat, then (4.6) reduces to

$$-x \cdot H = \frac{x_n \cdot x_n}{r^2}. \tag{4.7}$$

Consequently, we have

$$-\int_M (x \cdot H) dV = \int_M \left(\frac{x_n \cdot x_n}{r^2}\right) dV. \tag{4.8}$$

On the other hand, since M is compact, a well-known formula of Minkowski-Hsiung (cf. [6] for instance) gives

$$\int_M (x \cdot H) dV + \int_M dV = 0. \tag{4.9}$$

Combining (4.8) and (4.9) we find

$$\int_M \left(1 - \frac{x_n \cdot x_n}{r^2}\right) dV = 0.$$

Since $x_n \cdot x_n \leq r^2$, this implies that $x = x_n$. That is, the position vector x on M is always normal to M. Consequently, M must lie in a hypersphere of E^m centered at the origin. Thus the conformal mapping φ on E^m, restricted to M, gives a homothetic transformation on M. This proves Theorem 2.

5. **Remarks.**

1. In view of Theorem 1 and known results on total mean curvature, the author would like to make the following conjecture.

Conjecture. Every compact surface M of genus ≥ 1 in E^m satisfies

$$\int_M \alpha^2 \, dV \geq 2\pi^2 . \tag{1.3}$$

And the equality holds when and only when M is a conformal Clifford torus.

2. From the proof of Theorem 2, we also have the following.

COROLLARY. *A compact flat surface in E^m is spherical if and only if there exists an inversion on E^m which carries it into a compact flat surface.*

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, U. S. A.