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PARABOLIC EQUATIONS ON INFINITE DIMENSIONAL SPACES(*)

BY

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Abstract. Let (H, B) be an abstract Wiener space. In this paper, we investigate the existence and regularity properties of solutions for the initial value problem associated with the inifinite dimensional heat equation $u_t(x, t) = \frac{1}{2}\,\text{trace}_H D^2 u(x, t)$, where $t > 0$, $x \in B$. We extend a L. Gross' theorem so that the above initial value problem is solvable for a wider class of initial functions which contains both the class of bounbed Lip-1 functions and the class of real analytic functions of exponential type. The heat equation with constant coefficient is also considered in this paper.

1. **Introductions.** Let $(B, \|\ \|)$ be a real separable Banach space. There always exists a real separable Hilbert space $(H, |\ |)$ containning in B such that H is dense in B and that the B-norm $\|\ \|$ is measurable on H. The pair (H, B) is called an abstract Wiener space (see [1]). It is well-known that the Gauss cylinder set measure (with variance $t > 0$) on H extends to a (σ-additive) Borel measure p_t on B. p_t is called the Wiener measure (with variance t) (see [1]). Integration over B is then performed with respect to p_t.

If f is a real-valued function defined on B, we may regard f as a function g defined in a neighborhood of the origin of H by restricting f to the coset $x + H$ and defining $g(h) = f(x + h)$. If g is k-times Frechet-differentiable at 0, then we say that f is k-times H-differentiable at x and denote the derivative by $D^k f(x)$. After L. Gross [2], if f is a twice H-differentiable function on B such that $D^2 f(x)$ is a trace class operator, we define the Laplacian of f (at x) $\Delta f(x)$ by $\Delta f(x) = \text{trace}_H D^2 f(x)$. In [2], Gross showed that if f is a bounded Lip-1 function on B, then

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$$u(t, x) = p_t f(x) = \int_B f(x+y)\, p_t(dy)$$

solves the Cauchy problem associated with the heat equation:

$$u_t(t, x) = \frac{1}{2} \text{trace}\,_H D^2 u(t, x), \tag{1}$$

and

$$\lim_{t\to 0} u(t, x) = f(x) \quad \text{uniformly in } B. \tag{2}$$

In [7; §7], we showed that for each member of $\mathcal{E}_a(B)$—the space of expontial type real analytic functions on B (see [6] or [7]), the function $u(t, x) = p_t f(x)$ also solves equation (1) and $\lim_{t\to 0} u(t, x) = f(x)$ uniformly on every bounded set. Observe that $\mathcal{E}_a(B)$ is obviously different from the space $\mathcal{A}$ of bounded Lip-1 functions. It is desirable to find a larger class of $\mathcal{S}$ functions which contains both $\mathcal{E}_a(B)$ and $\mathcal{A}$ so that if $f \in \mathcal{S}$ $u(t,x)=p_t f(x)$ will solve the eqtation (1) and $u(t, x) \to f(x)$ (as $t \to 0$) in some sense.

For this purpose, we introduce a "weight function" $w(x)$ which satisfies the following conditions:

(C-1) $w(x)$ is measurable and $w(x) \geq 1$ for each x in B;
(C-2) $w(x+y) \leq w(x)w(y)$;
(C-3) $w(ax) \leq w(x)^{|a|} (a \in R)$;
(C-4) $w(x)$ is locally bounded in B.

(Examples of weight functions are $w(x) = 1$; $e^{\|x\|}$; $\exp[|(g, x)|]$ $(g \in B^*)$, etc.)

Let L_w be the class of measurable functions such that for x, y in B,

(C-5) $$|f(x) - f(y)| \leq c\cdot w(x)^{c'} w(y)^{c'} \|x - y\|,$$

for some constants $c, c' > 0$ which depend only on f.

It turns out that if $f \in L_w$, $u(t, x) = p_t f(x)$ solves equation (1) and that $u(t, x) \to f(x)$ uniformly on every set on which w is bounded. It follows that our results imply [2; Theorem 3] and [7; Theorem 7.7] by taking $w(x) = 1$ and $w(x) = e^{\|x\|}$, respectively.

2. **Solution of** $u_t = \frac{1}{2}\Delta u$. Through out this section, $\langle \cdot, \cdot \rangle$

will denote the inner product in H; $(\cdot, \cdot)$ will denote the (B^*, B) pairing; and B^* is embeded in $H^* \approx H$ in the sense that if $y \in B^*$, $x \in H$, then $(y, x) = \langle y, x \rangle$. (It is also worth to know that if $h \in H$, $\langle h, \cdot \rangle$ is define almost everywhere in B with respect to p_t.) If $w(x)$ is a weight function defined as in §1, a set on which $w(x)$ is bounded will be called a *w-bounded set*. By a C_H^1 function we mean a real-valued H-differentiable function f defined on B such that $Df(x)$ is continuous from B into H^*(with norm topology).

Our goal in this section is to show that the following Theorem holds.

THEOREM 1. *Let w be a weight function and f be a member of L_w. Then we have*

(a) *$D^2(p_t f)(x)$ is a trace class operator for each x in B and each $t > 0$.*

(b) *For each pair of positive numbers a, a' and every w-bounded set U, the map $(t, x) \to D^2(p_t f)(x)$ is uniformly continuous on $[a, a'] \times U$ into the Banach space of trace class operators on H.*

(c) *The function $v(t, x) = (p_t f)(x)$ is jointly uniformly continuous on $[0, \alpha] \times U$, where U is a w-bounded set and α is a finite real number.*

(d) *For each $t > 0$, $\partial v/\partial t$ exists uniformly in x on any w-bounded set and*

$$(\partial v/\partial t)(t, x) = \frac{1}{2} \operatorname{trace}_H D^2 v(t, x) .$$

Moreover, $\lim_{t \to 0} v(t, x) = f(x)$ *uniformly for x in every w-bounded set.*

REMARK 1. L. Gross' proof of [2; Theorem 3] has been simplified by Kuo [4] whose approach (after some suitable modifications) makes our proof of Theorem 1 possible. ///

REMARK 2. If $w(x)$ is a weight function, it is easy to see that

$$w(x) \leq e^{c\|x\|}$$

for some constant $c > 0$. Therefore, by Fernique's theorem (see, for example, [4; p. 159]), that $w(x) \in L^p(B)$ for all $p \geq 1$ and

$L_w \subset L_{\exp(\|x\|)}$ (this implies that $L_w \subset L^p(B)$ for all $p \geq 1$). It $f \in L_w$, it is easy to see that there are constants c, c' such that

$$|f(x)| \leq c \cdot e^{c'\|x\|} \quad \text{for all } x \; B. \tag{3}$$

Therefore, $p_t f(x)$ exists for all $f \in L_w$. ///

Our proof will be accomplished by the following Lemmas and propositions. We start with the

PROPOSITION 1. *Let w be a weight function and f a measurable function on B having the property* (3) (*Remark* 2). *Let $t > 0$ be fixed and $g(x) = (p_t f)(x)$. We have*

(i) *$g(x)$ also has the property* (3) *for some constants c, c'.*

(ii) *$g(x)$ is infinitely H-differentiable on B with first and second derivative given by*

$$\langle Dg(x), h\rangle = \frac{1}{t}\int_B f(x+y)\langle h, y\rangle\, p_t(dy) \qquad (h \in H), \tag{4}$$

$$\begin{aligned}\langle D^2 g(x)k, h\rangle = \frac{1}{t}\int_B f(x+y)\\ \cdot\left[\frac{1}{t}\langle h, y\rangle\langle k, y\rangle - \langle h, k\rangle\right] p_t(dy) \qquad (h, k \in H).\end{aligned} \tag{5}$$

(iii) For each n, there are constants c, c' such that

$$|D^n g(x)\, h_1 \cdots h_n| \leq (nt^{-1})^{(1/2)n} \cdot c[w(x)]^{c'} \cdot |h_1| \cdots |h_n|.$$

Proof. (i) is trivial.

(ii) The H-differentiability of $g(x)$ and equation (4) follow by Remark 2 and [3; Proposition 1]. Now we consider the function $u(k) = \langle Dg(x+k), h\rangle$. Using the arguments in the proof of [2; Prop. 9], we may write

$$\begin{aligned}u(k) = \frac{1}{t}\int_B f(x+y)\langle h, y\rangle\, p_t(k, dy)\\ - \frac{1}{t}\langle h, k\rangle \int_B f(x+y)\, p_t(k, dy)\end{aligned}$$

(though $\langle h, \cdot\rangle$ is not linear on B), where $p_t(x, dy) = p_t(dy - x)$. Anain, by [3; Prop. 1] and (4), we obtain that

$$\begin{aligned}u'(0)k = \frac{1}{t}\cdot\frac{1}{t}\int_B f(x+y)\langle h, y\rangle\langle k, y\rangle\, p_t(dy)\\ - \frac{1}{t}\langle h, k\rangle \int_B f(x+y)\, p_t(dy),\end{aligned}$$

and (5) follows.

The existence of higher order H-differentiations of g follow in the same way. (See [2; Prop. 9].)

(iii) follows by [5; Proposition 1 and 2].

COROLLARY 1. *In addition to the assumptions on f in Proposition 1, if we assume that f is continuous on B, then $(p_t f)(x)$ is a C_H^1 function.*

DEFINITION 1. A bounded linear operator of B with finite dimensional range contained in B^* is called a test operator [2].

PROPOSITION 2. *Let f a function as in Corollary 1. If a, b are positive numbers such that $a+b=1$, then we have*

(a) $\langle D(p_t f)(x), h\rangle$

$$= \int_B \langle D(p_{bt} f)(x+y), h\rangle\, p_{at}(dy) \qquad (h \in H);$$

(b) $\langle D^2(p_t f)(x)k, h\rangle$

$$= \frac{1}{at}\int_B \langle D(p_{bt} f)(x+y), h\rangle\langle k, y\rangle\, p_{at}(dy) \quad (h, k \in H);$$

(c) If T is a test operator and T' is the restriction of T on H, then the following equality holds:

$$\text{trace}_H[T' D^2(p_t f)(x)] = \frac{1}{at}\int_B (D(p_{bt} f)(x+y), Ty)\, p_{at}(dy) \tag{6}$$

Proof. First of all we write $(p_t f)(x) = \int_B (p_{bt} f)\ (x+y) p_{at}$ (dy).

(a) It is easy to estimate that

$$(p_{bt} f)(x+h) \leq \text{const. } w(x)^{c'} \cdot \exp(|h|^2/bt)$$

and

$$|D(p_{bt} f(x+k)| \leq \frac{1}{\sqrt{bt}} w(x)c' \cdot \exp(|k|^2/bt)\cdot(1+|k|)\cdot c(b, t),$$

where

$$c(b, t) = \text{const.}\left[\int_B w(y)^{4c'}\, p_{bt}(dy)\right]^{1/4}.$$

Thus (a) is a consequence of [3; Proposition 2].

(b) follows by Proposition 1 and (a).

(c) follows from (b). ///

LEMMA 1. *If g is a C_H^1 function in L_w, then for each x in B, $Dg(x) \in B^*$ and $\|Dg(x)\|_{B^*} \leq c[w(x)]^{c'}$ for some constant c, c'. Moreover, for each pair of x, $y \in B$,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (g(x + \varepsilon y) - g(x)) = (Dg(x), y).$$

Proof. Suppose that g satisfies (C–5) and $x \in B$ be fixed and h be an element in H. Then

$$\langle Dg(x), h \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (g(x + \varepsilon h) - g(x)),$$

and consequently.

$$|\langle Dg(x), h \rangle| \leq \lim_{\varepsilon \to 0} \frac{c}{\varepsilon} w(x)^{2c'} \cdot w(h)^{\varepsilon c'} \cdot \|\varepsilon h\| = c \cdot w(x)^{c'} \cdot \|h\| .$$

Thus $Dg(x) \in B^*$.

Now let x, $y \in B$ be fixed and η be any positive number. We can choose an $h \in H$ such that $\|y - h\| < \eta$. Then

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} (g(x + \varepsilon y) - g(y)) - (Dg(x), y) \right| \\
&\leq \frac{1}{\varepsilon} |g(x + \varepsilon y) - g(x + \varepsilon h)| \\
&\qquad + \left| \frac{1}{\varepsilon} (g(x + \varepsilon h) - g(x)) - (Dg(x), h) \right| \\
&\qquad + |(Dg(x), h) - (Dg(x), y)| \\
&\leq c \cdot w(x)^{2c'} \cdot w(y)^{\varepsilon c'} \cdot w(h)^{\varepsilon c'} \|y - h\| \\
&\qquad + \left| \frac{1}{\varepsilon} (g(x + \varepsilon h) - g(x)) - (Dg(x), h) \right| \\
&\qquad + c \cdot w(x)^{c'} \cdot \|y - h\| .
\end{aligned}$$

Let $\varepsilon \to 0$. We get

$$\lim_{\varepsilon \to 0} \left| \frac{1}{\varepsilon} (g(x + \varepsilon y) - g(x)) - (Dg(x), y) \right| \leq 2c \cdot w(x)^{2c'} \cdot \eta .$$

Since η is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(g(x+\varepsilon y)-g(x))=(Dg(x), y). \quad ///$$

PROPOSITION 3. *Let f be function in L_w which satisfies (C–5). Then we have*

(a) *$p_t f \in L_w$ for each $t>0$;*

(b) *$D(p_t f)(x) \in B^*$ and there are constants c, c' such that*

$$\|D(p_t f)(x)\|_{B^*} \le c\cdot w(x)^{c'}\left(\int_B w(z)^{2c'}\,p_t(dz)\right) \tag{7}$$

(c) *If $g(s)=(p_t f)(x+sy)$, then $(d/ds)g(s)=(D(p_t f)(x+sy), y)$.*

(d) *$D^2(p_t f)(x)$ is a trace class operator for each $x\in B$ and $t>0$. Moreover, we have*

$$\|D^2(p_t f)(x)\|_{\mathrm{tr}} \le \text{const. } t^{-1/2}\,w(x)^{c'}\cdot\left(\int_B w(y)^{2c'}\,p_{(1/2)t}(dy)\right)^{3/2},$$

where $\|\ \|_{\mathrm{tr}}$ denotes the trace norm.

Proof. (a) is trivial.

(b) and (c) follows by Lemma 1.

To prove (d), let T be a test operator on B. Applying Proposition 2(c) with $a=b=\frac{1}{2}$, we obtain

$$\begin{aligned}
&|\,\mathrm{trace}_H T'D^2(p_t f)(x)\,| \\
&\le \frac{2}{t}\cdot\int_B |(D(p_{(1/2)t}f)(x+y), Ty)|\,p_{(1/2)t}(dy) \\
&\le \left(\frac{2}{t}\right)^{1/2} c\left(\int_B w(y)^{2c'}\,p_{(1/2)t}(dy)\right)^{3/2} \\
&\quad\cdot\left(\int_B \|Ty\|^2 p_1(dy)\right)^{1/2} w(x)^{c'} \qquad \text{(by (7))} \\
&\le \left(\frac{2}{t}\right)^{1/2} c\left(\int_B w(y)^{2c'}\,p_{(1/2)t}(dy)\right)^{3/2} \\
&\quad\cdot\left(\int_B \|y\|^2\,p_1(dy)\right)^{1/2}\|T'\|_{H,H}\,w(x)^{c'}.
\end{aligned}$$

(The last inequality follows by [2; Equation. (36)].)

Thus we have proved that

$$\begin{aligned}
&|\,\mathrm{trace}_H T'D^2(p_t f)(x)\,| \\
&\le \text{const. }\frac{1}{\sqrt{t}}\,w(x)^{c'}\cdot\left[\int_B w(y)^{2c'}\,p_{(1/2)t}(dy)\right]^{3/2}\|T'\|_{H,H}
\end{aligned} \tag{8}$$

for every test operator T on B.

Since the restrictions of test operators to H are dense in the Banach space of compact operators of H, it follows form (8) that $D^2(p_t f)(x)$ is a trace class operator with trace class norm

$$\|D^2(p_t f)(x)\|_{tr} \leq \text{const.} \frac{1}{\sqrt{t}} \cdot w(x)^{c'} \cdot \left(\int_B w(y)^{2c'} p_{(1/2)t}(dy)\right)^{3/2}. \quad ///$$

To prove (b) of Theorem 1, we need the Corollary 4.2 of [4; Chapter 2] which states that in the abstract Wiener space (H, B), there exists another abstract Wiener space (H, B_0) such that the B_0-norm $\|\cdot\|_0 \geq \|\cdot\|$ and there exist an increasing sequence of finite rank projections $\{P_n\}$ on H such that (1) P_n converges to the identy operator on H strongly; (2) each P_n extends to a projection $\tilde{P}_n$ of B_0 such that $\tilde{P}_n$ converges strongly to the identy on B_0 (w. r. t. $\|\cdot\|_0$). It follows that we have the

LEMMA 2. $\lim_{n\to\infty} \int_{B_0} \|\tilde{P}_n y - y\|_0^r \, p_1(dy) = 0 \qquad (r \geq 1)$.

LEMMA 3. *Let f be a function in L_w and U be a w-bounded set. For each pair of a, $a' > 0$, the map $(t, x) \to D^2(p_t f)(x)$ is uniformly continuous from $[a, a'] \times U$ into the Banach space of Hilbert-Schmidt operator on H (we will denote the Hilbert-Schmidt norm by $\|\cdot\|_{H-S}$).*

Proof. By Proposition 1, we have, for any $h \in H$,

$$\langle D(p_t f)(x), h\rangle = \frac{1}{\sqrt{t}} \cdot \int_B f(x + \sqrt{t}\, y)\langle h, y\rangle p_1(dy).$$

It follows that if $s, t \in [a, a']$ and $x, x' \in U$ we obtain

$$\begin{aligned}
&|\langle D(p_t f)(x) - D(p_s f)(x'), h\rangle| \\
&\quad \leq c \cdot w(x)^{c'} \cdot |h| \\
&\qquad \cdot \left\{ |\sqrt{t} - \sqrt{s}| \left[\int_B w(y)^{\sqrt{a'}c'} \|y\| \left| \langle \frac{h}{|h|}, y\rangle \right| p_1(dy)\right] \right. \\
&\qquad \left. + \|x - x'\| \left[\int_B w(y)^{\sqrt{a'}c'} \left|\langle \frac{h}{|h|}, y\rangle \right| p_1(dy)\right]\right\} \\
&\qquad + \frac{1}{a} |\sqrt{s} - \sqrt{t}\,| c'' w(x)^{c'''} \\
&\qquad\quad \cdot \left[\int_B w(y)^{\sqrt{a'}c'''} \left|\langle \frac{h}{|h|}, y\rangle\right| p_1(dy)\right)\Big]|h|
\end{aligned}$$

so that

$$\begin{aligned}(9)\qquad &|D(p_t f)(x) - D(p_s f)(x')| \\ &\le C\cdot w(x)^{C'}(|\sqrt{t} - \sqrt{s}| + \|x - x'\|),\end{aligned}$$

where C, C' are constants.

Let T be a test operator. By the formula (6), we have

$$\text{trace}_H T'D^2(p_t f)(x) = \frac{\sqrt{2}}{\sqrt{t}}\int_B\left(D(p_{(1/2)t} f)\left(x + \frac{\sqrt{t}}{\sqrt{2}}y\right), Ty\right) p_1(dy).$$

Consequently,

$$\begin{aligned}(10)\qquad &|\text{trace}_H T'D^2(p_t f)(x) - \text{trace}_H T'D^2(p_s f)(x')| \\ &\le C_1\left|\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{s}}\right| w(x)^{C'} \\ &\quad \cdot\left\{\int_B w(y)^{\sqrt{2a'}C'} p_1(dy)\right\}^{1/2}\left\{\int_B \|Ty\|^2 p_1(dy)\right\}^{1/2} \\ &\quad + C\cdot\frac{1}{\sqrt{s}} w(x)^{C'}\cdot\left\{\int_B (|\sqrt{s} - \sqrt{t}| \right. \\ &\quad \left. + \sqrt{2}\|x - x'\| + |\sqrt{s} - \sqrt{t}|\|y\|)|Ty|\, p_1(dy)\right\}\end{aligned}$$

(by (9)), where C_1 is a constant. Recall that $\left(\int_B |Ty|^2 p_1(dy)\right)^{(1/2)} = \|T'\|_{H\text{-}S}$ and note that $\|T'\|_{H,H} \le \|T'\|_{H\text{-}S}$. It follows by (10) that

$$\begin{aligned}(11)\qquad &\|D^2(p_t f)(x) - D^2(p_s f)(x')\|_{H\text{-}S} \\ &\le \text{const. } w(x)^{C'}\cdot\left(\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{t}} + |\sqrt{s} - \sqrt{t}| + \|x - x'\|\right),\end{aligned}$$

where the constant "const." depends only on a, a' and f.

The Lemma now follows by (11). ///

Proof of Theorem 1. (a) has been proved in Proposition 3.

For a proof of (b), let $\{P_n\}$ and B_0 be as in Lemma 2. We have seen in Lemma 3 that $D^2(p_{(\cdot)} f)(\cdot)$ is uniformly continuous from $[a, a] \times U$ into the Banach space of Hilbert-Schmidt operators, hence so is $P_n D^2(p_{(\cdot)} f)(\cdot)$ for each n. Note that

$$(12)\qquad \|P_n A\|_{tr} \le [\dim(P_n(H))]^{1/2}\|A\|_{H\text{-}S}$$

provided that A is a Hilbert-Schmidt operator on H. Therefore, for each n the map $(t, x) \to P_n D^2(p_t f)(x)$ is uniformly continuous from $[a, a'] \times U$ into the Banach space of trace class operators on H. Thus, to finish the proof of (b), we need only to show that

$$\lim_{n\to\infty} \|P_n D^2(p_t f)(x) - D^2(p_t f)(x)\|_{tr} = 0 \quad \text{on } [a, a'] \times U.$$

Let T be a test operator on B. Write

$$\begin{aligned}&\operatorname{trace}_H T' D^2(p_t f)(x)\\ &\quad = \frac{\sqrt{2}}{\sqrt{t}} \int_{B_0} \left(D\left(p_{(1/2)t} f\left(x + \frac{\sqrt{t}}{\sqrt{2}}\, y\right), Ty\right)\tilde{p}_1(dy)\end{aligned}$$

and

$$\begin{aligned}&\operatorname{trace}_H T' P_n D^2(p_t f)(x)\\ &\quad = \frac{\sqrt{2}}{\sqrt{t}} \int_{B_0} \left(D\left(p_{(1/2)t} f\left(x + \frac{\sqrt{t}}{\sqrt{2}}\, y\right), T\tilde{P}_n y\right)\tilde{p}_1(dy),\end{aligned}$$

where $(\cdot, \cdot)$ denotes the (B^*, B) pairing. Since $\|D(p_t f)(x)\|_{B^*} \leq c\, w(x)^{c'}$ by Proposition 3, hence

$$\begin{aligned}&|\operatorname{trace}_H T'(P_n D^2(p_t f)(x) - D^2(p_t f)(x))|\\ &\quad \leq \frac{c''}{\sqrt{t}}\, w(x)^{c'} \left(\int_{B_0} \|\tilde{P}_n y - y\|_0^2\, \tilde{p}_1(dy)\right)^{1/2} \|T'\|_{H,H}\end{aligned}$$

so that

$$\begin{aligned}&\|P_n D^2(p_t f)(x) - D^2(p_t f)(x)\|_{tr}\\ &\quad \leq \frac{c''}{\sqrt{a}}\, w(x)^{c'} \left(\int_{B_0} \|\tilde{P}_n y - y\|_0^2\, \tilde{p}_1(dy)\right)^{1/2}\end{aligned}$$

which converges uniformly to 0 on $[a, a'] \times U$. This proves (b).

(c) follows by the following inequality

$$\begin{aligned}&|v(t, x) - v(s, y)|\\ &\quad \leq c[w(x) w(y)]^{c'}\\ &\qquad \cdot \left[\int_B w(z)^{c'\sqrt{a}} (\|x - y\| + |\sqrt{t} - \sqrt{s}|\, \|z\|)\, p_1(dz)\right]\end{aligned}$$

To prove (d), we use the approach given in [4; p. 180] by writing

$$(p_{t+\varepsilon} f)(x) = \int_B (p_{\varepsilon^2(t+\varepsilon)} f)(x + \sqrt{(1-\varepsilon^2)(t+\varepsilon)} \cdot y)\, p_1(dy)$$

and

$$(p_t f)(x) = \int_B (p_{\varepsilon^2 t} f)(x + \sqrt{(1-\varepsilon^2)\, t} \cdot y)\, p_1(dy).$$

Let

$$\theta(\varepsilon, x) = \int_B (p_{\varepsilon^2(t+\varepsilon)} f - p_{\varepsilon^2 t} f)(x + \sqrt{(1-\varepsilon^2)(t+\varepsilon)} \cdot y) p_1 (dy)$$

and

$$\begin{aligned}\gamma(\varepsilon, x) = \int_B [(p_{\varepsilon^2 t} f)(x + \sqrt{(1-\varepsilon^2)(t+\varepsilon)}\, y) \\ - (p_{\varepsilon^2 t} f)(x + \sqrt{(1-\varepsilon^2)t}\, y)] p_1 (dy).\end{aligned}$$

We have $(p_{t+\varepsilon} f)(x) - (p_t f)(x) = \theta(\varepsilon, x) + \gamma(\varepsilon, x)$.

For the sake of simplicity, we assume that $\varepsilon \leq 1$. We see easily that

$$\frac{1}{\varepsilon} |\theta(\varepsilon, x)| \leq c |\sqrt{t+\varepsilon} - \sqrt{t}| w(x)^{c'} \cdot \left(\int_B w(y)^{2c'\sqrt{t+1}} p_1 (dy)\right)^2$$

so that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |\theta(\varepsilon, x)| = 0 \quad \text{uniformly in } U.$$

By Proposition 3(c), we may write

$$\begin{aligned}\gamma(\varepsilon, x) = \int_0^1 \int_B \lambda(t, \varepsilon) D(p_{\varepsilon^2 t} f)(x + \sqrt{(1-\varepsilon^2)t}\, y) \\ + (s\lambda(t, \varepsilon) y, y) p_1 (dy)\, ds,\end{aligned}$$

where

$$\lambda(t, \varepsilon) = \sqrt{(1-\varepsilon^2)(t+\varepsilon)} - \sqrt{(1-\varepsilon^2)t}.$$

Let $\{P_n\}$ and B_0 be as in Lemma 2. Then

(13) $\qquad \text{trace}_H[(I - P_n) D^2(p_t f)(x)] \longrightarrow 0$ uniformly in U,

and

(14) $$\int_{B_0} \|(I - \tilde{P}_n) z\|_0^2\, \tilde{p}_1(dz) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Then

$$\left|\frac{1}{\varepsilon} \gamma(\varepsilon, x) - \frac{1}{2} \text{trace}_H D^2(p_t f)(x)\right| \leq \text{(I)} + \text{(II)} + \text{(III)},$$

where

$$\begin{aligned}\text{(I)} \quad = \Big| \int_0^1 \int_{B_0} \frac{\lambda(t, \varepsilon)}{\varepsilon} (D(p_{\varepsilon^2 t} f) \\ \cdot (x + \sqrt{(1-\varepsilon^2)t}\, y + s\lambda(t, \varepsilon) y, \tilde{P}_n y)\, \tilde{p}_1(dy)\, ds \\ - \frac{1}{2} \text{trace}_H P_n D^2(p_t f)(x) \Big|;\end{aligned}$$

$$\text{(II)} \quad = \left| \int_0^1 \int_{B_0} \frac{\lambda(t, \varepsilon)}{\varepsilon} (D(p_{\varepsilon^2 t} f) \right.$$

$$\cdot (x + \sqrt{(1-\varepsilon^2)t}\, y + s\lambda(t, \varepsilon)y,$$

$$\left. \cdot (I - \tilde{P}_n)y)\, \tilde{p}_1(dy)\, ds) \right| ;$$

$$\text{(III)} \quad = \left| \frac{1}{2} \operatorname{trace}_H (I - P_n)\, D^2(p_t f)(x) \right| .$$

Observe that

$$\text{(II)} \quad \leq \frac{1}{2\sqrt{t}} c'' \cdot w(x)^{c'} \cdot \left(\int_{B_0} \| (I - \tilde{P}_n) \|_0^2\, \tilde{p}_1(dy)^{1/2} \right).$$

Since $w(x)$ is bounded on U, given any $\delta > 0$, we may choose n so large that

$$\text{(II)} \quad < \frac{1}{2} \delta \tag{15}$$

and

$$\text{(III)} \quad < \frac{1}{2} \delta \tag{16}$$

for all x in U (by (13) and (14)).

Now, let n be such an integer so that (15) and (16) hold and write

$$\frac{1}{2} \operatorname{trace}_H P_n D^2(p_t f) \cdot (x) = \frac{1}{2\sqrt{(1-\varepsilon^2)\, t}} \int_{B_0} (D(p_{\varepsilon^2 t} f)$$

$$\cdot (x + \sqrt{(1-\varepsilon^2)t}\, y,\ \tilde{P}_n y)\, \tilde{p}_1(dy).$$

We have

$$\text{(I)} \quad \leq \left(\frac{|\lambda(t, \varepsilon)|^2}{\varepsilon} + \left| \frac{\lambda(t, \varepsilon)}{\varepsilon} - \frac{1}{2\sqrt{(1-\varepsilon^2)\, t}} \right| \right)$$

$$\cdot \frac{c''}{\sqrt{t}} \cdot w(x)^{c'} \cdot \|P_n\|_{H-S}$$

$$\longrightarrow 0 \quad \text{uniformly in } x \text{ on } U \text{ as } \varepsilon \to 0 .$$

Thus we have that, for arbitrary small positive δ,

$$\varlimsup_{\varepsilon \to 0} | \varepsilon^{-1} r(\varepsilon, x) - \frac{1}{2} \operatorname{trace}_H D^2(p_t f)(x) | < \delta;$$

in other words,

$$\lim_{\varepsilon\to 0}\left|\varepsilon^{-1}\gamma(\varepsilon, x) - \frac{1}{2}\operatorname{trace}_H D^2(p_t f)(x)\right| = 0 .$$

It follows that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}((p_{t+\varepsilon} f)(x) - (p_t f)(x)) = \frac{1}{2}\operatorname{trace}_H D^2(p_t f)(x)$$

uniformly for x in U.

Finally,

$$|(p_t f)(x) - f(x)| \leq c\cdot w(x)^{2c'}\left(\int_B w(y)^{c'\sqrt{\alpha}}\|y\|\, p_1(dy)\right)\cdot\sqrt{t}$$
$$\longrightarrow 0 \quad \text{uniformly in } x \text{ on } U.$$

This concludes the proof of Theorem 1. ///

REMARK 3. The results of Theorem also shows that, in [2, Theorem. 3], the assumption that f is bounded is superfluous. ///

3. **Solution of $u_t = \frac{1}{2}\operatorname{trace}[AD^2 u]$.** To conclude this paper, we would like to consider the heat equation with constant coefficients.

Let A be a fixed member of $L(H, H)$ (the space of bounded linear operators on H) such that (1) A is symmetric; (2) $A \geq \varepsilon I$ for some $\varepsilon > 0$; (3) $A = I + C$, where C is of Hilbert-Schmidt class. Let $q_t(x, dy)$ be a family of measures defined as follows

$$q_t(x, dy) = [\det(A)]^{-1/2}$$
$$\cdot \exp[-\langle[A^{-1} - I](x-y),\, x-y\rangle/2t]\, p_t(x, dy) .$$

By the same arguments as in [8], it is not hard to verify the following.

THEOREM 2. *If A is a bounded linear operator on H satisfying* (1), (2) *and* (3) *above, and $f \in L_w$, then*

$$V(t, x) = (q_t f)(x) = \int_B f(y)\, q_t(x, dy)$$

solves the equation

$$(\partial/\partial t)V(t, x) = \frac{1}{2}\operatorname{trace}_H[AD^2V(t, x)],$$

and

$$\lim_{t \to 0} V(t, x) = f(x) \quad \textit{uniformly on every w-bounded set.}$$

(The proof is almost identical to that of [8].) ///

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