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BEST POSSIBLE LENGTH ESTIMATES FOR NONLINEAR BOUNDARY VALUE PROBLEMS

BY

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Abstract. In this paper we have used shooting method to find best possible length estimates for nonlinear two point boundary value problems. The results are in terms of the zeros of the Cauchy function for the equations of constant coefficients.

1. **Introduction.** In this paper we shall consider the following nth order differential equation

$$x^{(n)}(t) = f(t, x, x', \cdots, x^{(q)}) \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} x^{(i)}(a_1) &= A_i, \qquad i = 0, 1, \cdots, n-2 \\ x^{(p)}(a_2) &= B_p \qquad (0 \le p \le n-1) \end{aligned} \tag{1.2}$$

or

$$\begin{aligned} x^{(p)}(a_1) &= A_p \qquad (0 \le p \le n-1) \\ x^{(i)}(a_2) &= B_i, \qquad i = 0, 1, \cdots, n-2. \end{aligned} \tag{1.3}$$

Throughout this paper we shall assume that the function f is continuous on $[a_1, a_2] \times R^{q+1}$ and satisfies the following condition

$$\begin{aligned} M_j(x_j - y_j) \le f(t, x_0, x_1, \cdots, x_j, \cdots, x_q) \\ - f(t, x_0, x_1, \cdots, y_j, \cdots, x_q) \\ \le K_j(x_j - y_j) \\ x_j \ge y_j, \qquad j = 0, 1, \cdots, q. \end{aligned} \tag{1.4}$$

The condition (1.4) is equivalent to the Lipschitz condition but more informative, particularly since there are no sign restrictions on the constants.

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The results obtained in this paper are in continuation to our work done in [1] – [4] and generalize the known results for second and third order obtained in [5], [6]. We have used here shooting method to find the best possible length of the interval in terms of the constants M_j and K_j, this is one of the main techniques used to construct the solutions of the nonlinear boundary value problems, for example see [7], [8].

2. **Cauchy function.** We shall denote $w(t)$ as the solution of the the following initial value problem

$$\begin{aligned} &D_n w(t) = w^{(n)}(t) + \sum_{i=0}^{q} b_i w^{(i)}(t) = 0 \\ &w^{(i)}(a) = 0, \qquad i = 0, 1, \cdots, n-2 \\ &w^{(n-1)}(a) = 1 \end{aligned} \tag{2.1}$$

where b_i $(i = 0, 1, \cdots, q)$ are constants. It is well known that the Cauchy function for (2.1) can be written as $w(t, s)(a \le s \le t)$ where

$$w(t, s) = w_1(t, s) = \sum_{i=1}^{n} \frac{e^{(t-s)\lambda_i}}{P'(\lambda_i)}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the distinct roots of the auxiliary equation

$$\lambda^n + \sum_{i=0}^{q} b_i \lambda^i = 0$$

and

$$P(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$

also in case the roots are $\lambda_1^*, \cdots, \lambda_k^*$ having multiplicity $r_1, \cdots, r_k$ respectively, with $\sum_{i=1}^{k} r_i = n$, then if we assume $\lambda_1^* = \lambda_1 = \lambda_2 = \cdots = \lambda_{r_1}$, $\lambda_2^* = \lambda_{r_1+1} = \lambda_{r_1+2} = \cdots = \lambda_{r_1+r_2}; \cdots;$ $\lambda_k^* = \lambda_{n-r_k+1} = \lambda_{n-r_k+2} = \cdots = \lambda_n$ the Cauchy function can be written as

$$\begin{aligned} w(t, s) = w_2(t, s) = \lim_{\lambda_2 \to \lambda_1^*} \quad & \lim_{\lambda_3 \to \lambda_1^*} \cdots \\ \lim_{\lambda_{r_1} \to \lambda_1^*} \quad & \lim_{\lambda_{r_1+2} \to \lambda_2^*} \cdots \\ & \lim_{\lambda_n \to \lambda_k^*} w_1(t, s). \end{aligned}$$

The following properties of $w(t, s)$ are immediate:

1. If we denote $t-s=X$ then $w(t, s)$ and its derivatives with respect to t can be written as a function of X alone, and we shall denote this by $w(X)$.

2. $w(0)=\dfrac{\partial w(0)}{\partial t}=\cdots=\dfrac{\partial^{n-2}w(0)}{\partial t^{n-2}}=0,\ \dfrac{\partial^{n-1}w(0)}{\partial t^{n-1}}=1.$

3. Let $X_0, X_1,\cdots, X_{n-1}$ be first right positive roots of $w(X)$, $\dfrac{\partial w(X)}{\partial t},\cdots,\dfrac{\partial^{n-1}w(X)}{\partial t^{n-1}}$ then $X_0\geq X_1\geq\cdots\geq X_{n-1}$.

4. If $\lambda_1=\lambda_2=\cdots=\lambda_n=\lambda$, then

$$w(X)=\frac{1}{(n-1)!}e^{\lambda X}\,x^{n-1}$$

and hence $X_0=\infty$.

3. Some comparison results.

THEOREM 3.1. *Let $u(t)$, $v(t)$ be two functions satisfying*

$$D_n u(t)\geq 0 \tag{3.1}$$

$$D_n v(t)\leq 0 \tag{3.2}$$

on $(a_1, a_2]$ with

$$u^{(i)}(a_1)=v^{(i)}(a_1),\qquad i=0, 1,\cdots, n-1.$$

Then

$$u^{(i)}(t)\geq v^{(i)}(t),\qquad t\in[a_1, a_1+X_i]. \tag{3.3}$$

Proof. Inequalities (3.1) and (3.2) with some $\phi_1(t)\geq 0$, $\phi_2(t)\geq 0$ can be written as

$$D_n u(t)-\phi_1(t)=0 \tag{3.4}$$

$$D_n v(t)+\phi_2(t)=0 \tag{3.5}$$

and hence if $u_0(t)$ is the solution of the homogeneous equation $D_n u_0(t)=0$ with $u_0^{(i)}(a_1)=u^{(i)}(a_1)=v^{(i)}(a_1)$ for $i=0, 1,\cdots, n-1$; then we can write the solutions of (3.4) and (3.5) as

$$u(t)=u_0(t)+\int_{a_1}^{t} w(t, s)\ \phi_1(s)\,ds.$$

$$v(t)=u_0(t)-\int_{a_1}^{t} w(t, s)\ \phi_2(s)\,ds.$$

Thus,

$$u^{(i)}(t) - v^{(i)}(t) = \int_{a_1}^{t} \frac{\partial^i w(t, s)}{\partial t^i} [\phi_1(s) + \phi_2(s)]\, ds.$$

But since $\partial^i w(t, s)/\partial t^i \geq 0$ as long as $a_1 \leq s \leq t \leq a_1 + X_i$ $(i = 0, 1, \cdots, n-1)$ the result follows immediately.

LEMMA 3.2. *Let $h(t)$ be a function $n-1$ times continuously differentiable on $[a_1, a_2]$. Then, if $h^{(n-1)}(a_2) > 0$, $h^{(i)}(a_2) = 0$, $i = 0, 1, \cdots, n-2$ there is some point $t^* < a_2$ such that $(-1)^{n-i+1}\cdot h^{(i)}(t) > 0$, $t \in (t^*, a_2)$, $i = 0, 1, \cdots, n-1$.*

Proof. Let $t^* < a_2$ be the first point wher $h^{(n-1)}(t) = 0$, then we have $h^{(n-1)}(t) > 0$, $t \in (t^*, a_2]$. For this t^*, the conclusion follows.

We shall denote $Y_0, Y_1, \cdots, Y_{n-1}$ as the distance between the point 0 and the first left zero (positive or negative according as $n-i+1$ is even or odd) of $w(X)$, $\partial w(X)/\partial t, \cdots, \partial^{n-1}w(X)/\partial t^{n-1}$ then we have from lemma 3.2, $Y_0 \geq Y_1 \geq Y_2 \geq \cdots \geq Y_{n-1}$.

THEOREM 3.3. *Let $u(t)$, $v(t)$ be two functions satisfying* (3.1) *and* (3.2) *on $[a_1, a_2)$ with $u^{(i)}(a_2) = v^{(i)}(a_2)$, $i = 0, 1, \cdots, n-1$. Then, $(-1)^{n-i+1} u^{(i)}(t) \leq (-1)^{n-i+1} v^{(i)}(t)$, $t \in [a_2 - Y_i, a_2]$.*

The proof of this theorm is same as of theorem 3.1.

4. **Uniqueness result.** Hereafter we shall use the following notations:

(a) $X_p(M_0, M_1, \cdots, M_q)$ in short $X_p(M)$ as the first positive zero of $\partial^p w(X)/\partial t^p$ replacing b_j by $-M_j (j = 0, 1, \cdots, q)$ in (2.1).

(b) for (1) n is even and p is odd, or (2) n is even and p is even we shall denote Y_p^* for $Y_p(M_0, K_1, M_2, K_3, \cdots, M_q$ (or K_q) according as q is even or odd) replacing b_j by $-M_j$ (or $-K_j$) in (2.1) according as j is even or odd.

(c) for (3) n is odd and p is even, or (4) n is odd and p is odd we shall denote Y_p^{**} for $Y_p(K_0, M_1, K_2, M_3, \cdots, K_q$ (or M_q) according as q is even or odd) replacing b_j by $-K_j$ (or $-M_j$) in (2.1) according as j is even or odd.

THEOREM 4.1. *Let $f(t, x, x', \cdots, x^{(q)})$ satisfy* (1.4). *Then, for*

$q \leq p$, (a) *the boundary value problem* (1.1), (1.2) *has atmost one solution if* $0 < a_2 - a_1 < X_p(M)$ (b) *the boundary value problem* (1.1), (1.3) *has atmost one solution if* $0 < a_2 - a_1 < Y_p^*$ *for the cases* (1) *and* (2) *and* $0 < a_2 - a_1 < Y_p^{**}$ *for the cases* (3) *and* (4).

Proof. We shall prove (b) and the proof of (a) follows similarly. Let $x_1(t)$, $x_2(t)$ be two solutions of (1.1), (1.3) and that they are distinct. Then, since solutions of initial values problems are unique we must have $x_1^{(n-1)}(a_2) \neq x_2^{(n-1)}(a_2)$. So, we can assume without loss of generality that $x_1^{(n-1)}(a_2) > x_2^{(n-1)}(a_2)$. Let, $h(t) = x_1(t) - x_2(t)$, this $h(t)$ satisfies the conditions of lemma 3.2 and hence $(-1)^{n-i+1} h^{(i)}(t) > 0$, $t \in (t^*, a_2)$. Let t_p be the first left zero of $h^{(p)}(t)$ (positive or negative according as $n - p + 1$ is even or odd) then naturally $a_1 \leq t_p \leq a_2$. Without loss of generality we may assume that $t_p = a_1$, then $(-1)^{n-i+1} h^{(i)}(t) \geq 0, t \in [a_1, a_2]$; $i = 0, 1, \cdots, q$ and hence $h(t)$ satisfies

$$h^{(n)}(t) \leq \sum_{\substack{j=\text{even}\\ \leq q}} M_j h^{(j)}(t) + \sum_{\substack{j=\text{odd}\\ \leq q}} K_j h^{(j)}(t), \text{ in case (1) and (2)}$$

$$h^{(n)}(t) \leq \sum_{\substack{j=\text{even}\\ \leq q}} K_j h^{(j)}(t) + \sum_{\substack{j=\text{odd}\\ \leq q}} M_j h^{(j)}(t), \text{ in case (3) and (4)}$$

$$h^{(i)}(a_2) = 0, \ i = 0, 1, \cdots, n-2$$

$$h^{(p)}(a_1) = 0$$

and

$$(-1)^{n-p+1} h^{(p)}(t) > 0, \ t \in (a_1, a_2).$$

Let $u(t)$ be the function defined by

$$u^{(n)}(t) = \sum_{\substack{j=\text{even}\\ \leq q}} M_j u^{(j)}(t) + \sum_{\substack{j=\text{odd}\\ \leq q}} K_j u^{(j)}(t), \text{ in case (1) and (2)}$$

$$u^{(n)}(t) = \sum_{\substack{j=\text{even}\\ \leq q}} K_j u^{(j)}(t) + \sum_{\substack{j=\text{odd}\\ \leq q}} M_j u^{(j)}(t), \text{ in case (3) and (4)}$$

$$u^{(i)}(a_2) = 0, \quad i = 0, 1, \cdots, n-2$$

$$u^{(n-1)}(a_2) = h^{(n-1)}(a_2).$$

By theorem 3.3, we have $(-1)^{n-p+1} h^{(p)}(t) \geq (-1)^{n-p+1} u^{(p)}(t) \geq 0$ for $t \in [a_2 - Y_p^*, a_2]$ in case (1) and (2) and $t \in [a_2 - Y_p^{**}, a_2]$ in case (3) and (4). Hence, in case (1), we find $h^{(p)}(t) \geq u^{(p)}(t) \geq 0$, in case (2) $h^{(p)}(t) \leq u^{(p)}(t) \leq 0$ for $t \in [a_2 - Y_p^*, a_2]$ and in case (3) $h^{(p)}(t) \geq u^{(p)}(t) \geq 0$ in case (4) $h^{(p)}(t) \leq u^{(p)}(t) \leq 0$ for $t \in [a_2 - Y_p^{**}, a_2]$. Since $h^{(p)}(a_1) = 0$ and $a_2 < a_1 + Y_p^*$ for (1) and (2) also $a_2 < a_1 + Y_p^{**}$ for (3) and (4), it follows that $u^{(p)}(t) = 0$ for some t in $[a_1, a_r)$. But this is a contradiction. Hence the proof is complete.

5. **Main result.** In what follows. we shall put $f(t, 0, 0, \cdots, 0) = \phi_q(t)$ and $X_p(K)$ for $X_p(M)$ replacing M_j to K_j $(j = 0, 1, \cdots, q)$.

LEMMA 5.1. *Suppose that* $a_1 < T < a_1 + X_p^*$ *where* $X_p^* = \min\{X_p(M), X_p(K)\}$ *and for* $a_1 \leq t \leq T$ *the functions* $u_1(t)$, $u_2(t)$ *and their derivatives up to order* p *are nonnegative and satisfy*

$$\begin{aligned} & u_1^{(n)}(t) - \sum_{j=0}^{q} M_j u_1^{(j)}(t) - \phi_q(t) = 0 \\ (5.1) \qquad & u_2^{(n)}(t) - \sum_{j=0}^{q} K_j u_2^{(j)}(t) - \phi_q(t) = 0 \\ & u_1^{(i)}(a_1) = u_2^{(i)}(a_1) = 0,\ i = 0, 1, \cdots, n-2 \\ & u_1^{(n-1)}(a_1) = u_2^{(n-1)}(a_1) > 0 \end{aligned}$$

and that $x(t)$ *satisfies*

$$\begin{aligned} & x^{(n)}(t) = f(t, x(t), x'(t), \cdots, x^{(q)}(t)) \\ (5.2) \qquad & x^{(i)}(a_1) = 0,\ i = 0, 1, \cdots, n-2 \\ & x^{(n-1)}(a_1) = u_1^{(n-1)}(a_1) = u_2^{(n-1)}(a_1) \end{aligned}$$

where f *satisfies* (1.4). *Then,*

$$(5.3) \qquad u_1^{(j)}(t) \leq x^{(j)}(t) \leq u_2^{(j)}(t),\ j = 0, 1, \cdots, p.$$

Proof. As in theorem 3.1, if $x(t)$ and $u_i(t)(i - 1, 2)$ satisfy the same initial conditions, then

$$x(t) - u_1(t) = \int_{a_1}^{t} w_M(t, s) \cdot \Big[- \sum_{j=0}^{q} M_j x^{(j)}(s) + f(s, x(s), \cdots, x^{(q)}(s)) - \phi_q(s) \Big] ds$$

and

$$x(t) - u_2(t) = \int_{a_1}^{t} w_k(t, s) \cdot \Big[-\sum_{j=0}^{q} K_j x^{(j)}(s)$$

$$+ f(s, x(s), \cdots, x^{(q)}(s)) - \phi_q(s) \Big] ds$$

where $w_M(t, s)$ is $w(t, s)$ replacing b_j to $-M_j$ and $w_K(t, s)$ is $w(t, s)$ replacing b_j to $-K_j$. But since $\partial^i w(t, s)/\partial t^i$ $(i = 0, 1, \cdots, q)$ are each nonnegative for $a_1 \leq s \leq t \leq a_1 + X_p^*$, it follows from (1.4)

$$x^{(i)}(t) - u_1^{(i)}(t) \geq 0 \quad \text{as long as } x^{(i)}(t) \geq 0; \quad i = 0, 1, \cdots, p$$

$$x^{(i)}(t) - u_2^{(i)}(t) \leq 0 \quad \text{as long as } x^{(i)}(t) \geq 0; \quad i = 0, 1, \cdots, p.$$

Thus (5.3) hold as long as $x^{(i)}(t) \geq 0$ and obviously $x^{(i)}(t) \geq 0$ as long as (5.3) hold. This proves the lemma.

REMARK 1. In lemma 5.1, $X_p^* = X_p(M)$ if we consider only the left half inequality, i. e. $u_1^{(j)}(t) \leq x^{(j)}(t)$, $j = 0, 1, \cdots, p$.

REMARK 2. From (1.4)

$$f(t, x, x', \cdots, x^{(q)}) - \phi_q(t) \leq \sum_{j=0}^{q} M_j x^{(j)}(t)$$

as long as $x^{(j)}(t) \leq 0$, $(j = 0, 1, \cdots, q)$. Thus, if $u_1(t)$ and all its derivatives up to order p are nonpositive for $a_1 \leq t \leq T$ where $a_1 < T < a_1 + X_p(M)$ and satisfy (5.1) with $u_1^{(n-1)}(a_1) < 0$ and that $x(t)$ satisfies (5.2), then $x^{(j)}(t) \leq u_1^{(j)}(t)$, $j = 0, 1, \cdots, p$.

LEMMA 5.2. *If $0 < a_2 - a_1 < X_p(M)$, there is a unique solution to the problem*

(5.4) $$u^{(n)}(t) - \sum_{j=0}^{q} M_j u^{(j)}(t) - \phi_q(t) = 0$$

$$u^{(i)}(a_1) = 0, \quad i = 0, 1, \cdots, n-2$$

(5.5) $$u^{(p)}(a_2) = m \in R.$$

Proof. It can easily be verified that the only solution of the problem (5.4), (5.5) is

$$u(t) = [u_0^{(p)}(a_2)]^{-1}\left[m - \int_{a_1}^{a_2} \frac{\partial^p w_M(t, s)}{\partial t^p} \phi_q(s)\right] u_0(t) + \int_{a_1}^{t} w_M(t, s)\ \phi_q(s)\, ds$$

where $u_0(t)$ is the solution of

$$u_0^{(n)}(t) = \sum_{j=0}^{q} M_j u_0^{(j)}(t)$$

$$u_0^{(i)}(a_1) = 0,\ i = 0, 1, \cdots, n-2$$

$$u_0^{(n-1)}(a_1) = 1.$$

Using this lemma we can take $m_2 > m$ and sufficiently large and positive so that $u^{(i)}(t) \geq 0$ on $[a_1, a_2]$ and $u^{(n-1)}(a_1) > 0$ also $m > m_1$ and sufficiently large and negative so that $u^{(i)}(t) \leq 0$ on $[a_1, a_2]$ and $u^{(n-1)}(a_1) < 0$ for all $i = 0, 1, \cdots, p$.

THEOREM 5.3. *In theorem 4.1. "atmost" can be replaced by "there exist atmost". Also this result is best possible.*

Proof. In theorem 4.1, we have already proved the uniqueness part so now we must prove atleast one part only. We shall give the proof for (a) only and for (b) it follows analogously. For simplicity we assume $A_i = 0$, $i = 0, 1, \cdots, n-2$.

Let $m_2 > m > B_p$ be sufficiently large in the sense of lemma 5.2, so that there is a unique solution $u(t, m_2)$ of (5.4) (5.5). As a result of remark 1 and standard theorem on the continuation of solutions of differential equations the solution $x_2(t)$ of (1.1) with the initial conditions $x_2^{(i)}(a_1) = 0$, $i = 0, 1, \cdots, n-2$, $x_2^{(n-1)}(a_1) = u^{(n-1)}(a_1, m_2)$ can be continued to $t = a_2$ and has

$$B_p < m < m_2 \leq x_2^{(p)}(a_2).$$

Likewise if $m_1 < m < B_p$ and is sufficiently large and negative in the sense of lemma 5.2, so that there is a unique solution $u(t, m_1)$ of (5.4), (5.5). By remark 2, the solution $x_1(t)$ of (1.1) with the initial conditions $x_1^{(i)}(a_1) = 0$, $i = 0, 1, \cdots, n-2$; $x_1^{(n-1)}(a_1) = u^{(n-1)}(a_1, m_1)$ exists as far as $t = a_2$ and has

$$x_1^{(p)}(a_2) \leq m_2 < m < B_p.$$

Now let $x(t)$ be any solution of (1.1) satisfying $x^{(i)}(a_1) = 0$,

$i = 0, 1, \cdots, n-2$ and $x_1^{(n-1)}(a_1) < x^{(n-1)}(a_1) < x_2^{(n-1)}(a_1)$. From the uniquness of solutions of the boundary value problem (1.1), (1.2) it follows that

$$x_1^{(i)}(t) < x^{(i)}(t) < x_2^{(i)}(t), \qquad i = 0, 1, \cdots, p$$

for $a_1 < t < a_2$, which guarantees that $x(t)$ exists as far as $t = a_2$. Now, by standard theorems on continuity with respect to initial conditions and the intermediate value theorem, it follows that there exists a unique solution of (1.1), (1.2).

From the definition of $X_p(M)$, when

$$f(t, x, \cdots, x^{(q)}) = \sum_{j=0}^{q} M_j x^{(j)}$$

$a_2 - a_1 = X_p(M)$, $x^{(p)}(a_2) = 0$, the problem (1.1), (1.2) has a non-trivial as well as the trivial solution also if $x^{(p)}(a_2) \neq 0$, this problem fails to have a solution. Hence the result is best possible.

THEOREM 5.4. *Let $q = 0$ in* (1.1) *and f satisfy* (1.4), *with $M_0 = 0$* (*i.e. nondecreasing in x*). *Then, there exists a unique solution of the problem* (1.1), (1.2) ($p = 0$).

The proof follows from the property (4) of $w(t, s)$. This, result was previously obtained by Schrader *et. al.* [9].

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