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ON DISCRETE GENERALIZATIONS OF GRONWALL'S INEQUALITY

BY

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Abstract. This paper deals with several new discrete generalizations of Gronwall's type inequality. To justify the results obtained here some applications are also given.

1. **Introduction.** The theory of difference equations is in a process of continuous development and it has become significant for its various applications in numerical analysis, control systems, engineering and so on. The role played by the discrete inequalities in the theory of difference equations is well known. One of the most used discrete inequality is the analogue of the celebrated Gronwall-Bellman-Reid inequality established by Jones [6] and Sugiyama [18]. On the basis of various motivations this inequality has been extended and used in various contexts. The discrete analogue of Bihari's inequality is due to Hull and Luxemburg [5].

For the finite system of discrete inequalities

$$\phi(t) \le a(t) + \sum_{s=0}^{t-1} A(s)\,\phi(s)$$

where the components of $\phi(t)$, $a(t)$ and the elements of the matrix $A(t)$ are nonnegative, then the upper estimate can be obtained on using the known results in literature provided $A(t)$ is upper or lower triangular matrix.

For the general $A(t)$ two different approaches have been used: first deals with the solution of the related difference system (component-wise inequalities as in [1]), and another is by using any convenient norm as in [18].

For the second order system

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$$\phi_1(t) \le a_1(t) + \sum_{s=0}^{t-1} f_{11}(s)\,\phi_1(s) + \sum_{s=0}^{t-1} f_{12}(s)\,\phi_2(s)$$

$$\phi_2(t) \le a_2(t) + \sum_{s=0}^{t-1} f_{21}(s)\,\phi_1(s)$$

one can find the explicit genuine upper estimate for $\phi_1(t)$ and $\phi_2(t)$ provided an upper estimate for the inequality

$$\phi_1(t) \le a_1(t) + \sum_{s=0}^{t-1} f_{11}(s)\,\phi_1(s) + \sum_{s=0}^{t-1} f_{12}(s)\Big(a_2(s) + \sum_{\tau=0}^{s-1} f_{21}(\tau)\,\phi_1(\tau)\Big)$$

is known. Thus if the matrix $A(t)$ is of some particular nature, the study of the discrete inequalities of the above type involving n summations is necessary. In this paper we shall study mainly this type of inequalities in section 2 and in section 3 we shall present some nonlinear generalizations. Finally in section 4 several applications of these results are given.

The following notations we shall use throughout the paper. N denotes the set $\{0, 1, \cdots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents a solution of the linear difference equation $\Delta x(t) = b(t)$ for all $t \in N$ under the initial condition $x(0) = 0$, where Δ is the operator defined by $\Delta x(t) = x(t+1) - x(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} c(s)$ represents the solution of the linear difference equation $x(t+1) = c(t)x(t)$ for all $t \in N$ under the initial condition $x(0) = 1$. It is supposed that $\prod_{s=0}^{-1} c(s) = 1$.

In what follows we shall assume that the functions appear in the inequalities are real—valued, non-negative and defined on N.

2. **Linear generalizations.**

THEOREM 1. *Let the following inequality be satisfied*

(1) $$u(t) \le p(t) + q(t) \sum_{\Omega=1}^{n} E_\Omega(t, u)$$

where

(2) $$E_\Omega(t, u) = \sum_{t_1=0}^{t-1} f_{\Omega_1}(t_1) \sum_{t_2=0}^{t_1-1} f_{\Omega_2}(t_2) \cdots \sum_{t=0}^{t_{\Omega-1}-1} f_{\Omega\Omega}(t_\Omega)\, u(t_\Omega)$$

for all $t \in N$. Then for all $t \in N$

$$u(t) \leq p(t) + q(t) \sum_{s=0}^{t-1} \left(\sum_{\Omega=1}^{n} \Delta E_{\Omega}(s, p) \right) \times \prod_{\tau=s+1}^{t-1} \left[1 + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(\tau, q) \right]. \tag{3}$$

Proof. Define $m(t)$ as follows

$$m(t) = \sum_{\Omega=1}^{n} E_{\Omega}(t, u), \qquad m(0) = 0$$

and hence

$$\Delta m(t) = \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, u)$$

where

$$\Delta E_{\Omega}(t, u) = f_{\Omega_1}(t) \sum_{t_2=0}^{t-1} f_{\Omega_2}(t_2) \cdots \sum_{t_{\Omega}=0}^{t_{\Omega-1}-1} f_{\Omega\Omega}(t_{\Omega}) u(t_{\Omega}).$$

From the assumptions on the functions $\Delta m(t) \geq 0$, hence $m(t)$ is nondecreasing on N. Hence we find

$$\begin{aligned} \Delta m(t) &\leq \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, p + qm) \\ &\leq \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, p) + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, qm) \\ &\leq \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, p) + m(t) \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, q) \end{aligned}$$

and thus

$$m(t+1) - \left[1 + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, q) \right] m(t) \leq \sum_{\Omega=1}^{n} \Delta E_{\Omega}(t, p).$$

Multiplying the above inequality by $\prod_{s=0}^{t} [1 + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(s, q)]^{-1}$ and summing over from 0 to $t-1$, we get

$$m(t) \prod_{s=0}^{t-1} \left[1 + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(s, q) \right]^{-1} \leq \sum_{s=0}^{t-1} \left(\sum_{\Omega=1}^{n} \Delta E_{\Omega}(s, p) \right) \times \prod_{\tau=0}^{s} \left[1 + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(\tau, q) \right]^{-1}$$

and hence finally

$$m(t) \leq \sum_{s=0}^{t-1} \left(\sum_{\Omega=1}^{n} \Delta E_{\Omega}(s, p) \right) \times \prod_{\tau=s+1}^{t-1} \left[1 + \sum_{\Omega=1}^{n} \Delta E_{\Omega}(\tau, q) \right].$$

Substituting this estimate in (1), we obtain the desired inequality (3).

REMARK 1. Let in inequality (1), $n=1$, $q(t)=1$; then the estimate (3) is same as obtained earlier by Sugiyama [18].

REMARK 2. Let in inequality (1), $n=1$; then the estimate (3) is same as obtained by Pachpatte [9].

REMARK 3. Several particular cases of theorem 1 have been considered by Pachpatte [13]—[16] but the results are not comparable.

THEOREM 2. *Let the following inequality be satisfied*

$$u(t) \le u_0 + \sum_{s=0}^{t-1} f(s)\left[u(s) + \sum_{\tau=0}^{s-1} g(\tau)\, u(\tau)\right] \tag{4}$$

for all $t \in N$. *Then for all* $t \in N$

$$u(t) \le u_0\left[1 + \sum_{s=0}^{t-1} f(s)(1-\phi(s)) \prod_{\tau=0}^{s-1} [1 + f(\tau) + g(\tau)]\right] \tag{5}$$

where

$$\phi(t) = \sum_{s=0}^{t-1} g(s) \left(\prod_{\tau=0}^{s} [1 + f(\tau) + g(\tau)]\right)^{-1} \sum_{\tau=0}^{s-1} g(\tau).$$

Proof. As in the proof of theorem 1, we define $m(t)$ as the right member of (4). Then

$$\begin{aligned} \Delta m(t) &= f(t)\left[u(t) + \sum_{\tau=0}^{t-1} g(\tau) u(\tau)\right] \\ &\le f(t)\left[m(t) + \sum_{\tau=0}^{t-1} g(\tau)\, m(\tau)\right]. \end{aligned} \tag{6}$$

Define $n(t)$ as follows

$$n(t) = m(t) + \sum_{\tau=0}^{t-1} g(\tau)\, m(\tau)$$

then, we have

$$\Delta n(t) = \Delta m(t) + g(t) m(t) \tag{7}$$

and

$$m(t) \leq n(t) - u_0 \sum_{\tau=0}^{t-1} g(\tau)\,. \tag{8}$$

Using (6) and (8) in (7), we get

$$\Delta n(t) \leq f(t)\, n(t) + g(t) n(t) - u_0\, g(t) \sum_{\tau=0}^{t-1} g(\tau)$$

or

$$n(t+1) - [1 + f(t) + g(t)]\, n(t) \leq -u_0\, g(t) \sum_{\tau=0}^{t-1} g(\tau)\,.$$

Multiplying the above inequality by $\Pi_{s=0}^{t}(1 + f(s) + g(s))^{-1}$ and summing over from 0 to $t-1$, we get

$$n(t) \leq u_0\, [1 - \phi(t)] \prod_{s=0}^{t-1} (1 + f(s) + g(s))\,.$$

On substituting the above estimate in (6) and summing over from 0 to $t-1$, we obtain the desired result.

REMARK 4. For $\phi(t) = 0$ in (5), the estimate is same as obtained in [10]. Infact almost all the results obtained in [13] – [16] can be improved uniformly using the same arguments as in the proof of theorem 2.

Our next result is the discrete analogue of Willett's inequality [22].

THEOREM 3. *Let the following inequality be satisfied*

$$u(t) \leq p_0(t) + \sum_{i=1}^{n} p_i(t) \left(\sum_{s=0}^{t-1} v_i(s)\, u(s) \right) \tag{9}$$

for all $t \in N$. *Then for all* $t \in N$

$$u(t) \leq E_n\, p_0(t) \tag{10}$$

where

$$E_i = D_i\, D_{i-1} \cdots D_0$$

$$D_0\, \omega = \omega$$

$$D_j\, \omega = \omega + (E_{j-1}\, p_j) \left(\sum_{s=0}^{t-1} v_j\, \omega \prod_{\tau=s+1}^{t-1} (1 + v_j\, E_{j-1}\, p_j) \right)$$

$$j = 1, 2, \cdots, n\,.$$

Proof. For $n=1$, we obtain from theorem 1

$$u(t) \le p_0(t) + p_1(t) \sum_{s=0}^{t-1} v_1(s)\, p_0(s) \prod_{\tau=s+1}^{t-1} (1 + v_1(\tau)\, p_1(\tau))$$
$$= E_1\, p_0(t)\,.$$

Now, assume that the result is true for some k such that $1 \le k \le n-1$, then for $k+1$, we are given

$$u(t) \le p_0(t) + \sum_{i=1}^{k} p_i(t) \sum_{s=0}^{t-1} v_i(s)\, u(s) + p_{k+1}(t) \sum_{s=0}^{t-1} v_{k+1}(s)\, u(s)$$

and, we find

$$u(t) \le E_k\, p^*(t)$$

where

$$p^*(t) = p_0(t) + p_{k+1}(t) \sum_{s=0}^{t-1} v_{k+1}(s)\, u(s)\,.$$

Form the definition of E_k, we have

$$u(t) \le E_k\, p_0(t) + E_k\, p_{k+1}(t) \left(\sum_{s=0}^{t-1} v_{k+1}(s)\, u(s)\right)$$

which is obtained on using the fact that $\sum_{s=0}^{t-1} v_{k+1}(s)\, u(s)$ is nondecreasing for all $t \in N$.

Once again on using theorem 1, we obtain

$$.u(t) \le E_k\, p_0(t) + E_k\, p_{k+1}(t) \sum_{s=0}^{t-1} v_{k+1}(s)\, E_k\, p_0(s)$$
$$\times \prod_{\tau=s+1}^{t-1} [1 + v_{k+1}(\tau) E_k\, p_{k+1}(\tau)]$$
$$= D_{k+1}(E_k\, p_0(t))$$
$$= E_{k+1}\, p_0(t)\,.$$

Hence the result follows from finite iduction.

Corollary 4. *Let in inequality* (9), $p_i(t) \ge 1$ *for all* $i=1,2,\cdots,n$ *and* $t \in N$. *Then*

$$u(t) \le \prod_{i=1}^{n} p_i(t) \Big[p_0(t) + \sum_{s=0}^{t-1} p_0(s) \Big(\sum_{i=1}^{n} h_i(s) \prod_{j=1}^{n} p_j(s) \Big)$$
$$\times \prod_{\tau=s+1}^{t-1} [1 + \sum_{\Omega=1}^{n} h_\Omega(\tau) \prod_{i=1}^{n} p_i(\tau)] \Big].$$

COROLLARY 5. *Let the inequality* (9) *be satisfied for all* $t \in N$, *where* (i) $p_0(t)$ *is positive and nondecreasing* (ii) $p_i(t) \geq 1$ *for all* $i = 1, \cdots, n$ *and nondecreasing for all* $n \geq i \geq 2$. *Then*

$$u(t) \leq E_n\, p_0(t)$$

where

$$E_0\, \omega = \omega$$

$$E_k\, \omega = \omega(E_{k-1}\, p_k) \prod_{s=0}^{t-1} [1 + h_k(s)\, E_{k-1}(p_k(s))]$$

$$k = 1, 2, \cdots, n.$$

The proof of corollaries 4 and 5 are similar to theorem 1 and theorem 3 respectively.

3. Nonlinear Generalizations.

THEOREM 6. *Let the following inequality be satisfied*

$$u(t) \leq p(t)\left[u_0 + \sum_{\Omega=1}^{n-1} E_\Omega(t, u) + E_n(t, u^\alpha)\right] \tag{11}$$

for all $t \in N$, *where* $u_0 \geq 0$ *and* $0 \leq \alpha < 1$. *Then for all* $t \in N$

$$u(t) \leq p(t)\, e^{-1}(t) \left\{u_0^{1-\alpha} + (1-\alpha) \sum_{s=0}^{t-1} \Delta E_n(s, p^\alpha) \times [e(s+1)]^{1-\alpha}\right\}^{1/1-\alpha} \tag{12}$$

where

$$e(t) = \prod_{s=0}^{t-1} \left[1 + \sum_{\Omega=1}^{n-1} \Delta E_\Omega(s, p)\right]^{-1}.$$

Proof. Let $R(t)$ be the term inside the bracket of right side of (11). Then

$$u(t) \leq p(t)\, R(t)$$

$$\begin{aligned} \Delta R(t) &= \sum_{\Omega=1}^{n-1} \Delta E_\Omega(t,u) + \Delta E_n(t, u^\alpha) \\ &\leq \sum_{\Omega=1}^{n-1} \Delta E_\Omega(t, pR) + \Delta E_n(t, p^\alpha R^\alpha) \\ &\leq \sum_{\Omega=1}^{n-1} \Delta E_\Omega(t, p) R(t) + \Delta E_n(t, p^\alpha)\, R^\alpha(t) \end{aligned}$$

or

$$R(t+1)-\left[1+\sum_{\Omega=1}^{n-1}\Delta E_\Omega(t,\ p)\right]R(t)\le \Delta E_n(t,\ p^\alpha)\,R^\alpha(t)\,.$$

On multiplying the above inequality by $e(t+1)$, we obtain

$$\begin{aligned}\Delta[R(t)\,e(t)] &= R(t+1)\,e(t+1)-R(t)\,e(t)\\ &\le \Delta E_n(t,\ p^\alpha)\times e^{1-\alpha}(t+1)[R(t)\,e(t+1)]^\alpha\,.\end{aligned}\tag{13}$$

For all $t\in N$ when $\Delta[R(t)\,e(t)]\ge 0$, we have

$$\begin{aligned}\frac{\Delta[R(t)\ e(t)]^{1-\alpha}}{1-\alpha} &= \int_t^{t+1}\frac{d[R(s)\ e(s)]}{[R(s)\ e(s)]^\alpha}\\ &\le \frac{\Delta[R(t)\ e(t)]}{[R(t)\ e(t)]^\alpha}\end{aligned}$$

and from (13), we obtain

$$\frac{\Delta[R(t)\ e(t)]^{1-\alpha}}{1-\alpha}\le \Delta E_n(t,\ p^\alpha)\ e^{1-\alpha}(t+1)\,.\tag{14}$$

Similarly for all $t\in N$ when $\Delta[R(t)\ e(t)]\le 0$, we have $\Delta[R(t)\ e(t)]^{1-\alpha}/1-\alpha\le 0$. Hence obviously (14) follows. Summing up both the sides of (14) from 0 to $t-1$, we get the result.

COROLLARY 7. *Let the following inequality be satisfied*

$$u(t)\le u_0+\sum_{\Omega=1}^{n}\overset{*}{E}_\Omega(t,\ u)$$

where

$$\overset{*}{E}_\Omega(t,\ u)=\sum_{t_1=0}^{t-1}f_{\Omega 1}(t_1)\,u^{k_{\Omega 1}}(t_1)\sum_{t_2=0}^{t_1-1}f_{\Omega 2}(t_2)\,u^{k_{\Omega 2}}(t_2)\cdots\sum_{t_\Omega=0}^{t_{\Omega-1}-1}f_{\Omega\Omega}(t_\Omega)\,u^{k_{\Omega\Omega}}(t_\Omega)$$

for all $t\in N$*, where* $k_{\Omega p}$ *are nonnegative numbers and* $u_0>0$. *Then for all* $t\in N$

$$u(t)\le u_0\left[1+(1-\alpha)\sum_{s=0}^{t-1}\sum_{\Omega=1}^{n}\Delta\overset{*}{E}_\Omega(t,\ 1)\ u_0^{\alpha_\Omega-1}\right]^{1/1-\alpha}$$

where $\alpha_\Omega=\sum_{i=1}^{\Omega}k_{\Omega i}$ *and* $\alpha=\max\{\alpha_\Omega:\Omega=1,2,\cdots,n\}$ $(\ne 1)$, *and if* $\alpha=1$, *then*

$$u(t)\le u_0\left[1+\sum_{s=0}^{t-1}\sum_{\Omega=1}^{n}\Delta\overset{*}{E}_\Omega(t,\ 1)u_0^{\alpha_\Omega-1}\right].$$

REMARK 5. Several particular cases of theorem 6 and corollary 7 have been discussed in [10] – [21], however the results are not comparable, but following the same lines as in theorem 2 most of his results can be improved uniformly.

In our next result the following class of functions will be needed.

DEFINITION. A function W: $[0, \infty) \to [0, \infty)$ is said to belong to the class S if

(i) $W(u)$ is positive, nondecreasing and continuous for $u \geq 0$

(ii) $(1/v)\, W(u) \leq W(u/v)$ for all $u \geq 0$ and $v > 0$

THEOREM 8. *Let the following inequality be satisfied*

$$u(t) \leq p(t) + \sum_{\Omega=1}^{n} g_\Omega(t) \sum_{s=0}^{t-1} h_\Omega(s)\, W(u(s)) \tag{15}$$

for all $t \in N$. *In* (15) $p(t)$ *is positive, nondecreasing and* $W_\Omega \in S$, $g_\Omega(t) \geq 1$ *for all* $\Omega = 1, 2, \cdots, n$. *Then for all* $t \in N$

$$u(t) \leq p(t) \prod_{\Omega=1}^{n} g_\Omega(t) \prod_{k=1}^{n} E_k(t) \tag{16}$$

where

$$E_1(t) = G_1^{-1}\Big[G_1(1) + \sum_{s=0}^{t-1} h_1(s) \prod_{\Omega=1}^{n} g_\Omega(s)\Big]$$

$$E_k(t) = G_k^{-1}\Big[G_k(1) + \sum_{s=0}^{t-1} h_k(s) \prod_{\Omega=1}^{n} g_\Omega(s) \prod_{i=1}^{k-1} E_i(s)\Big]$$

$$k = 2, 3, \cdots, n$$

$$G_k(u) = \int_{u_0}^{u} \frac{ds}{W_k(s)} \qquad 0 < u_0 \leq u$$

as long as

$$\Big[G_k(1) + \sum_{s=0}^{t-1} h_k(s) \prod_{\Omega=1}^{n} g_\Omega(s) \sum_{i=1}^{k-1} E_i(s)\Big] \in \operatorname{dom} G_k^{-1}.$$

Proof. From inequality (15) it follows that

$$v(t) \leq 1 + \sum_{\Omega=1}^{n} \sum_{s=0}^{t-1} h_\Omega(s) \prod_{i=1}^{n} g_i(s)\, W_\Omega(v(s))$$

where

$$v(t) = \frac{u(t)}{p(t) \prod_{i=1}^{n} g_i(t)} \,.$$

Thus it is sufficient to show that $v(t) \le \prod_{k=1}^{n} E_k(t)$, and this we shall prove by induction. For $\Omega = 1$, we have

$$v(t) \le 1 + \sum_{s=0}^{t-1} h_1(s) \prod_{i=1}^{n} g_i(s)\, W_1(v(s))\,. \tag{17}$$

Let $m(t)$ be the right member of (17), then we obtain

$$\Delta\, m(t) \le h_1(t) \prod_{i=1}^{n} g_i(t)\, W_1(m(t))\,.$$

Now from the definition of G_1 and the above inequality, we have

$$\begin{aligned} G_1(m(t+1)) - G_1(m(t)) &= \int_{m(t)}^{m(t+1)} \frac{ds}{W_1(s)} \\ &\le \frac{1}{W_1(m(t))}\, \Delta\, m(t) \\ &\le h_1(t) \prod_{i=1}^{n} g_i(t)\,. \end{aligned}$$

Summing up the above inequality from 0 to $t-1$, we find

$$m(t) \le E_1(t),$$

and hence

$$v(t) \le E_1(t)\,.$$

Now, assume that the result is true for some k such that $1 \le k \le n-1$, then for $k+1$, we are given

$$\begin{aligned} v(t) \le & \left[1 + \sum_{s=0}^{t-1} h_{k+1}(s) \prod_{i=1}^{n} g_i(s)\, W_{k+1}(v(s))\right] \\ & + \sum_{\Omega=1}^{k} \sum_{s=0}^{t-1} h_\Omega(s) \prod_{i=1}^{n} g_i(s)\, W_\Omega(v(s)) \end{aligned}$$

since the part in the bracket in the right side is non-decreasing, we find

$$v(t) \le \left[1 + \sum_{s=0}^{t-1} h_{k+1}(s) \prod_{i=1}^{n} g_i(s)\, W_{k+1}(v(s))\right] \prod_{\Omega=1}^{k} E_\Omega(t)$$

or

$$\frac{v(t)}{\prod_{\Omega=1}^{k} E_\Omega(t)} \le 1 + \sum_{s=0}^{t-1} h_{k+1}(s) \prod_{i=1}^{n} g_i(s) \prod_{\Omega=1}^{k} E_\Omega(s) \times W_{k+1}\left(\frac{v(s)}{\prod_{\Omega=1}^{k} E_\Omega(s)}\right)$$

now $v(t) \le \prod_{\Omega=1}^{k+1} E_\Omega(t)$ follows on using the same arguments as in the case $\Omega = 1$.

COROLLARY 9. *Let the hypothesis of theorem* 8 *holds and in addition* $g_i(t)$ *for all* $i = 2, \cdots, n$ *are nondecreasing. Then for all* $t \in N$

$$u(t) \le p(t) \prod_{i=1}^{n} F_i(t)$$

where

$$F_1(t) = g_1(t)\, G_1^{-1}\left[G_1(1) + \sum_{s=0}^{t-1} h_1(s)\, g_1(s)\right]$$

$$F_k(t) = g_k(t)\, G_k^{-1}\left[G_k(1) + \sum_{s=0}^{t-1} h_k(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s)\right]$$

as long as

$$G_k(1) + \sum_{s=0}^{t-1} h_k(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s) \in \operatorname{dom} G_k^{-1}.$$

4. **Some applications.** As we have mentioned in section 1, for several particular systems of discrete inequalities, explicit upper estimates can be obtained on using the results of sections 2 and 3. For example, for two dimensional discrete inequalities

$$|x_i(t)| \le |k_i| + \sum_{s=0}^{t-1} |f_i(s, x_1(s), x_2(s))| \qquad (i = 1, 2)$$

(which appear in the study of two dimensional differential systems using Euler's method), if

$$|f_i(t, x_1(t), x_2(t))| \le b_i(t) + a_{i1}(t)|x_1(t)| + a_{i2}(t)|x_2(t)|$$

then it follows from corollary 3.2 [1] that $|x_i(t)| \le u_i(t)$ where $u_1(t)$ and $u_2(t)$ is the solution of the following discrete system

$$\begin{aligned} \Delta u_i(t) &= b_i(t) + a_{i1}(t)\, u_1(t) + a_{i2}(t)\, u_2(t) \\ u_i(0) &= |k_i| \end{aligned} \tag{18}$$

now, theorem 1 can be used to obtain upper estimates atleast. Infact from (18) we find

$$u_2(t) = \prod_{s=0}^{t-1}(1+a_{22}(s))\Big[|k_2| + \sum_{s=0}^{t-1}(b_2(s)+a_{11}(s)\,u_1(s)) \cdot \prod_{\tau=0}^{s}[1+a_{22}(\tau)]^{-1}\Big]$$

now substituting this in the first equation of (18), we find for $u_1(t)$ the exact form as in theorem 1.

Next we shall make a comparative study of some known results, Following the same notations as in [11], we consider the linear stochastic discrete system

$$(19)\qquad y_{n+1}(\omega) = A(\omega)y_n(\omega), \qquad y_0(\omega) = x_0$$

and the perturbed system including an operator T as

$$(20)\qquad \begin{aligned} x_{n+1}(\omega) &= A(\omega)\,x_n(\omega) + f_n(\omega,\, x_n(\omega),\, (Tx_n)(\omega)) \\ x_0(\omega) &= x_0\,. \end{aligned}$$

Let $Y_n(\omega)$ denotes the stochastic fundamental matrix solution of the homogeneous system (19) such that $Y_0(\omega)$ is the unit matrix.

The following modified versions of his [11] theorems 2–4 which require weaker conditions can be proved using the results obtained in sections 2 and 3 here and the parallel arguments he has used.

THEOREM 2′. *Suppose that*

$$|Y_n(\omega)\,Y_{s+1}^{-1}(\omega)\,f_s(\omega,\, x_s(\omega),\, (Tx_s)\,(\omega))| \le a_s(\omega)\,|x_s(\omega)| + b_s(\omega)\,|(Tx_s)(\omega)|$$

where $a_s(\omega)$, $b_s(\omega)$ *are non-negative random functions defined for* $s \in N$, $\omega \in \Omega$. *Further, suppose that the operator* T *satisfies the inequality*

$$|(Tx_n)(\omega)| \le \sum_{s=0}^{n-1} c_s(\omega)\,|\,x_s(\omega)|$$

where $c_n(\omega)$ *is a non-negative random function defined for* $n \in N$, $\omega \in \Omega$. *Then to every bounded random solution* $x_n(\omega)$ *of* (19) *on* N, *the corresponding random solution* $x_n(\omega)$ *of* (20) *is bounded on* N *provided*

$$\prod_{s=0}^{\infty}\left[1+a_s(\omega)+b_s(\omega)\sum_{\tau=0}^{s-1}c_\tau(\omega)\right]<\infty .$$

THEOREM 3′. *Let us assume*

$$|Y_n(\omega)Y_{s-1}^{-1}(\omega)|\le Me^{-\alpha(n-s)}, \qquad |Y_n(\omega)|\le Me^{-\alpha n}$$

$$|f_n(\omega, x_n(\omega), (Tx_n)(\omega))|\le a_n(\omega)|x_n(\omega)|+b_n(\omega)|(Tx_n)(\omega)|$$

$$|(Tx_n)(\omega)|\le e^{-\alpha n}\sum_{s=0}^{n-1}c_s(\omega)|x_s(\omega)|$$

where $M>0$, $\alpha>0$ *are constants and* $a_n(\omega)$, $b_n(\omega)$, $c_n(\omega)$ *are defined in theorem* 2′. *Then all random solutions of* (20) *approach zero as* $n\to\infty$, *provided*

$$K=\prod_{s=0}^{\infty}\left[1+a_s(\omega)+b_s(\omega)\sum_{\tau=0}^{s-1}c_\tau(\omega)\,e^{-\alpha\tau}\right]<\infty .$$

THEOREM 4′. *In theorem* 3′ *let* $-\alpha=\varepsilon$ *and* $K\le c$ *where* $c>0$ *is a constant, then the conclusion of his theorem* 4 *follows.*

Several other applications of these reults obtained here will be published seperately.

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