BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 9, Number 2, June 1981

SOME EXPONENTIAL FORMULAS FOR m-PARAMETER SEMIGROUPS

BY

SEN-YEN SHAW(*)

Abstract. Several new exponential formulas for m-parameter semigroups of operators are obtained through a unified approach using a method of approximation. When $m=1$, these reduce to the well-known formulas of Kendal and Chung for one-parameter semi-groups.

1. **Introduction.** Among existing proofs of the exponential formulas in semigroup theory, a remarkable one is the efficient probability-theoretic proof given by K. L. Chung [1]. His method provides a unified approach to such well-known formulas for one-parameter semigroups as those of Hille, Phillips, Kendal, Widder (cf.[3]) and Chung himself. The purpose of this paper is to report another unified approach to exponential formulas for not only one-parameter but also m-parameter semigroups.

Our method is based on a generalized Korovkin-type approximation theorem established by the author in [4]; a particular version of it which is suitable for use in this paper will be quoted in Theorem 1. By using this theorem, together with some suitably chosen approximation operators, we can reduce the proofs of formulas to just routine verifications. In fact, we have practiced in that paper to revise all formulas mentioned above. Besides, we also obtained therein the two formulas (for m-parameter semigroups) to be quoted in Theorem 2.

In the present paper, we shall introduce two new approximation operators which will deliver two new formulas (Theorems 3 and 4). Also, a new formula for one-parameter contraction semigroups is

Received by the editors July 10, 1980.

(*) Communicated by Charles S. C. Lin.

AMS (MOS) Subject classifications (1980): Primary 47D05, 41A35.

Key words and phrases: exponential formulas, m-parameter (C_0)-semigroups, infinitesimal generator, approximation operator, Chernoff's product formula.

obtained by using Chernoff's product theorem. The results will be mentioned in section 2 and the proofs given in section 3.

2. **Approximation theorem and exponential formulas.** In the following, X is a Banach space, $B(X)$ is the set of all bounded linear operators on X, R^m_{0+} denotes the set $\{t \in R^m;\ t_i \geq 0, i = 1, 2, \cdots, m\}$, $BC(R^m_{0+})$ denotes the set of all bounded, strongly continuous $B(X)$-valued functions on R^m_{0+}, and $C(S)$ denotes the set of all strongly continuous $B(X)$-valued functions on a set $S \subset R^m_{0+}$. We first mention two theorems established in [4].

THEOREM 1. *Let* L_n: $BC(R^m_{0+}) \to C(S)$, $n = 1, 2, \cdots$, *be linear operators defined by* $(L_n f)(t)\, x \equiv \int_{R^m_{0+}} g_n(u, t) f(u)\, x du (x \in X,\ t \in S)$ *with* $g_n(u, t)$ *assumed to be nonnegative functions on* R^m_{0+}. *If* $(L_n 1)(t) \equiv \int g_n(u, t)\, du \to 1$, $(L_n u_i)(t) \equiv \int g_n(u, t)\, u_i\, du \to t_i$ *and* $(L_n u_i^2)(t) \equiv \int g_n(u, t)\, u_i^2\, du \to t_i^2$, $i = 1, 2, \cdots, m$, *uniformly for* t *in a compact subset* K *of* S *while* $n \to \infty$, *then for any* f *in* $BC(R^m_{0+})$ *and any* $x \in X$,

$$\lim_{n\to\infty} (L_n f)(t)\, x = f(t) x \tag{1}$$

uniformly for t *in* K.

THEOREM 2. *Let* $\{T(t);\ t \in R^m_{0+}\}$ *be a uniformly bounded m-parameter* (C_0)-*semigroup and* $T_i(t_i)$ *be the restriction of* $T(t)$ *to the* t_i-*axis. If* $\{\alpha_n\}$ *is a sequence of positive numbers such that* $\lim_{n\to\infty} \alpha_n / n = 0$ *and* $\liminf \alpha_n > 0$, *then*

$$T(t)\, x = \lim_{n\to\infty} \left\{ I + \sum_{i=1}^{m} (t_i/\alpha_n)\, [T_i(\alpha_n/n) - I] \right\}^n x\ (x \in X), \tag{2}$$

$$T(t)\, x = \lim_{n\to\infty} \left\{ I - \sum_{i=1}^{m} (t_i/\alpha_n)\, [T_i(\alpha_n/n) - I] \right\}^{-n} x\ (x \in X); \tag{3}$$

(2) holds for those t which are contained in the set $K_n = \{t \in R^m_{0+};\ \bar{t} = \sum_{i=1}^m t_i \leq \alpha_n\}$ for all large n, and (3) holds for all t in R^m_{0+}. Furthermore, the convergence of each limit is uniform on any compact set of t for which the limit exists.

The proofs of the above two theorems were given in [4]. The following theorems are our main results in this paper. In the formulation, A_i denotes the infinitesimal generator of the one-parmeter semigroup $T_i(t_i)$.

THEOREM 3. *Let* $\{T(t);\ t\in R^m_{0+}\}$ *be a uniformly bounded m-parameter* (C_0)*-semigroup of operators on* X, $\{\alpha_n\}$ *be a sequence of positive numbers such that* $\lim_{n\to\infty}\alpha_n/n=0$ *and* $\liminf \alpha_n>0$. *Then for all* $x\in X$ *and for all* t *which are contained in* K_n *for almost all* n,

$$(4)\qquad T(t)x=\lim_{n\to\infty}\left\{I+\sum_{i=1}^{m}(t_i/\alpha_n)\left[(I-(\alpha_n/n)A_i)^{-1}-I\right]\right\}^n x,$$

and the convergence is uniform on any compact set of t *for which the limit* (4) *exists.*

THEOREM 4. *Under the assumptions of Theorem* 3, *we have*

$$(5)\qquad T(t)x=\lim_{n\to\infty}\left\{I-\sum_{i=1}^{m}(t_i/\alpha_n)\left[(I-(\alpha_n/n)A_i)^{-1}-I\right]\right\}^{-n} x$$

for all $x\in X$ *and all* $t\in R^m_{0+}$, *and the convergence is uniform on any compact subset of* R^m_{0+}.

When $m=1$, formulas (2), (3), (4) and (5) actually hold for any (unbounded) one-parameter (C_0)-semigroup; this has been shown in [4]. In this case, if we put $\alpha_n=1$, then (2), (4) and (5) reduce to Kendall's formula and Chung's formulas.

On the other hand, if we put $m=1$ and $\alpha_n=t_1=t$, then (4) and (5) yield the pointwise convergence of Widder's formula:

$$(6)\qquad T(t)x=\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}x$$

and the pointwise convergence of the formula:

$$(7)\qquad T(t)x=\lim_{n\to\infty}\left[2I-\left(I-\frac{t}{n}A\right)^{-1}\right]^{-n}x$$

In fact, the convergence of Widder's formula is uniform on compact sets of $[0,\infty)$. (cf. [1], [3] or [4].) This is also the case for (7) at least when $T(t)$ is contractive.

THEOREM 5. *If* $T(t)$ *is a one-parameter contraction* (C_0)*-semigroup, then* (7) *holds uniformly for* t *in any compact subset of* $[0,\infty)$.

The proofs of Theorems 3, 4 and 5 will be given in section 3.

3. **The proofs.** We shall need in the proofs the following two kinds of approximation operators; in the formulations of them, $\bar{t}$ denotes for $t=(t_1,\cdots,t_m)$ the number $\sum_{i=1}^m t_i$, and given an integer n and $k=(k_1,\cdots,k_m)$ with nonnegative integer components, $\binom{n}{k}$ denotes the number $n(n-1)\cdots(n-\bar{k}+1)/k_1!\,k_2!\cdots k_m!$.

$$(L_n f(\cdot))(t)\,x$$
$$=\sum_{0\le\bar{k}\le n}\binom{n}{k}\Big[\prod_{i=1}^m\Big(\frac{t_i}{\alpha_n}\Big)^{k_i}\Big]\Big(1-\frac{\bar{t}}{\alpha_n}\Big)^{n-\bar{k}}$$
$$\cdot\int_{R_{0+}^m}\Big[\prod_{i=1}^m\frac{(n/\alpha_n)^{k_i}}{(k_i-1)!}\exp\Big(-\frac{nu_i}{\alpha_n}\Big)u_i^{k_i-1}\Big]f(u)\,x\,du,$$

defined for those t in K_n.

$$(M_n f(\cdot))(t)\,x$$
$$=\sum_{0\le\bar{k}<\infty}\binom{-n}{k}\Big[\prod_{i=1}^m\Big(-\frac{t_i}{\alpha_n}\Big)^{k_i}\Big]\Big(1+\frac{\bar{t}}{\alpha_n}\Big)^{-n-\bar{k}}$$
$$\cdot\int\Big[\prod_{i=1}^m\frac{(n/\alpha_n)^{k_i}}{(k_i-1)!}\exp\Big(-\frac{nu_i}{\alpha_n}\Big)u_i^{k_i-1}\Big]f(u)\,x\,du$$

where the integral is taken over the region R_{0+}^m and t may be any element of R_{0+}^m.

First we prove the following two lemmas.

LAMMA 6. *Let L_n, M_n, $n=1,2,\cdots$, be defined as above. Then*

$$(L_n1)(t)=(M_n1)(t)=1, \tag{8}$$

$$(L_nu_i)(t)=(M_nu_i)(t)=t_i,\qquad i=1,2,\cdots,m, \tag{9}$$

$$(L_nu_i^2)(t)=\frac{n-1}{n}t_i^2+2\frac{\alpha_n}{n}t_i,\qquad i=1,2,\cdots,m, \tag{10}$$

$$(M_nu_i^2)(t)=\frac{n+1}{n}t_i^2+2\frac{\alpha_n}{n}t_i,\qquad i=1,2,\cdots,m. \tag{11}$$

Proof. By symmetry, we need only to verify the identities for the case $i=1$. Thus, using the easily proved identities

$$\lambda^{j+1}\int_0^\infty e^{-\lambda s}s^j\,ds=j!\qquad(\lambda>0,\ j=0,1,2,\cdots) \tag{12}$$

we carry out the following computations.

$$
\begin{aligned}
(L_n 1)(t) &= \sum_{0\le \bar{k}\le n} \binom{n}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-\bar{k}} \prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i} \frac{(n/\alpha_n)^{k_i}}{(k_i-1)!} \\
&\quad \cdot \int_0^\infty \exp\left(-\frac{nu_i}{\alpha_n}\right) u_i^{k_i-1}\, du_i \\
&= \sum_{0\le \bar{k}\le n} \binom{n}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-\bar{k}} \prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i} \\
&= \left(1-\frac{\bar{t}}{\alpha_n}+\frac{\bar{t}}{\alpha_n}\right)^{n} = 1 .
\end{aligned}
$$

$$
\begin{aligned}
(L_n u_1)(t) &= \sum_{0\le \bar{k}\le n} \binom{n}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-\bar{k}} \left[\prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i}\right] \frac{(n/\alpha_n)^{k_1}}{(k_1-1)!} \\
&\quad \cdot \int_0^\infty \exp\left(-\frac{nu_1}{\alpha_n}\right) u_1^{k_1}\, du_1 \\
&= \sum_{0\le \bar{k}\le n} \binom{n}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-\bar{k}} \left[\prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i}\right] \frac{k_1\alpha_n}{n} \\
&= t_1 \sum_{0\le \bar{k}\le n-1} \binom{n-1}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-1-\bar{k}} \prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i} = t_1 .
\end{aligned}
$$

$$
\begin{aligned}
(L_n u_1^2)(t) &= \sum_{0\le \bar{k}\le n} \binom{n}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-\bar{k}} \left[\prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i}\right] k_1(k_1+1)\left(\frac{\alpha_n}{n}\right)^2 \\
&= t_1 \sum_{0\le \bar{k}\le n-1} \binom{n-1}{k}\left(1-\frac{\bar{t}}{\alpha_n}\right)^{n-1-\bar{k}} \prod_{i=1}^{m}\left(\frac{t_i}{\alpha_n}\right)^{k_i} (k_1+2)\frac{\alpha_n}{n} \\
&= \frac{n-1}{n} t_1^2 \left(1-\frac{\bar{t}}{\alpha_n}+\frac{\bar{t}}{\alpha_n}\right)^{n-2} \\
&\quad + 2\frac{\alpha_n}{n} t_1 \left(1-\frac{\bar{t}}{\alpha_n}+\frac{\bar{t}}{\alpha_n}\right)^{n-1} .
\end{aligned}
$$

Hence (8), (9) and (10) hold for L_n, $n=1, 2, \cdots$, and the validity of (8), (9) and (11) for M_n can be verified in a similar way.

LEMMA 7. *For each* $t \in R_{0+}^m$,

$$
B_n(t) \equiv I - \sum_{i=1}^{m} \frac{t_i}{\alpha_n}\left[\left(I-\frac{\alpha_n}{n}A_i\right)^{-1} - I\right]
$$

is invertible and we have

$$(13)\quad (B_n(t))^{-n} = \sum_{0\le \bar{k}<\infty} \binom{-n}{k} \Big\{\prod_{i=1}^{m}\Big[-\frac{t_i}{\alpha_n}\Big(I-\frac{\alpha_n}{n}A_i\Big)^{-1}\Big]^{k_i}\Big\}\Big(1+\frac{\bar{t}}{\alpha_n}\Big)^{-n-\bar{k}}$$

where the convergence is uniform on any compact set $K \subset R^m_{0+}$.

Proof. Let M be an upper bound of $\|T(t)\|$. It follows from the Hille-Yosida theorem that $\|(I-\lambda A_i)^{-j}\| \le M$ for all $\lambda \ge 0$, $i=1, 2,\cdots, m$ and $j=1, 2,\cdots$. Therefore the series in (13) is dominated by the series

$$\sum_{0\le \bar{k}<\infty} \binom{-n}{k}(-1)^{\bar{k}}\prod_{i=1}^{m}\Big(\frac{t_i}{\alpha_n}\Big)^{k_i}\Big\|\Big(I-\frac{\alpha_n}{n}A_i\Big)^{-k_i}\Big\|\Big(1+\frac{\bar{t}}{\alpha_n}\Big)^{-n-\bar{k}}$$

$$\le M\sum_{\nu=0}^{\infty}\binom{-n}{\nu}(-1)^{\nu}\Big[\sum_{\bar{k}=\nu}\binom{\nu}{k}\prod_{i=1}^{m}\Big(\frac{t_i}{\alpha_n}\Big)^{k_i}\Big]\Big(1+\frac{\bar{t}}{\alpha_n}\Big)^{-n-\nu}$$

$$= M\sum_{\nu=0}^{\infty}\binom{-n}{\nu}\Big[\sum_{i=1}^{m}\Big(-\frac{t_i}{\alpha_n}\Big)\Big]^{\nu}\Big(1+\frac{\bar{t}}{\alpha_n}\Big)^{-n-\nu}$$

$$= M\Big(-\frac{\bar{t}}{\alpha_n}+1+\frac{\bar{t}}{\alpha_n}\Big)^{-n} = M.$$

Since the last series converges uniformly for t in K, so does the series in (13). That the limit is indeed $(B_n(t))^{-n}$ follows from the routine negative polynomial expansion of the latter.

Proof of Theorem 3. Let K be a compact set such that $K \subset K_n$ for all $n \ge N$. Then for any such n, $L_n: BC(R^m_{0+}) \to C(K)$ has the property that the corresponding function $g_n(u, t)$ is nonnegative. Since (8), (9) and (10) imply the uniform convergence on K of $(L_n 1)(t)$ to 1, $(L_n u_i)(t)$ to t_i, $(L_n u_i^2)(t)$ to t_i^2, respectively, it follows from the mentioned Theorem 1 that for each x $(L_n T(\cdot))(t)x$ converges to $T(t)x$ uniformly on K. As a semigroup, $T(t)$ can be written as $\prod_{i=1}^m T_i(t_i)$, and we have

$$\Big(\frac{n}{\alpha_n}-A_i\Big)^{-j}x = \frac{1}{(j-1)!}\int_0^\infty \exp\Big(-\frac{n}{\alpha_n}u_i\Big)u_i^{j-1}T(u_i)\,x\,du_i$$

for each $i=1, 2,\cdots, m$ and each $j=1, 2,\cdots$. (cf. [3, p. 360].) Hence

$$\int \prod_{i=1}^{m} \left[\frac{(n/\alpha_n)^{k_i}}{(k_i - 1)!} \exp\left(-\frac{nu_i}{\alpha_n}\right) u_i^{k_i - 1} \right] T(u)\, x\, du$$
$$= \left[\prod_{i=1}^{m} \left(\frac{n}{\alpha_n}\right)^{k_i} \right] \left\{ \frac{1}{(k_m - 1)!} \int_0^\infty \exp\left(-\frac{nu_m}{\alpha_n}\right) u_m^{k_m - 1} T_m(u_m) \cdots \right.$$
$$\left. \cdots \left[\frac{1}{(k_1 - 1)!} \int_0^\infty \exp\left(-\frac{nu_1}{\alpha_n}\right) u_1^{k_1 - 1} T_1(u_1)\, x\, du_1 \right] du_2 \cdots du_m \right\}$$
$$= \left[\prod_{i=1}^{m} \left(I - \frac{\alpha_n}{n} A_i \right)^{-k_i} \right] x,$$

and therefore

$$(L_n T(\cdot))(t)$$
$$= \sum_{0 \le \bar{k} \le n} \binom{n}{k} \left(1 - \frac{\bar{t}}{\alpha_n}\right)^{n - \bar{k}} \prod_{i=1}^{m} \left[\frac{t_i}{\alpha_n} \left(I - \frac{\alpha_n}{n} A_i \right)^{-1} \right]^{k_i}$$
$$= \left[\left(1 - \frac{\bar{t}}{\alpha_n}\right) I + \sum_{i=1}^{m} \frac{t_i}{\alpha_n} \left(I - \frac{\alpha_n}{n} A_i \right)^{-1} \right]^n.$$

Thus the proof is completed.

By using Lemmas 6 and 7, one can verify Theorem 4 in quite the same way as above. We shall omit its proof.

Proof of Theorem 5. Let $T(t)$, $t \ge 0$, be a one-parameter contraction (C_0)-semigroup with the infinitesimal generator A. The conclusion of Theorem 5 will follows from Chernoff's product formula (cf. [2]) once we show that $V(t) \equiv [2I - (I - tA)^{-1}]^{-1} (t \ge 0)$ has the following properties:

(a) $V(0) = I$.

(b) $V(t)$ is contractive. In fact, the inequalities $\|(I - tA)^{-k}\| \le 1$ $(t \ge 0,\ k = 0, 1, 2, \cdots)$ in the Hille-Yosida theorem ensure the estimate:

$$\|V(t)\| = \left\| \frac{1}{2} \sum_{k=0}^{\infty} \left[\frac{1}{2} (I - tA)^{-1} \right]^k \right\| \le \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1.$$

(c) $V(t)$ is strongly continuous. Actually, it is even continuous in the uniform operator topology since

$$\|V(s) - V(t)\| = \|V(s)[(V(t))^{-1} - (V(s))^{-1}]\, V(t)\|$$
$$= \|V(s)[(I - sA)^{-1} - (I - tA)^{-1}]\, V(t)\|$$
$$= \|V(s)(I - sA)^{-1}[(I - tA) - (I - sA)](I - tA)^{-1} V(t)\|$$
$$\le \|V(s)\| \|(I - sA)^{-1}\| |s - t| \|A(I - tA)^{-1}\| \|V(t)\|$$
$$\le |s - t| \|A(I - tA)^{-1}\|.$$

(d) $$V'(0)\,x = \lim_{t\to 0+} \frac{V(t)-I}{t}\,x = \lim_{t\to 0+} \frac{V(t)[(I-tA)^{-1}-I]}{t}\,x$$
$$= \lim_{t\to 0+} V(t)(I-tA)^{-1}\,Ax$$
$$= Ax$$

for all x in the domain of A.

REFERENCES

1. K. L. Chung, *On the exponential formulas of semi-group theory*, Math. Scand. **10** (1962), 153–162.

2. J. A. Goldstein, *Semigroup-theoretic proofs of the central limit theorem and other theorems of analysis*, Semigroup Forum **13** (1976), 189–206.

3. E. Hille and R. S. Phillips, "*Functional analysis and semigroups*", Amer. Math. Soc. Colloquium Publ. vol. **31**, New York, 1957.

4. S. Y. Shaw, *Approximation of unbounded functions and applications to representations of semigroups*, Journal of Approximation Theory **28** (1980), 238–259.

NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN, R. O. C.