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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

BY

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Abstract. Inequalities obtained in the first part of this paper entitled 'Error estimates in polynomial interpolation' are u:ed to find sufficient conditions to prove the existence and uniqueness for the nonlinear boundary value problems of higher order with deviating arguments.

1. **Introduction.** In this paper we shall consider the following nth order differential equation with deviating arguments

$$x^{(n)}(t) = f(t, x(t), x(t-\theta_0(t)), x'(t), x'(t-\theta_1(t)), \cdots, x^{(q)}(t), x^{(q)}(t-\theta_q(t))) \tag{1.1}$$

where f is a real-valued continuous function defined on $[a_1, a_r] \times R^{2q+1}$ and θ_i $(i=0, 1, \cdots, q)$ are non-negative continuous functions with domain $[a_1, a_r]$.

Let, the initial function $\phi(t)$ be a $C^{(n-1)}$ function which is bounded together with its all derivatives up to order $n-1$ on $[a_0, a_1]$ where

$$a_0 = \min\left\{\min_{a_1 \le t \le a_r} (t-\theta_i(t)),\ \ i = 0, 1, \cdots, q\right\}.$$

With equation (1.1), we shall consider the following boundary conditions

$$\begin{aligned} & x(a_1) = \phi(a_1), \quad x'(a_1) = \phi'(a_1), \cdots, x^{(k_1)}(a_1) = \phi^{(k_1)}(a_1) \\ & x(a_i) = A_{1,i}, \quad x'(a_i) = A_{2,i}, \cdots, x^{(k_i)}(a_i) = A_{k_i+1,i} \quad (2 \le i \le r) \\ & a_1 < a_2 < \cdots < a_r, \quad 0 \le k_i, \quad \sum_{i=1}^{r} k_i + r = n \end{aligned} \tag{1.2}$$

and seek a function $x(t)$, satisfying (1.1), (1.2). For this, we

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shall use the inequalities obtained in [1]. We define

$$(1.3)\qquad \mathfrak{l}^*_{(1.2)}(t) = \begin{cases} \phi(t), & \text{if } a_0 \le t \le a_1 \\ \mathfrak{l}_{(1.2)}(t), & \text{if } a_1 \le t \le a_r \end{cases}$$

and

$$(1.4)\qquad g^*_{(1.2)}(t, s) = \begin{cases} g_{(1.2)}(t, s), & \text{if } a_1 \le t \le a_r \\ 0, & \text{otherwise} \end{cases}$$

where $\mathfrak{l}_{(1.2)}(t)$ is the $(n-1)$th degree polynomial satisfying the boundary conditions (1.2) and $g_{(1.2)}(t, s)$ is the Green's function for the homogeneous problem: $x^{(n)}(t) = 0$, zero boundary condition (1.2).

Several other boundary conditions as in [2] for the equation (1.1) can be considered analogously. The results obtained here generalize the known results obtained in [3]–[7] for second and third order differential equations and in a special case for nth order in [8].

2. Existence Theorem.

THEOREM 2.1. *Let $K_i > 0$, $i = 0, 1, \cdots, q$ be given real numbers and let Q be the maximum of $|f(t, u_0, u_1, \cdots, u_{2q+1})|$ on the compact set*

$$\{(t, u_0, u_1, \cdots, u_{2q+1}) : a_1 \le t \le a_r, \quad |u_{2i}|, \quad |u_{2i+1}| \le 2K_i, \quad i = 0, 1, \cdots, q\}.$$

Then, if

$$(2.1)\qquad \max_{a_0 \le t \le a_r} |\mathfrak{l}^{*(i)}_{(1.2)}(t)| \le K_i, \qquad i = 0, 1, \cdots, q$$

and

$$(2.2)\qquad (a_r - a_1) \le (K_i/Q\, C^{**}_{n,i})^{1/n-i}, \qquad i = 0, 1, \cdots, q$$

the boundary value problem (1.1), (1.2) *has a solution.*

Proof. Define the set

$$B[a_0, a_r] = \{x(t) \in C^{(q)}[a_0, a_r] : \|x^{(j)}\| \le 2K_j, \quad j = 0, 1, \cdots, q\}$$

where

$$\|x^{(j)}\| = \max_{a_0 \le t \le a_r} |x^{(j)}(t)|.$$

The set $B[a_0, a_r]$ is a closed convex subset of the Banach space $C^{(q)}[a_0, a_r]$. Now define a mapping $T: C^{(q)}[a_0, a_r] \to C^{(n-1)}[a_0, a_r] \cap C^{(n)}[a_1, a_r]$ as follows

$$
(2.3)\qquad \begin{aligned}(Tx)(t) = l^*_{(2.1)}(t) + \int_{a_1}^{a_r} g^*_{(2.1)}(t, s) f[s, x(s),\\ x(s-\theta_0(s)),\ x'(s),\ x'(s-\theta_1(s)), \cdots,\\ x^{(q)}(s),\ x^{(q)}(s-\theta_q(s))]\, ds.\end{aligned}
$$

The following properties of T may be easily established

(a) $(Tx)(t) = \phi(t)$, if $a_0 \leq t \leq a_1$

(b) $(Tx)(t)$ is n times continuously differentiable on $a_1 \leq t \leq a_r$

(c) $(Tx)^{(n)}(t) = f(t, x(t), x(t-\theta_0(t)), \cdots, x^{(q)}(t), x^{(q)}(t-\theta_q(t)))$, if $a_1 \leq t \leq a_r$

(d) $(Tx)(t) - l^*_{(2.1)}(t)$ satisfies conditions (1.1.1) (see [1])

(e) fixed points of T are solutions of the boundary value problem (1.1), (1.2)

(f) T is a continuous operator.

Now, we shall show that T maps $B[a_0, a_r]$ into itself. For this, if $a_0 \leq t \leq a_1$ then we have from property (a) and (2.1) $\|(Tx)^{(i)}\| \leq K_i$ and hence the conclusion. If $a_1 \leq t \leq a_r$ then from the property (c) we have $\|(Tx)^{(n)} - l^{(n)}_{(2.1)}\| = \|(Tx)^{(n)}\| \leq Q$, also on using theorem 2.1 [1], we find

$$
|(Tx)^{(i)}(t)| \leq K_i + Q\, C^{**}_{n,i}(a_r - a_1)^{n-i}, \qquad i = 0, 1, \cdots, q.
$$

Thus, condition (2.2) implies that T maps $B[a_0, a_r]$ into itself. It then follows from the Schauder's fixed point theorem that T has a fixed point in $B[a_0, a_r]$.

3. **Uniqueness Theorem.** Let $K_0, K_1, \cdots, K_q$ be given positive constants such that $|\phi^{(i)}(t)| \leq K_i$, $i = 0, 1, \cdots, q$ for all $a_0 \leq t \leq a_1$ and let D defined by

$$
D = \{(u_0, u_1, \cdots, u_{2q+1}) : |u_{2i}|,\quad |u_{2i+1}| \leq K_i,\quad i = 0, 1, \cdots, q\}.
$$

We shall assume that f satisfies the Lipschitz condition

$$|f(t, u_0, u_1, \cdots, u_{2q+1}) - f(t, \bar{u}_0, \bar{u}_1, \cdots, \bar{u}_{2q+1})| \le \sum_{i=0}^{2q+1} L_i |u_i - \bar{u}_i|$$

where L_i $(i = 0, 1, \cdots, 2q + 1)$ are Lipschitz constants, for all $a_1 \le t \le a_r$ and $(u_0, u_1, \cdots, u_{2q+1})$, $(\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_{2q+1}) \in D$.

THEOREM 3.1. *Let* $K_i > 0$, $i = 0, 1, \cdots, q$ *be given positive constants such that* $|\phi^{(i)}(t)| \le K_i$ *for all* $a_0 \le t \le a_1$. *Let* f *satisfy the Lipschitz condition on* $[a_1, a_r] \times D$. *Then, if*

$$\left(\sum_{i=0}^{2q+1} L_i\right) \max_{0 \le i \le q} \{C^{**}_{n,i}(a_r - a_1)^{n-i}\} < 1$$

the boundary value problem (1.1), (1.2) *has at most one solution* $x(t)$ *with* $|x^{(i)}(t)| \le K_i$, $i = 0, 1, \cdots, q$.

Proof. Define M as the set of q times continuously differentiable functions on $[a_0, a_r]$ with the norm

$$\|x\| = \max_{0 \le i \le q} \left(\max_{a_0 \le t \le a_r} |x^{(i)}(t)| \right).$$

Let us assume that there are two solution $x_1(t)$ and $x_2(t)$ of the boundary value problem (1.1), (1.2) with $|x_1^{(i)}(t)|$, $|x_2^{(i)}(t)| \le K_i$, $i = 0, 1, \cdots, q$. Then, we have

$$\begin{aligned} x_1(t) - x_2(t) = \int_{a_1}^{a_r} g^*_{(2.1)}(t, s)\,[&f(s, \; x_1(s), \; x_1(s - \theta_0(s)), \cdots, \\ & x_1^{(q)}(s), \; x_1^{(q)}(s - \theta_q(s))) \\ & - f(s, \; x_2(s), \; x_2(s - \theta_0(s)), \cdots, \\ & x_2^{(q)}(s), \; x_2^{(q)}(s - \theta_q(s)))]\, ds. \end{aligned}$$

Thus, $x_1(t) - x_2(t)$ satisfies the hypothesis of Theorem 2.1 [1] and we have on using the Lipschitz condition over D

$$\begin{aligned} &|x_1^{(k)}(t) - x_2^{(k)}(t)| \\ &\quad \le C^{**}_{n,k}(a_r - a_1)^{n-k} \Big[L_0 \max_{a_1 \le t \le a_r} |x_i(t) - x_2(t)| \\ &\qquad + L_1 \max_{a_1 \le t \le a_r} |x_i(t - \theta_0(t)) - x_2(t - \theta_0(t))| + \cdots \\ &\qquad + L_{2q+1} \max_{a_1 \le t \le a_r} |x_1^{(q)}(t - \theta_q(t)) - x_2^{(q)}(t - \theta_q(t))| \Big] \\ &\quad \le C^{**}_{n,q}(a_r - a_1)^{n-k} \left(\sum_{i=0}^{2q+1} L_i \right) \|x_1 - x_2\|, \qquad k = 0, 1, \cdots, q. \end{aligned}$$

Thus, we have

$$\|x_1 - x_2\| \leq \left(\sum_{i=0}^{2q+1} L_i\right) \max_{0\leq k\leq q} \{C_{n,k}^{**}(a_r - a_1)^{n-k}\} \|x_1 - x_2\|$$

which is same as $\|x_1 - x_2\| < \|x_1 - x_2\|$. This contradiction proves the result.

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