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SUBBUNDLES OF THE TANGENT BUNDLE OF HYPERSURFACE OF COMPLEX PROJECTIVE SPACE*

BY

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1. Introduction. The purpose of this note is to study the holomorphic line subbundles and holomorphic rank $n-1$ subbundles of the tangent bundle of a smooth hypersurface M of degree d in the complex projective $n+1$-space $\boldsymbol{P}^{n+1}$ $(n \geq 2)$.

For $d=1$, Roan [4] has shown that there is no line subbundle for all n, and no rank $n-1$ subbundle except odd n, in which case, there exists an unique one (up to the action of automorphisms group). Hence in this work, we mainly devote ourselves to the case for $d \geq 2$.

So far the result we have is:

MAIN THEOREM. For the tangent of smooth hypersurface M defined by a homogeneous polynomial of degree $d \geq 2$ in $\boldsymbol{P}^{n+1}$ $(n \geq 2)$,

(1) There is no line subbundle and holomorphic rank $n-1$ subbundle for all $n \geq 3$.

(2) In the case $n=2$,

$d=2$, $M \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, there are exactly two line subbundles.
$d=3$, there is no line subbundle.
$d \geq 4$, there is no line subbundle for generic hypersurface.

The remaining case is still unknown.

We shall prove the result in section 2. Now we give an outline of the method. First we examine the relation of the Chern classes

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between the tangent bundle T of M and its subbundle E so that we can exclude all the cases for $n \geq 3$ except

n: odd

$E = H$ or $T/E = H$, H: hyperplane bundle over P^{n+1}, and for $n = 2$, both $E = H^k$ and $T/E = H^k$ are impossible for all $k \in Z$. Next, we examine the bundle homomorphism from T to H and bundle homomorphism from H to T. We find that $\Gamma(M, T^* \otimes H) = 0 = \Gamma(M, H^{-1} \otimes T)$, thus gives (1). For (2) $n = 2$, we discuss it case by case.

2. **Preliminaries.** In this section we list some theorems we need in the sequel and the following notations are used.

M: a complex manifold, all bundles are holomorphic.

$\pi' : L \to M$, a line bundle over M.

$X : L -$ {zero section}.

$\pi \doteqdot \pi'_{\text{res}} : X \to M$, the C^*-bundle associated to L.

$\sigma : X \times C^* \to X$, $\sigma(x, \alpha) = \alpha x$.

$\tilde{\sigma}_k$ $(k \in Z)$: the C^*-linearization of $X \times C$

$$\begin{array}{ccc} (X \times C) \times C^* & \xrightarrow{\tilde{\sigma}_k} & X \times C \\ \downarrow & \circlearrowleft & \downarrow \\ X \times C^* & \xrightarrow{\sigma} & X \end{array}$$

defined by $\tilde{\sigma}_k((x, \beta), \alpha) = (\alpha x, \alpha^k \beta)$.

For a vector bundle $\rho : E \to M$ over M, let

$\pi^* E = \{(x, e) \in X \times E \mid \pi(x) = \rho(e)\}$, the pullback of E via π.

Σ_E: the C^*-linearization of $\pi^* E$

$$\begin{array}{ccc} \pi^* E \times C^* & \xrightarrow{\Sigma_E} & \pi^* E \\ \downarrow & \circlearrowleft & \downarrow \\ X \times C^* & \xrightarrow{\sigma} & X \end{array}$$

defined by $\Sigma_E((x, e), \alpha) = (\alpha x, e)$.

If V is a vector bundle over X and Σ is a C^*-linearization of V

$$\begin{array}{ccc} V \times C^* & \xrightarrow{\Sigma} & V \\ \downarrow & \circlearrowleft & \downarrow \\ X \times C^* & \xrightarrow{\sigma} & X \end{array}$$

then $V/\Sigma \to X/\sigma = M$ is a vector bundle over M whose fibre over a point p of M is

$$(V/\Sigma)_p = \Gamma(\pi^{-1}(p),\ V|_{\pi^{-1}(p)})^{\Sigma}$$
$$\overset{\text{def.}}{=} \left\{ s \in \Gamma(\pi^{-1}(p),\ V|_{\pi^{-1}(p)}) \,\middle|\, \begin{array}{l} s(\sigma(x,\ \alpha)) = \Sigma(s(x),\ \alpha) \\ \text{for}\quad (x,\ \alpha) \in \pi^{-1}(p) \times C^* \end{array} \right\}$$

whose sections over an open subset U of M are

$$\Gamma(U,\ V/\Sigma) = \Gamma(\pi^{-1}(U),\ V|_{\pi^{-1}(U)})^{\Sigma}$$
$$\overset{\text{def.}}{=} \left\{ s \in \Gamma(\pi^{-1}(U),\ V|_{\pi^{-1}(U)}) \,\middle|\, \begin{array}{l} s(\sigma(x,\ \alpha)) = \Sigma(s(x),\ \alpha) \\ \text{for}\quad (x,\ \alpha) \in \pi^{-1}(U) \times C^* \end{array} \right\}.$$

In particular, for $k \in Z$

$$(X \times C/\tilde{\sigma}_k)_p$$
$$= \{(\text{id.},\ f) : \pi^{-1}(p) \longrightarrow \pi^{-1}(p) \times C \mid f(\alpha x) = \alpha^k f(x)\}$$

$$\Gamma(U,\ X \times C/\tilde{\sigma}_k)$$
$$= \{(\text{id.},\ f) : \pi^{-1}(U) \longrightarrow \pi^{-1}(U) \times C \mid f(\alpha x) = \alpha^k f(x)\}$$

where $p \in M$ and U is an open subset of M.

For a vector bundle E over M, we have a bundle isomorphism between E and $\pi^* E/\Sigma_E$ under which an element e of E_p corresponds to the element (id., f) in $(\pi^* E/\Sigma_E)_p$ with $f(x) = e$ for all $x \in \pi^{-1}(p)$, and we shall identify them $E = \pi^* E/\Sigma_E$ via this isomorphism.

For each $k \in Z$, thus we can define the natural isomorphism of line bundles $\iota_k : X \times C/\tilde{\sigma}_k \simeq L^{-k}$.

If $\rho : E \to M$ is a vector bundle over M and $\tilde{\sigma}_{E,k}$ ($k \in Z$) is the C^*-linearization of $\pi^* E$

$$\begin{array}{ccc} \pi^* E \times C^* & \xrightarrow{\tilde{\sigma}_{E,k}} & \pi^* E \\ \downarrow & \circlearrowleft & \downarrow \\ X \times C^* & \xrightarrow{\sigma} & X \end{array}$$

defined by $\tilde{\sigma}_{E,k}((x,\ e),\ \alpha) = (\alpha x,\ \alpha^k e)$.

Then $\pi^* E/\tilde{\sigma}_{E,k}$ is isomorphic to $E \otimes L^{-k}$ as vector bundles over M. For details, see Roan [4].

3. **Proof of the result.** Let $M \subseteq^i P^{n+1}$ be a smooth hypersurface of complex projective $n+1$-space defined by a homogeneous polynomial of degree $d \geq 1$ with tangent bundle T.

By Lefschetz's theorem, we know that $H^j(P^{n+1}, Z) \xrightarrow{i^*} H^j(M, Z)$ is isomorphic for $j \leq n-1$, injective for $j = n$ and $H^n(M, Z)/i^* H^n(P^{n+1}, Z)$ has no torsion.

Now, if we have a short exact sequence of holomorphic vector bundles over M, i.e.,

$$0 \longrightarrow L \longrightarrow T \longrightarrow Q \longrightarrow 0 \qquad \begin{array}{l} L\text{: line subbundle of } T \\ Q\text{: quotient bundle} \end{array}$$

then the total Chern classes of T, L and Q satisfy the relation $C(T) = C(L) \cdot C(Q)$, i.e.,

$$(*) \qquad \begin{aligned} &(1+t)^{n+2}(1+dt)^{-1} \\ &\quad = (1+at)(1+x_1+x_2+\cdots+x_{n-1}) \qquad \text{for } n \geq 2 \end{aligned}$$

where $t = i^*(c_1(H)) \in H^2(M, Z)$ is the pullback of first Chern class of hyperplane bundle over P^{n+1} and $x_i \in H^{2i}(M, Z)$. (In the case $n = 2$, we also assume that $L = i^* H^a$, $a \in Z$).

LEMMA. (*) *holds only if* n *is odd and* $\begin{cases} a=1 \\ d=2 \end{cases}$ *or* $\begin{cases} a=2 \\ d=1 \end{cases}$.

Proof. Rewrite (*) as

$$(1+t)^{n+2}(1-dt+\cdots+(-d)^n t^n)(1-at+\cdots+(-a)^n t^n) = 1+x_1+\cdots+x_{n-1}$$

$$(1+c_1 t+\cdots+c_n t^n)(1-at+\cdots+(-a)^n t^n) = 1+x_1+\cdots+x_{n-1}$$

$$\therefore \quad x_i = [c_i - a c_{i-1} + \cdots + (-a)^{i-1} c_1 + (-a)^i] t^i, \qquad 1 \leq i \leq n-1$$

and

$$(**) \qquad c_n - a c_{n-1} + a^2 c_{n-2} + \cdots + (-a)^{n-1} c_1 + (-a)^n = 0$$

where

$$c_k = \binom{n+2}{k} - d\binom{n+2}{k-1} + \cdots + (-d)^k, \qquad 1 \le k \le n$$

substitute c_k into (**), we have

$$0 = \frac{1}{-a+d}\left\{\frac{1}{-a}[(1-a)^{n+2} + (n+2)a - 1]\right.$$
$$\left. - \frac{1}{-d}[(1-d)^{n+2} + (n+2)d - 1]\right\} \qquad \text{for } a \neq d$$

$$(**) = 0 \Longleftrightarrow d(1-a)^{n+2} - a(1-d)^{n+2} - d + a = 0$$
$$\Longleftrightarrow d(1+b)^{n+2} + b(1-d)^{n+2} - d - b = 0,$$
$$a = -b, \quad b > 0.$$

(I) $a \neq 0$. If $a = 0$,

$$0 = \binom{n+2}{n} + (-d)\binom{n+2}{n-1} + \cdots + (-d)^n\binom{n+2}{0}$$
$$= d^{-2}[-1 + (n+2)d + (1-d)^{n+2}]$$

$$\therefore \quad -1 + (n+2)d + (1-d)^{n+2} = 0, \qquad (d \ge 1)$$

$$\therefore \quad n \text{ must be odd and } d \neq 1, 2.$$

It is also clear that $-1 + (n+2)d < (d-1)^{n+2}$ for $d \ge 3$.

(II) $a > 0$. If $a < 0$.

For n is even, $b(1-d)^{n+2} \ge 0$,

$$d(1 + (n+2)b + \cdots + b^{n+2}) - d - b + b(1-d)^{n+2} > 0,$$

hence $(**) \neq 0$.

For n is odd, set $n + 2 = 2k + 1$, $k \ge 2$,

(1) $d = 1$, $(1+b)^{2k+1} - b - 1 > 0$,

(2) $d = 2$, $2(1+b)^{2k+1} - 2b - 2 > 0$,

(3) $d > 2$, $(**) = 0$ if and only if

$$[(1+b)^{2k} + \cdots + (1+b) + 1]$$
$$- [(1-d)^{2k} + \cdots + (1-d) + 1] = 0.$$

Thus the proof of (II) will be completed if we prove the following sublemma.

SUBLEMMA. $g: \boldsymbol{R} \to \boldsymbol{R}$, $x \mapsto g(x) = x^{2k} + \cdots + x + 1$, $k \ge 2$, $k \in \boldsymbol{N}$, *then* $g(m) \neq g(-p)$ *for all* M, $p \in \boldsymbol{N}$, $m \ge 2$, $p \ge 2$.

Proof. $$g(x)=\frac{x^{2k+1}-1}{x-1}, \qquad x\neq 1$$

$$g(m)\neq g(-p)\Longleftrightarrow \frac{m^{2k+1}-1}{m-1}\neq\frac{p^{2k+1}+1}{p+1}.$$

(1) $m=p+h,\ h\geq 0,\ h\in Z$

$$\frac{p^{2k+1}+1}{p+1}-p^{2k}=\frac{1-p^{2k}}{1+p}<0$$

$$\frac{m^{2k+1}-1}{m-1}-m^{2k}=\frac{m^{2k}-1}{m-1}>0$$

$$\frac{m^{2k+1}-1}{m-1}=\frac{(p+h)^{2k+1}-1}{(p+h)-1}>(p+h)^{2k}\geq p^{2k}>\frac{p^{2k+1}+1}{p+1}.$$

(2) $m=p+h',\ h'<0$, i.e., $p=m+h,\ h>0,\ h,\ h'\in Z.$ Let

$$f(x)=\frac{(m+x)^{2k+1}+1}{m+x+1} \qquad x\in \boldsymbol{R},\quad x\geq 0.$$

$$\because\quad f'(x)=\frac{(m+x)^{2k}\left[(2k)(m+x+1)+1\right]-1}{(m+x+1)^2}>0, \qquad x\geq 0$$

$$\therefore\quad f(x) \text{ is strictly increasing on } x\geq 0.$$

Hence

$$\frac{(m+h)^{2k+1}+1}{m+h+1}\geq\frac{(m+1)^{2k+1}+1}{m+2} \qquad \text{for}\quad h\geq 1.$$

From (1), we have

$$\frac{m^{2k+1}-1}{m-1}>\frac{m^{2k+1}+1}{m+1}.$$

In order to show

$$\frac{(m+h)^{2k+1}+1}{m+h+1}\neq\frac{m^{2k+1}-1}{m-1}, \qquad h\geq 1,$$

it suffices to show

$$\frac{(m+1)^{2k+1}+1}{m+2}>\frac{m^{2k+1}-1}{m-1}$$

which is equivalent to $A>B$, where

$$A=\frac{(m+1)^{2k+1}+1}{m+2}-\frac{m^{2k+1}+1}{m+1}$$

$$\begin{aligned}
B &= \frac{m^{2k+1}-1}{m-1} - \frac{m^{2k+1}+1}{m+1} \\
&= \frac{2m(m^{2k}-1)}{m^2-1} = 2m(m^{2(k-1)}+\cdots+m^2+1)
\end{aligned}$$

$$\begin{aligned}
A - B &= \frac{1}{(m+1)(m+2)}\{(m+1)[(m+1)^{2k+1}+1] \\
&\quad - (m+2)(m^{2k+1}+1) \\
&\quad - (2m^3+6m^2+4m)(m^{2(k-1)}+\cdots+m^2+1)\} \\
&= \frac{1}{(m+1)(m+2)}\Big\{m\Big[m^{2k+1}+(2k+1)\,m^{2k} \\
&\quad + \binom{2k+1}{2}m^{2k-1}+\cdots+(2k+1)\,m+1\Big] \\
&\quad + (m+1)^{2k+1}+m+1-m^{2k+2}-m-2m^{2k+1}-2 \\
&\quad - [2m^{2k+1}+6(m^{2k}+m^{2k-1}+\cdots+m^2)+4m]\Big\} \\
&= \frac{1}{(m+1)(m+2)}\Big\{[(2k-2)\,m^{2k+1}-1] \\
&\quad + \Big[\binom{2k+1}{2}+\binom{2k+1}{1}-6\Big]m^{2k}+\cdots \\
&\quad + \Big[\binom{2k+1}{j-1}+\binom{2k+1}{j}-6\Big]m^{j}+\cdots \\
&\quad + [1+2k+1-4]\,m+1\Big\}, \qquad 2\le j\le 2k
\end{aligned}$$

$$\because \quad k\ge 2, \qquad 2+2k-4\ge 0$$

and

$$\binom{2k+1}{j-1}+\binom{2k+1}{j}-6=\binom{2k+2}{j}-6\ge 9 \qquad \text{for} \quad 2\le j\le 2k.$$

Hence $A-B>0$, i.e., $A>B$. This completes the proof of the sublemma.

(III) $a>0$.

case 1. $a=d$

Then (**) can be written as

$$\begin{aligned}
0 &= \binom{n+2}{0}((n+1)(-a)^n)+\binom{n+2}{1}(n(-a)^{n-1})+\cdots \\
&\quad + \binom{n+2}{n-1}(2(-a))+\binom{n+2}{n}.
\end{aligned}$$

Consider

$$h(x)=\binom{n+2}{0}x^{n+1}+\binom{n+2}{1}x^{n}+\cdots+\binom{n+2}{n}x$$
$$=\frac{1}{x}[(1+x)^{n+2}-(n+2)x-1]$$

$$h'(x)=\frac{1}{x^2}[(1+x)^{n+1}((n+1)x-1)+1].$$

If $a\in N$ and $h'(-a)=0$, i.e., $(1-a)^{n+1}((n+1)a+1)=1$,

$$\therefore\quad 1=|(1-a)|^{n+1}((n+1)a+1)\geq 7,$$

which gives the contradiction.

case 2. $a\neq d$

From $(**)=0$, we have $d(1-a)^{n+2}-a(1-d)^{n+2}-d+a=0$,

$$\therefore\quad [(1-a)^{n+1}+(1-a)^{n}+\cdots+(1-a)+1]$$
$$-[(1-d)^{n+1}+\cdots+(1-d)+1]=0.$$

Consider

$$g(x)=x^{n+1}+x^{n}+\cdots+x+1=\frac{x^{n+2}-1}{x-1}$$

set $u=d-1$, $v=a-1$, $u\geq 0$, $v\geq 0$, $u\neq v$. Suppose that $u>0$ and $v>0$. $g(-u)=g(-v)$ implies

$$(u-v)[1+(-1)^{n+2}(u^{n+1}v+\cdots$$
$$+uv^{n+1}+u^{n+1}+u^{n}v+\cdots+uv^{n}+v^{n+1})]=0$$

$$\therefore\quad 1+(-1)^{n+2}[u^{n+1}v+\cdots$$
$$+uv^{n+1}+u^{n+1}+u^{n}v+\cdots+uv^{n}+v^{n+1}]=0.$$

But the absolute value of left hand side is no less than 8, this is absurd,

$$\therefore\quad u=0\quad\text{or}\quad v=0.$$

If $u=0$,

$$g(0)=1=g(-v)=\frac{(-v)^{n+2}-1}{-v-1},$$

then

$$(-v)((-v)^{n+1}-1)=0.$$

So n must be odd and $v=1$, hence $d=1$ and $a=2$. Same argument for $v=0$, we get $d=2$ and $a=1$. Thus the proof of the lemma is completed. Q.E.D.

Roan [4] has ruled out the case $a=2$, $d=1$. For the other case $a=1$ and $d=2$, we may assume that $M\subseteq \boldsymbol{P}^{n+1}$ is defined by the equation $f=z_0^2+z_1^2+\cdots+z_n^2+z_{n+1}^2=0$ for $n\geq 3$.

In order to examine $\mathrm{Hom}(\boldsymbol{H}, \boldsymbol{T})\simeq\Gamma(M, \boldsymbol{H}^{-1}\otimes \boldsymbol{T})\simeq\Gamma(M, \boldsymbol{T}\otimes\boldsymbol{H}^{-1})$ we are going to compute $\Gamma(M, \boldsymbol{T}\otimes\boldsymbol{H}^{-1})$.

The following notations are used in the computation.

$\tilde{X}=\boldsymbol{C}^{n+2}-\{0\}$.

$X=C(M)-\{0\}$, $\quad C(M)=\{z=(z_0,\cdots, z_{n+1})\in \boldsymbol{C}^{n+2}\mid f(z)=0\}$

$\pi: X\longrightarrow M$, $\quad (z_0,\cdots, z_{n+1})\xrightarrow{\pi}[z_0,\cdots, z_{n+1}]$

$X\xrightarrow{i}\tilde{X}$, $\quad$ inclusion map.

First we have $\boldsymbol{T}\otimes\boldsymbol{H}^{-1}\simeq\pi^*\boldsymbol{T}/\tilde{\sigma}_{T,-1}$ and the map $\boldsymbol{T}(X)\xrightarrow{\pi_*}\pi^*\boldsymbol{T}$. In terms of local coordinates, π_* can be describe as follows: For $z_0\neq 0$, $z_1,\cdots, z_{n+1}$ are local coordinates of X and $w_1=z_1/z_0,\cdots, w_{n+1}=z_{n+1}/z_0$ are local coordinates of $\boldsymbol{P}^{n+1}$. Hence we may take $w_2,\cdots, w_{n+1}$ as local coordinates of M, if $w_1\neq 0$.

$$\pi_*\begin{bmatrix}\dfrac{\partial}{\partial z_1}\\ \vdots\\ \dfrac{\partial}{\partial z_{n+1}}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial w_2}{\partial z_1}, & \cdots, & \dfrac{\partial w_{n+1}}{\partial z_1}\\ \vdots, & \cdots, & \vdots\\ \dfrac{\partial w_2}{\partial z_{n+1}}, & \cdots, & \dfrac{\partial w_{n+1}}{\partial z_{n+1}}\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial w_2}\\ \vdots\\ \dfrac{\partial}{\partial w_{n+1}}\end{bmatrix}$$

$$=\frac{1}{z_0^3}\begin{bmatrix}z_1z_2, & z_1z_3, & \cdots, & z_1z_{n+1}\\ z_0^2+z_2^2, & z_2z_3, & \cdots, & z_2z_{n+1}\\ z_2z_3, & z_0^2+z_3^2, & \cdots, & z_3z_{n+1}\\ \vdots & \vdots & & \vdots\\ z_2z_{n+1}, & z_3z_{n+1}, & \cdots, & z_0^2+z_{n+1}^2\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial w_2}\\ \dfrac{\partial}{\partial w_3}\\ \dfrac{\partial}{\partial w_4}\\ \vdots\\ \dfrac{\partial}{\partial w_{n+1}}\end{bmatrix}$$

Define a map $\alpha: X\times C\rightarrow T(X)$

if $z_0\neq 0$, $((z_0,\cdots, z_{n+1}), c)\longmapsto\left(cz_1\dfrac{\partial}{\partial z_1}+\cdots+cz_{n+1}\dfrac{\partial}{\partial z_{n+1}}\right)_{(z_0,\cdots,z_{n+1})}$

α is well-defined. For $z_0 \neq 0$ and $z_1 \neq 0$, we have local coordinates $(z_1, \cdots, z_{n+1})$ and $(z_0, z_2, \cdots, z_{n+1})$ respectively, and

$$((z_0, \cdots, z_{n+1}), c) \longmapsto \left(cz_1 \frac{\partial}{\partial z_1} + cz_2 \frac{\partial}{\partial z_2} + \cdots + cz_{n+1} \frac{\partial}{\partial z_{n+1}}\right)_{(z_0, \ldots, z_{n+1})}$$

if $z_0 \neq 0$, and

$$((z_0, \cdots, z_{n+1}), c) \longmapsto \left(cz_0 \frac{\partial}{\partial z_0} + cz_2 \frac{\partial}{\partial z_2} + \cdots + cz_{n+1} \frac{\partial}{\partial z_{n+1}}\right)_{(z_0, \ldots, z_{n+1})}$$

if $z_1 \neq 0$.

But the transition matrix of $T(X)$ for these two local trivializations is

$$\begin{bmatrix} \dfrac{\partial}{\partial z_1} \\ \dfrac{\partial}{\partial z_2} \\ \vdots \\ \dfrac{\partial}{\partial z_{n+1}} \end{bmatrix} = \begin{bmatrix} -\dfrac{z_1}{z_0}, & 0 \\ -\dfrac{z_2}{z_0}, & \\ \vdots & I_n \\ -\dfrac{z_{n+1}}{z_0}, & \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial z_0} \\ \dfrac{\partial}{\partial z_2} \\ \vdots \\ \dfrac{\partial}{\partial z_{n+1}} \end{bmatrix}.$$

Hence

$$(z_1, z_2, \cdots, z_{n+1}) \begin{bmatrix} -\dfrac{z_1}{z_0}, & 0 \\ -\dfrac{z_2}{z_0}, & \\ \vdots & I_n \\ -\dfrac{z_{n+1}}{z_0}, & \end{bmatrix} = (z_0, z_2, \cdots, z_{n+1})$$

$$\because \quad z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\,.$$

This shows that α is well-defined. And we have a short exact sequence of holomorphic vector bundles over X

$$0 \longrightarrow X \times C \xrightarrow{\alpha} T(X) \xrightarrow{\pi_*} \pi^* T \longrightarrow 0\,.$$

The exactness of this sequence is easily to check.

Consider the C^*-linearization, the following diagram is commutative

$$\begin{array}{ccccc}
(X \times C) \times C^* & \xrightarrow{(\alpha,\ \mathrm{id.})} & T(X) \times C^* & \xrightarrow{(\pi_*,\ \mathrm{id.})} & \pi^* T \times C^* \\
\downarrow \tilde{\sigma}_{-1} & \circlearrowleft & \downarrow \tilde{\omega}_0^{n+1} & \circlearrowleft & \downarrow \tilde{\sigma}_{T,-1} \\
X \times C & \xrightarrow{\alpha} & T(X) & \xrightarrow{\pi_*} & \pi^* T
\end{array}$$

where $\tilde{\omega}_k^{n+1}$, $k \in Z$, is defined as follows:

For each point $p \in X$, we have a local trivialization near p, i.e., there exists a neighborhood U of p such that

$$T(X)_U \simeq U \times C^{n+1}.$$

Under the identification, define

$$\tilde{\omega}_k^{n+1} : (U \times C^{n+1}) \times C^* \longrightarrow U \times C^{n+1}$$

by

$$(p,\ (\zeta_1, \cdots, \zeta_{n+1}),\ \alpha) \longmapsto (\alpha p,\ (\alpha^k \zeta_1, \cdots, \alpha^k \zeta_{n+1})).$$

It is easy to see that $\tilde{\omega}_k^{n+1}$ is well-defined.

So we have an exact sequence of holomorphic vector bundles over M

$$0 \longrightarrow X \times C/\tilde{\sigma}_{-1} \longrightarrow T(X)/\tilde{\omega}_0^{n+1} \longrightarrow \pi^* T/\tilde{\sigma}_{T,-1} \longrightarrow 0$$

$$X \times C/\tilde{\sigma}_{-1} \simeq H^{-1} \quad \text{and} \quad \pi^* T/\tilde{\sigma}_{T,-1} \simeq T \otimes H^{-1}$$

$$0 \longrightarrow H^{-1} \longrightarrow T(X)/\tilde{\omega}_0^{n+1} \longrightarrow T \otimes H^{-1} \longrightarrow 0.$$

From the long exact sequence

$$0 \longrightarrow \Gamma(M,\ H^{-1}) \longrightarrow \Gamma(M,\ T(X)/\tilde{\omega}_0^{n+1}) \longrightarrow \Gamma(M,\ T \otimes H^{-1}) \longrightarrow H^1(M,\ \mathcal{O}(H^{-1}))$$

and by Kodaira vanishing theorem $\Gamma(M, H^{-1}) = H^1(M, \mathcal{O}(H^{-1})) = 0$

$$\therefore\ \Gamma(M,\ T(X)/\tilde{\omega}_0^{n+1}) \simeq \Gamma(M,\ T \otimes H^{-1}).$$

For $z_0 \neq 0$, the map $T(X) \xrightarrow{i_*} T(\tilde{X})_{|X}$ can be described as

$$i_* \begin{bmatrix} \dfrac{\partial}{\partial z_1} \\ \dfrac{\partial}{\partial z_2} \\ \vdots \\ \dfrac{\partial}{\partial z_{n+1}} \end{bmatrix} = \left[\begin{array}{c:c} -\dfrac{z_1}{z_0} & \\ -\dfrac{z_2}{z_0} & I_{n+1} \\ \vdots & \\ -\dfrac{z_{n+1}}{z_0} & \end{array}\right] \begin{bmatrix} \dfrac{\partial}{\partial z_0} \\ \dfrac{\partial}{\partial z_1} \\ \vdots \\ \dfrac{\partial}{\partial z_{n+1}} \end{bmatrix}.$$

Define $\beta : T(\tilde{X})_{|X} \to X \times C$ by

$$\beta\left(c_0\left(\frac{\partial}{\partial z_0}\right)_z + \cdots + c_{n+1}\left(\frac{\partial}{\partial z_{n+1}}\right)_z\right)$$
$$= (z,\ c_0 z_0 + \cdots + c_{n+1} z_{n+1}) \qquad \text{for} \quad z = (z_0 \cdots z_{n+1}) \in X.$$

Clearly there is a short exact sequence

$$0 \longrightarrow T(X) \xrightarrow{i_*} T(\tilde{X})_{|X} \xrightarrow{\beta} X \times C \longrightarrow 0 .$$

Consider the C^*-linearization, the following diagram is commutative.

$$\begin{array}{ccccc} T(X) \times C^* & \xrightarrow{(i_*,\ \mathrm{id.})} & T(\tilde{X})_{|X} \times C^* & \xrightarrow{(\beta,\ \mathrm{id.})} & (X \times C) \times C^* \\ \downarrow \tilde{\omega}_0^{n+1} & \circlearrowleft & \downarrow \tilde{\sigma}_0^{n+2} & \circlearrowleft & \downarrow \tilde{\sigma}_1 \\ T(X) & \xrightarrow{i_*} & T(\tilde{X})_{|X} & \xrightarrow{\beta} & X \times C \end{array}$$

$$0 \longrightarrow T(X)/\tilde{\omega}_0^{n+1} \longrightarrow T(\tilde{X})_{|X}/\tilde{\sigma}_0^{n+2} \longrightarrow X \times C/\tilde{\sigma}_1 \longrightarrow 0$$

is exact over M and $X \times C/\tilde{\sigma}_1 \simeq H$. The long exact sequence is

$$\begin{array}{ccccc} 0 \longrightarrow \Gamma(M,\ T(X)/\tilde{\omega}_0^{n+1}) \longrightarrow & \Gamma(M,\ T(\tilde{X})_{|X}/\tilde{\sigma}_0^{n+2}) & \longrightarrow & \Gamma(M,\ X \times C/\tilde{\sigma}_1) \\ & \wr\| & \circlearrowleft & \wr\| \\ & C^{n+2} & \xrightarrow{\eta} & C[z]_1 \end{array}$$

where

$$\eta : C^{n+2} \longrightarrow C[z]_1 , \qquad (c_0, \cdots,\ c_{n+1}) \longmapsto c_0 z_0 + \cdots + c_{n+1} z_{n+1} ,$$
$$z = (z_0, \cdots,\ z_{n+1}) ,$$
$$C[z]_1 = \{\text{homogeneous polynomials of degree } 1\} ,$$
$$\therefore \quad \Gamma(M,\ T \otimes H^{-1}) \simeq \Gamma(M,\ T(X)/\tilde{\omega}_0^{n+1}) \simeq \mathrm{Ker}\, \eta = 0 .$$

Therefore, we cannot have a short exact sequence

$$0 \longrightarrow H \longrightarrow T \longrightarrow Q \longrightarrow 0 \qquad \text{over} \quad M .$$

Next, we examine $\mathrm{Hom}\,(T,\ H) \simeq \Gamma(M,\ T^* \otimes H)$. We have a short exact sequence of holomorphic vector bundles over X

$$0 \longrightarrow \pi^* T^* \xrightarrow{\pi^*} T^*(X) \xrightarrow{\beta} X \times C \longrightarrow 0 .$$

If $z_0 \neq 0$, $\beta : T^*(X) \to X \times C$, defined by

$$\beta((c_1\, dz_1 + \cdots + c_{n+1}\, dz_{n+1})_{(z_0, \dots, z_{n+1})})$$
$$= ((z_0, \cdots,\ z_{n+1}),\ c_1 z_1 + \cdots + c_{n+1} z_{n+1}) .$$

The following diagram commutes,

$$\begin{array}{ccccc}
\pi^* T^* \times C^* & \xrightarrow{(\pi^*,\ \mathrm{id.})} & T^*(X) \times C^* & \xrightarrow{(\beta,\ \mathrm{id.})} & (X \times C) \times C^* \\
\downarrow \tilde{\sigma}_{T^*,1} & \circlearrowleft & \downarrow \tilde{\omega}_0^{n+1} & \circlearrowleft & \downarrow \tilde{\sigma}_1 \\
\pi^* T & \xrightarrow{\pi^*} & T^*(X) & \xrightarrow{\beta} & X \times C
\end{array} .$$

Since $T^* \otimes H \simeq \pi^* T^* / \tilde{\sigma}_{T^*,1}$,

$$0 \longrightarrow T^* \otimes H \longrightarrow T^*(X)/\tilde{\omega}_0^{n+1} \longrightarrow X \times C/\tilde{\sigma}_1 \longrightarrow 0$$

exact over M. We have

$$0 \longrightarrow \Gamma(M,\ T^* \otimes H) \longrightarrow \Gamma(M,\ T^*(X)/\tilde{\omega}_0^{n+1}) \longrightarrow \Gamma(M,\ X \times C/\tilde{\sigma}_1) .$$

Again, we have a short exact sequence over X

$$0 \longrightarrow X \times C \xrightarrow{\alpha} T^*(\tilde{X})_{|X} \xrightarrow{i^*} T^*(X) \longrightarrow 0$$

$$\alpha : X \times C \longrightarrow T^*(\tilde{X})_{|X}$$

defined by

$$((z_0, \cdots, z_{n+1}),\ c) \longmapsto (cz_0\, dz_0 + \cdots + cz_{n+1}\, dz_{n+1})_{(z_0, \ldots, z_{n+1})}$$

and the following diagram is commutative,

$$\begin{array}{ccccc}
(X \times C) \times C^* & \xrightarrow{(\alpha,\ \mathrm{id.})} & T^*(\tilde{X})_{|X} \times C^* & \xrightarrow{(i^*,\ \mathrm{id.})} & T^*(X) \times C^* \\
\downarrow \tilde{\sigma}_{-1} & \circlearrowleft & \downarrow \tilde{\sigma}_0^{n+2} & \circlearrowleft & \downarrow \tilde{\omega}_0^{n+1} \\
X \times C & \xrightarrow{\alpha} & T^*(\tilde{X})_{|X} & \xrightarrow{i^*} & T^*(X)
\end{array}$$

$$0 \longrightarrow H^{-1} \longrightarrow T^*(\tilde{X})_{|X}/\tilde{\sigma}_0^{n+2} \longrightarrow T^*(X)/\tilde{\omega}_0^{n+1} \longrightarrow 0$$

is exact over M.

$$\begin{aligned}
0 \longrightarrow \Gamma(M,\ H^{-1}) &\longrightarrow \Gamma(M,\ T^*(\tilde{X})_{|X}/\tilde{\sigma}_0^{n+2}) \\
&\longrightarrow \Gamma(M,\ T^*(X)/\tilde{\omega}_0^{n+1}) \longrightarrow H^1(M,\ \mathcal{O}(H^{-1}))
\end{aligned}$$

$$\because \quad \Gamma(M,\ H^{-1}) = H^1(M,\ \mathcal{O}(H^{-1})) = 0$$

$$\therefore \quad \Gamma(M,\ T^*(\tilde{X})_{|X}/\tilde{\sigma}_0^{n+2}) \simeq \Gamma(M,\ T^*(X)/\tilde{\omega}_0^{n+1})$$

$$\begin{array}{ccccc}
0 \longrightarrow \Gamma(M,\ T^* \otimes H) & \longrightarrow & \Gamma(M,\ T^*(X)/\tilde{\omega}_0^{n+1}) & \longrightarrow & \Gamma(M,\ X \times C/\tilde{\sigma}_1) \\
 & & \wr\| & \circlearrowleft & \wr\| \\
\Gamma(M,\ T^*(X)_{|X}/\tilde{\sigma}_0^{n+2}) & = & C^{n+2} & \xrightarrow{\eta} & C[z]_1
\end{array}$$

$$\eta : C^{n+2} \longrightarrow C[z]_1, \qquad (c_0, \cdots, c_{n+1}) \longmapsto c_0 z_0 + \cdots + c_{n+1} z_{n+1},$$

$$z = (z_0, \cdots, z_{n+1})$$

$$\therefore \quad \Gamma(M, T^* \otimes H) \simeq \operatorname{Ker} \eta = 0$$

Again, we cannot have a short exact sequence

$$0 \longrightarrow E \longrightarrow T \longrightarrow H \longrightarrow 0 \text{ over } M \text{ for } n \geq 3.$$

Hence we get the result (1) of our main theorem.

Now consider the case $n = 2$.

(1) $d = 2$, $M \simeq P^1 \times P^1$

Let $\pi_i : P^1 \times P^1 \rightarrow P^1$ be the projection onto ith factor, $i = 1, 2$. $H^2(M, Z) \simeq Z \oplus Z$ with generators $\pi_1^* a$ and $\pi_2^* a$, $a = c_1(H) \in H^2(P^1, Z)$, $\pi_i^* a \cup \pi_i^* a = 0$ for $i = 1, 2$ and $\langle \pi_1^* a \cup \pi_2^* a, \mu_{P^1 \times P^1} \rangle = 1$. In this case, $c : H^1(M, \mathcal{O}^*) \rightarrow H^2(M, Z)$ is an isomorphism and the restriction of hyperplane bundle H on M is $\pi_1^* H \otimes \pi_2^* H$. The tangent bundle of $P^1 \times P^1$ is isomorphic to $\pi_1^* T \oplus \pi_2^* T$, where T is the tangent bundle of P^1.

$$(\sharp) \qquad 0 \longrightarrow \pi_i^* T \longrightarrow \pi_1^* T \oplus \pi_2^* T \longrightarrow \pi_j^* T \longrightarrow 0$$

is exact over M, for $(i, j) = (1, 2)$ or $(2, 1)$.

If we have a short exact sequence over M

$$0 \longrightarrow L \longrightarrow T(M) \longrightarrow L' \longrightarrow 0,$$

L, L': line bundles over M.

$$x^2 - 2tx + 2t^2 = 0,$$

$$x = c_1(L) = m(\pi_1^* a) + n(\pi_2^* a),$$

$$t = c_1(\pi_1^* H \otimes \pi_2^* H) \in H^2(M, Z),$$

$$mn - m - n + 2 = 0, \qquad (m-1)(n-1) + 1 = 0,$$

$$(m, n) = (2, 0) \text{ or } (0, 2).$$

That is $L = \pi_1^* T$, $L' = \pi_2^* T$ or $L = \pi_2^* T$, $L' = \pi_1^* T$.

Now, if we have a short exact sequence

$$(\sharp\sharp) \qquad 0 \longrightarrow \pi_1^* T \xrightarrow{\alpha} \pi_1^* T \oplus \pi_2^* T \xrightarrow{\beta} \pi_2^* T \longrightarrow 0 \text{ over } M.$$

We want to describe α and β. Since $\Gamma(M, M\times C)=C$ and $\Gamma(M, \pi_1^* H^{-2}\otimes\pi_2^* H^2)=0$, we have

$$\begin{aligned}\mathrm{Hom}\,(\pi_1^* T, \pi_1^* T\oplus\pi_2^* T) &= \Gamma(M, (M\times C)\oplus(\pi_1^* H^{-2}\otimes\pi_2^* H^2)) \\ &\simeq \Gamma(M, M\times C)\oplus\Gamma(M, \pi_1^* H^{-2}\otimes\pi_2^* H^2) \\ &= \Gamma(M, M\times C)=C .\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}\,(\pi_1^* T\oplus\pi_2^* T, \pi_2^* T) &= \Gamma(M, \pi_1^* H^{-2}\otimes\pi_2^* H^2)\oplus\Gamma(M, M\times C) \\ &\simeq \Gamma(M, M\times C)=C .\end{aligned}$$

Hence $\alpha:(x, m)\mapsto(x, (am, 0))$ and $\beta:(x, (m, n))\mapsto(x, bn)$ for some $a, b\in C-\{0\}$.

$$\begin{aligned}\mathrm{Hom}\,(T(M), T(M)) &= \Gamma(M, (\pi_1^* H^{-2}\oplus\pi_2^* H^{-2})\otimes(\pi_1^* H^2\oplus\pi_2^* H^2)) \\ &= \Gamma(M, M\times C)\oplus\Gamma(M, \pi_2^* H^{-2}\otimes\pi_1^* H^2) \\ &\quad\oplus\Gamma(M, \pi_1^* H^{-2}\otimes\pi_2^* H^2)\oplus\Gamma(M, M\times C) \\ &= C\oplus C .\end{aligned}$$

Thus a bundle isomorphism of $T(M)=\pi_1^* H^2\oplus\pi_2^* H^2$ is given by φ

$$\varphi:\pi_1^* H^2\oplus\pi_2^* H^2\longrightarrow\pi_1^* H^2\oplus\pi_2^* H^2$$

$$(x, (m, n))\longmapsto(x, (\delta m, \gamma n))\qquad\text{for some}\quad \delta, \gamma\in C^* .$$

So (##) is obtained from (#) by composing a bundle isomorphism of $\pi_1^* H^2\oplus\pi_2^* H^2$.

Hence there are exactly two line subbundles of $T(M)$.

(2) $d=3$

M is biholomorphic to the blow up of P^2 at six points $p_1,\cdots, p_6$ satisfying 1) not all lying on a conic curve, 2) no three of them lying on a line. Let $\pi: M\to P^2$ be the monoidal transformation.

Let $E_1,\cdots, E_6$ be exceptional divisors, then $c_1([E_1]),\cdots, c_1([E_6])$ and $c_1(\pi^* H)$ are generators of $H^2(M, Z)\simeq\underbrace{Z\oplus\cdots\oplus Z}_{7}$ with the following properties:

$$[E_i]\cdot[E_i]=-1, \qquad i=1,\cdots, 6$$

$$[E_i]\cdot[E_j]=0, \qquad i\neq j$$

$$[E_i]\cdot\pi^* \boldsymbol{H}=0\,,\qquad i=1,\cdots,6$$

$$\pi^*\boldsymbol{H}\cdot\pi^*\boldsymbol{H}=1\,.$$

In this case, $c: H^1(M, \mathcal{O}^*)\to H^2(M, Z)$ is also an isomorphism and the restriction of hyperplane bundle $\boldsymbol{H}$ on M is

$$\pi^*\boldsymbol{H}^3\otimes[-E_1]\otimes\cdots\otimes[-E_6]\,.$$

If we have a short exact sequence of holomorphic vector bundles over M

$$0\longrightarrow \boldsymbol{L}\longrightarrow T(M)\longrightarrow \boldsymbol{L}'\longrightarrow 0\,,$$

$\boldsymbol{L}, \boldsymbol{L}'$: line bundles over M. Then the following relation holds

$$x^2-xt+3t^2=0\,,$$

$$x=c_1(\boldsymbol{L})\,,$$

$$t=c_1(\pi^*\boldsymbol{H}^3\otimes[-E_1]\otimes\cdots\otimes[-E_6])\in H^2(M, Z)\,,$$

$$x=\alpha c_1(\pi^*\boldsymbol{H})+\beta_1 c_1([E_1])+\cdots+\beta_6 c_1([E_6])$$
$$\alpha,\ \beta_1,\cdots,\ \beta_6\in Z\,,$$

$$t=3c_1(\pi^*\boldsymbol{H})+(-1)\,c_1([E_1])-\cdots-c_1([E_6])\,.$$

$$\therefore\quad \alpha^2-\beta_1^2-\cdots-\beta_6^2-(3\alpha+\beta_1+\cdots+\beta_6)+9=0\,,$$

$$\alpha(\alpha-3)+9=\beta_1(\beta_1+1)+\cdots+\beta_6(\beta_6+1)\,,$$

$$\therefore\quad \text{odd}=\text{even}\,,$$

which is a contradiction.

Hence there is no line subbundles in this case.

(3) For the generic hypersurface of degree $d\geq 4$, we know that the restriction map $r: H^1(\boldsymbol{P}^3, \mathcal{O}^*)\to H^1(M, \mathcal{O}^*)$ is an isomorphism, i. e., each line bundle of M is of the form $i^*\boldsymbol{H}^m$ for some $m\in Z$.

If we have a short exact sequence over M

$$0\longrightarrow \boldsymbol{L}\longrightarrow T(M)\longrightarrow \boldsymbol{L}'\longrightarrow 0\,,\ \ \boldsymbol{L}, \boldsymbol{L}'\text{: line bundles over } M.$$

So $\boldsymbol{L}=i^*\boldsymbol{H}^m$ and $\boldsymbol{L}'=i^*\boldsymbol{H}^n$ for some $m, n\in Z$, which contradicts our lemma. Hence there is no line subbundles of $T(M)$ for the generic hypersurface M.

Therefore, we complete the proof of our main theorem.

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