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BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

BY

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Abstract. Inequalities obtained in the first part of this paper entitled 'Error estimates in polynomial interpolation' are used to prove the existence and uniqueness for the nonlinear boundary value problems of higher orders, some lower estimates for the iterative scheme quasilinearization to converge are also given.

1. **Introduction.** In this paper we shall consider the following nth order differential equation

$$x^{(n)} = f(t, x, x', \cdots, x^{(q)}) \qquad (0 \leq q \leq n-1) \tag{1.1}$$

with several different boundary conditions. The function f we shall assume *continuous* on $[a_1, a_r] \times R^{q+1}$ throughout the paper without mention.

In the second section, Schauder's fixed point theorem is employed to obtain the conditions on the length of the interval to prove the existence of solutions for a given boundary value problem. For the Lipschitz class of equations contraction mapping theorem is used which provides the condition on the length of the interval in terms of the Lipschitz constants.

In the third section, for the iterative scheme: quasilinearization a lower bound on the length of the interval is given so that it converges to the solution of the original problem. This will have a great advantage in solving the problems using this method.

The importance of this paper is that, not only the known results for some particular cases are extended to nth order differential equations but also the proofs are new. The Green's function technique is used without knowing the explicit form of it, and hence we do not require the estimates

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$$a_1 \leq \max_{a_1 \leq t \leq a_r} \leq a_r \int_{a_1}^{a_r} \left| \frac{\partial^{(i)} g(t, s)}{\partial t^i} \right| ds \quad (i = 0, 1, \cdots, q)$$

which plays a major role in the proof of the previous known results.

A particular equation $(\alpha.\beta)$ of the paper [1] will be refered here as $(1.\alpha.\beta)$ also we shall follow the same notations.

2. **Existence and uniqueness.** Here we shall consider the following boundary value problems: equation (1.1) and the boundary conditions

$$x(a_i) = A_{1,i}, \quad x'(a_i) = A_{2,i}, \cdots, \quad x^{(k_i)}(a_i) = A_{k_{i+1},i} \quad (1 \leq i \leq r) \tag{2.1}$$

$$a_1 < a_2 < \cdots < a_r, \quad 0 \leq k_i, \quad \sum_{i=1}^{r} k_i + r = n$$

which are same as (1.1.1.) with non-zero conditions. Similarly we shall consider equation (1.1) with other conditions considered in [1] taking them to be not necessarily zero.

We shall dnote $\ell_{(1.\alpha.\beta)}(t)$ as $(n-1)$th degree polynomial satisfying the boundary conditions $(1.\alpha.\beta)$. The Green's function of the boundary value problem: $x^{(n)} = 0$, zero boundary conditions $(1.\alpha.\beta)$, we shall represent as $g_{(1.\alpha.\beta)}(t, s)$.

THEOREM 2.1. *Suppose that*

(i) *let* $K_i > 0$, $i = 0, 1, \cdots, q$ *be given real numbers and let* Q *be the maximum of* $|f(t, u_0, u_1, \cdots, u_q)|$ *on the compact set*

$$\{(t, u_0, u_1, \cdots, u_q): t \in I, \ |u_i| \leq 2K_i, \ i = 0, 1, \cdots, q\}$$

(ii) $\max_{t \in I} |\ell^{(i)}_{(1.\alpha.\beta)}(t)| \leq K_i, \quad i = 0, 1, \cdots, q$

where I *is the interval on which the boundary conditions* $(1.\alpha.\beta)$ *are defined, for example* $[a_1, a_r]$ *for* (2.1) *and* $[a_2, a_3]$ *for* (1.2.17). *Then, if*

(1) $$(a_r - a_1) \leq (K_i / Q C^{**}_{n,i})^{1/n-i}, \quad i = 0, 1, \cdots, q \tag{2.2}$$

the boundary value problem (1.1), (2.1) *has a solution.*

(2) $$(a_j - a_1) \leq (K_i / Q \alpha_{n,i})^{1/n-i}, \quad i = 0, 1, \cdots, q \tag{2.3}$$

the boundary value problem (1.1), (1.2.8) *or* (1.2.10) *has a solution if* $j = 2$, *also the problem* (1.1), (1.2.14) *has a solution if* $j = 3$.

(3) $$(a_{j+1}-a_j) \leq (K_i/Q\,\beta_{n,i})^{1/n-i}, \qquad i=0,1,\cdots,q \tag{2.4}$$

the boundary value problem (1.1), (1.2.16) *has a solution if* $j=1$ *also the problem* (1.1), (1.2.17) *has a solution if* $j=2$.

(4) $$(a_{j+1}-a_j) \leq (K_i/Q\,\gamma_{n,i})^{1/n-i}, \qquad i=0,1,\cdots,q \tag{2.5}$$

the boundary value problem (1.1), (1.2.20) *has a solution if* $j=1$, *also the problem* (1.1), (1.2.21) *has a solution if* $j=2$.

(5) $$(a_{j+1}-a_j) \leq (K_i/Q\,\delta_{n,i})^{1/n-i}, \qquad i=0,1,\cdots,q \tag{2.6}$$

the boundary value problem (1.1), (1.2.24) *has a solution if* $j=1$ *also the problem* (1.1), (1.2.25) *has a solution if* $j=2$.

Proof. We shall prove (1) and for other cases it will follow analogously.

The set

$$B[a_1, a_r] = \{x(t) \in C^{(q)}[a_1, a_r] : \|x^{(j)}\| \leq 2K_j, \quad j=0,1,\cdots,q\}$$

where

$$\|x^{(j)}\| = \max_{a_1 \leq t \leq a_r} |x^{(j)}(t)|$$

is a closed convex subset of the Banach space $C^{(q)}[a_1, a_r]$. The mapping $T: C^{(q)}[a_1, a_r] \to C^{(n)}[a_1, a_r]$ defined by

$$T(x)(t) = \ell_{(2.1)}(t) + \int_{a_1}^{a_r} g_{(2.1)}(t,s) f(s, x(s), \cdots, x^{(q)}(s))\, ds \tag{2.7}$$

is completely continuous. Also, $(Tx)(t) - \ell_{(2.1)}(t)$ satisfies conditions (1.1.1) and $(Tx)^{(n)}(t) - \ell^{(n)}_{(2.1)}(t) = f(t, x(t), \cdots, x^{(n-1)}(t))$, hence $\|(Tx)^{(n)} - \ell^{(n)}_{(2.1)}\| = \|(Tx)^{(n)}\| \leq Q$, over $B[a_1, a_r]$. Now, using theorem 2.1 [1], for $x(t) \in B[a_1, a_r]$ we find

$$|(Tx)^{(i)}(t) - \ell^{(i)}_{(2.1)}(t)| \leq Q\, C^{**}_{n,i}(a_r - a_1)^{n-i}$$

and hence

$$|(Tx)^{(i)}(t)| \leq \max_{a_1 \leq t \leq a_r} |\ell^{(i)}_{(2.1)}(t)| + Q\, C^{**}_{n,i}(a_r - a_1)^{n-i}$$

$$i = 0, 1, \cdots, q.$$

Thus, (ii) and (2.2) implies that T maps $B[a_1, a_r]$ into itself. It then follows from the Schauder's fixed point theorem that T has a fixed point in $B[a_1, a_r]$. The fixed point is a solution of (1.1), (2.1).

Several particular cases of theorem 2.1 have been given in [2] – [9].

COROLLARY 2.2. *Assume that the function* $f(t, u_0, u_1, \cdots, u_q)$ *satisfies the following condition*

$$|f(t, u_0, u_1, \cdots, u_q)| \leq C_0 + \sum_{j=0}^{q} C_{j+1} |u_j|^{\alpha(j)} \tag{2.8}$$

where $0 < \alpha(j) < 1$ *for* $j = 0, 1, \cdots, q$. *Then each of the boundary value problems considered in theorem* 2.1 *has a solution.*

Theorem 2.1 is a local existence theorem whereas corollary 2.2 does not require any condition on the length of the interval or the boundary conditions. The question: What happens if $\alpha(j) = 1$ $(j = 0, 1, \cdots, q)$? is considered in the next theorem. We shall state and prove only for the boundary value problem (1.1), (2.1) and for all other problems discussed in theorem 2.1 the result follows analogously.

THEOREM 2.3. *Let* $f(t, u_0, u_1, \cdots, u_q)$ *satisfy the condition*

$$|f(t, u_0, u_1, \cdots, u_q)| \leq L + \sum_{j=0}^{q} L_i \, |u_j| \tag{2.9}$$

for all $(t, u_0, u_1, \cdots, u_q) \in [a_1, a_r] \times R^{q+1}$, *where* L *is any number, and let* L_j $(j = 0, 1, \cdots, q)$ *satisfy the inequality*

$$\theta = \sum_{i=0}^{q} C_{n,i}^{**} \, L_i \, (a_r - a_1)^{n-i} < 1. \tag{2.10}$$

Then, the boundary value problem (1.1), (2.1) *has at least one solution for any* $A_{i,k}$.

Proof. Let

$$\ell = \max_{a_1 \leq t \leq a_r} \sum_{i=0}^{q} L_i \, |\ell_{(2.1)}^{(i)}(t)|.$$

Define M as the set of functions n times continuously differentiable on $[a_1, a_r]$ and satisfying the boundary conditions (2.1). If we introduce in M the metric

$$\rho(x, y) = \max_{a_1 \leq t \leq a_r} |x^{(n)}(t) - y^{(n)}(t)|$$

for all $x, y \in M$, then M becomes a complete metric space. Define the mapping $T: M \to M$ as in (2.7). We shall show that the mapping T maps a sphere of radius $(L+\ell)/(1-\theta)$ of the space M into itself. Indeed, if $x \in M$ and $\rho(x, \ell_{(2.1)}) \leq (L+\ell)/(1-\theta)$, then

$$\begin{aligned}
\rho(Tx, \ell_{(2.1)}) &\leq \max_{a_1 \leq t \leq a_r} |f(t, x(t), x'(t), \cdots, x^{(q)}(t))| \\
&\leq L + \max_{a_1 \leq t \leq a_r} \sum_{i=0}^{q} L_i |(x(t) - \ell_{(2.1)}(t))^{(i)} + \ell^{(i)}_{(2.1)}(t)| \\
&\leq L + \ell + \max_{a_1 \leq t \leq a_r} \sum_{i=0}^{q} L_i |(x(t) - \ell_{(2.1)}(t))^{(i)}| \\
&\leq L + \ell + \theta \max_{a_1 \leq t \leq a_r} |x^{(n)}(t)| \\
&\leq L + \ell + \theta \frac{L+\ell}{1-\theta} = \frac{L+\ell}{1-\theta}.
\end{aligned}$$

Then, it follows by Schauder's fixed point theorem that T has at least one fixed point. The problem (1.1), (2.1) has therefore at least one solution $x(t)$ satisfying the condition

$$|x^{(n)}(t)| \leq \frac{L+\ell}{1-\theta} \quad (a_1 \leq t \leq a_r).$$

Hence, from theorem 2.1 [1], we obtain the inequalities

$$|x^{(i)}(t) - \ell^{(i)}_{(2.1)}(t)| \leq C^{**}_{n,i} \frac{L+\ell}{1-\theta} (a_r - a_1)^{n-i}$$
$$i = 0, 1, \cdots, n-1 \quad (a_1 \leq t \leq a_r).$$

DEFINITION. The function $f(t, u_0, u_1, \cdots, u_q)$ is said to be of Lipschitz class, if for all $(t, u_0, u_1, \cdots, u_q)$, $(t, v_0, v_1, \cdots, v_q) \in I \times R^{q+1}$, the following is satisfied

$$|f(t, u_0, u_1, \cdots, u_q) - f(t, v_0, v_1, \cdots, v_q)| \leq \sum_{i=0}^{q} L_i |u_i - v_i| \qquad (2.11)$$

where I is defined in (ii) of theorem 2.1.

THEOREM 2.4. *Let $f(t, u_0, u_1, \cdots, u_q)$ satisfy the Lipschitz condition* (2.11). *Then, if*

(1) $\theta < 1$, *where θ is defined in* (2.10) *the boundary value problem* (1.1), (2.1) *has a unique solution for any $A_{i,k}$.*

(2) $$\alpha_j = \sum_{i=0}^{q} \alpha_{n,i} L_i (a_j - a_1)^{n-i} < 1 \qquad (2.12)$$

the boundary value problem (1.1), (1.2.8) *or* (1.2.10) *has a unique* *solution if* $j=2$, *also the problem* (1.1), (1.2.14) *has a unique solution if* $j=3$.

(3) $$\beta_j=\sum_{i=0}^{q}\beta_{n,i}L_i(a_{j+1}-a_j)^{n-i}<1 \tag{2.13}$$

the boundary value problem (1.1), (1.2.16) *has a unique solution if* $j=1$, *also the problem* (1.1), (1.2.17) *has a unique solution if* $j=2$

(4) $$\gamma_j=\sum_{i=0}^{q}\gamma_{n,i}L_i(a_{j+1}-a_j)^{n-i}<1 \tag{2.14}$$

the boundary value problem (1.1), (1.2.20) *has a unique solution if* $j=1$, *also the problem* (1.1), (1.2.21) *has a unique solution if* $j=2$

(5) $$\delta_j=\sum_{i=0}^{q}\delta_{n,i}L_i(a_{j+1}-a_j)^{n-i}<1 \tag{2.15}$$

the boundary value problem (1.1), (1.2.24) *has a unique solution if* $j=1$, *also the problem* (1.1), (1.2.25) *has a unique solution if* $j=2$

(6) *for* $n=3$, $q=2$

$$\frac{1}{6}L_0(a_{j+1}-a_j)^3+\frac{1}{2}L_1(a_{j+1}-a_j)^2+L_2(a_{j+1}-a_j)<1 \tag{2.16}$$

the boundary value problem (1.1), (1.2.28) *has a unique solution if* $j=1$, *also the problem* (1.1), (1.2.29) *has a unique solution if* $j=2$

(7) *for* $n=3$, $q=2$

$$\frac{1}{3}L_0(a_{j+1}-a_j)^3+\frac{1}{2}L_1(a_{j+1}-a_j)^2+L_2(a_{j+1}-a_j)<1 \tag{2.17}$$

the boundary value problem (1.1), (1.2.31) *has a unique solution if* $j=1$, *also the problem* (1.1), (1.2.32) *has a unique solution if* $j=2$

(8) *for* $n=3$, $q=2$

$$\frac{\sqrt{3}}{27}L_0(a_{j+1}-a_j)^3+\frac{1}{3}L_1(a_{j+1}-a_j)^2+L_2(a_{j+1}-a_j)<1 \tag{2.18}$$

the boundary value problem (1.1), (1.2.34) *has a unique solution if* $j=1$, *also the problem* (1.1), (1.2.35) *has a unique solution if* $j=2$.

Proof. We shall prove (1) and for other cases it will follow analogously. We shall show that the mapping T defined on the metric space M in theorem 2.3 is contracting. Indeed, we find that for $x_1, x_2\in M$

$$\rho(Tx_1, Tx_2) = \max_{a_1 \le t \le a_r} |f(t, x_1(t), x_1'(t), \cdots, x_1^{(q)}(t)) - f(t, x_2'(t), x_2'(t), \cdots, x_2^{(q)}(t))|$$
$$\le \max_{a_1 \le t \le a_r} \sum_{i=0}^{q} L_i |x_1^{(i)}(t) - x_2^{(i)}(t)|$$
$$\le \theta \max_{a_1 \le t \le a_r} |x_1^{(n)}(t) - x_2^{(n)}(t)|$$
$$\le \theta \rho(x_1, x_2).$$

Thus, the mapping T in M, has one fixed point, and this is equivalent to the existence and uniqueness of the solution for the problem (1.1), (2.1).

The fact that T is in M a contraction mapping means, among other things, that under the conditions of theorem 2.4 for the existence of a unique solution of the boundary value problem under consideration, the method of successive approximations can be applied. The rate of convergence of the solution will be not less than the rate of convergence of geometric progression with common multiplier, for example θ for (1.1), (2.1).

The results of theorem 2.4 can be used to prove the existence and uniqueness results for several other boundary value problems on using 'matching' technique, see [10], [11].

THEOREM 2.5. *Let* (i) $f(t, u_0, u_1, \cdots, u_{n-1})$ $(q = n-1)$ *satisfy the Lipschitz condition* (2.11) (ii) f *satisfies the following monotonicity condition*

$$u_{n-2} < v_{n-2}, \quad (-1)^{n-j-1}(u_j - v_j) \ge 0 \quad (j = 0, 1, \cdots, n-3)$$

implies

$$f(t, u_0, u_1, \cdots, u_{n-2}, u_{n-1}) < f(t, v_0, v_1, \cdots v_{n-2}, u_{n-1}), \quad t \in (a_1, a_2]$$

and

$$u_{n-2} < v_{n-2}, \ u_j \le v_j \quad (j = 0, 1, \cdots, n-3)$$

implies

$$f(t, u_0, u_1, \cdots, u_{n-2}, u_{n-1}) < f(t, v_0, v_1, \cdots, v_{n-2}, u_{n-1}), \quad t \in [a_2, a_3)$$

(iii) $\delta_j < 1$ $(j = 1, 2)$. *Then there exists a unique solution to the boundary value problem*: (1.1) $(q = n-1)$, *satisfying*

$$\begin{aligned} x(a_1) = \lambda_1, \quad x^{(n-1)}(a_2) = \lambda_{n-1}, \quad x(a_3) = \lambda_3, \quad n \ge 3 \\ x^{(j)}(a_2) = \lambda_{j+2} \quad (j = 0, 1, \cdots, n-4), \quad n > 3. \end{aligned} \tag{2.19}$$

THEOREM 2.6. *Let the conditions* (i) *and* (ii) *of theorem* 2.5 *are satisfied. Then there exists a unique solution to the boundary value problem*: (1.1) $(q=n-1)$, *satisfying*

$$x^{(\mu)}(a_1)=\lambda_1,\quad x^{(j)}(a_2)=\lambda_{j+2}\quad (j=0,1,\cdots,n-3);\qquad x^{(\tau)}(a_3)=\lambda_n,\quad \mu,\ \tau\in\{0,1\} \tag{2.20}$$

provided $\beta_j<1$ $(j=1,2)$ *for* $(\mu=0,\ \tau=0)$ *and* $\tau_j<1$ $(j=1,2)$ *for other cases.*

The proof of these theorems follows from our theorem 2.4 and theorems 3.2 and 3.3 of [11] and generalizes their results theorem 4.1 – 4.8. In fact for a particular case $n=3$, $q=2$ even the condition (ii) in theorems 2.5 and 2.6 is not necessary and we obtain the following result.

THEOREM 2.7. *Let* $f(t, u_0, u_1, u_2)$ *satisfy the Lipschitz condition* (2.11). *Then the boundary value problem* (i), (1.1) $(n=3,\ q=2)$, *satisfying* (2.19) $(n=3)$ *has a unique solution provided the inequality* (2.17) *is satisfied for* $j=1$ *and* 2. (ii) (1.1) $(n=3,\ q=2)$, *satisfying* (2.20) $(n=3)$ *has a unique solution for* $\mu=1$, $\tau=1$ *or* $\mu=1$, $\tau=0$ *or* $\mu=0$, $\tau=1$ *provided the inequality* (2.17) *is satisfied for* $j=1$ *and* 2. (iii), (1.1) $(n=3,\ q=2)$, *satisfying* (2.20) $(n=3)$ *has a unique solution for* $\mu=0=\tau$ *provided the inequality* (2.18) *is satisfied for* $j=1$ *and* 2.

The proof of this theorem follows from our theorem 2.4 and theorem 2.1 and 2.2 of [11]. In [12] we have shown that not only condition (ii) required in their theorems 4.6 – 4.8 can be dispensed for $\mu=0=\tau$ in (2, 20) $(n=3)$ but also the inequality (2.18) can be improved to

$$\frac{9}{160}L_0(a_{j+1}-a_j)^3+\frac{1}{6}L_1(a_{j+1}-a_j)^2+\frac{1}{2}L_2(a_{j+1}-a_j)<1.$$

One can use the same technique as in [12] to improve the results for other cases also. It will be interesting to find whether condition (ii) in theorems 2.5 and 2.6 can be weakened as it is not required for $n=3$ to find the estimates on the length of the interval.

In general, if the function $f(t, x, x', \cdots, x^{(q)})$ satisfies the Lipschitz condition over a compact region, then the boundary value problem under consideration may not have a unique solution.

For example, $x'' + \exp(x) = 0$, $x(0) = 0 = x(1)$ has exactly two solutions [2]. In the next result we shall show that the function f need not satisfy Lipschitz condition on $I \times R^{q+1}$ but it is sufficient if it satisfies over a proper compact set (defined in the result). For this, we shall need the following:

LEMMA 2.8. *Let T map a ball $B = \{w : \|w - y_0\| \leq \mu\}$ of a complete normed linear space (Banach space) S into S. If there is an $\alpha \in (0, 1)$ such that for all $u, v \in B$*

$$\|Tu - Tv\| \leq \alpha \|u - v\| \tag{2.21}$$

and if

$$\|Ty_0 - y_0\| \leq \mu\ (1 - \alpha) \tag{2.22}$$

then T has a unique fixed point y in B. If T maps the Ball B into itself, then the condition (2.22) *can be omitted.*

In the following theorem, without loss of generality we shall onsider only the zero boundary conditions i.e. all the constants $A_{i,k}$ appearing in (2.1) are zero. We shall prove the result only for the boundary value problem (1.1), (2.1) and for other problems it follows analogously.

THEOREM 2.9. *Let the function $f(t, u_0, u_1, \cdots, u_q)$ satisfy Lipschitz condition* (2.11) *on*

$$D = \left\{(t, u_0, u_1, \cdots, u_q): \quad a_1 \leq t \leq a_r, \ |u_j| \leq N \frac{C^{**}_{n,j}}{C^{**}_{n,0}\,(a_r - a_1)^j}, \quad j = 0, 1, \cdots, q\right\} \tag{2.23}$$

where N satisfies either

$$m(a_r - a_1)^n\, C^{**}_{n,0} \leq N(1 - \theta) \tag{2.24}$$

if $m = \max |f(t, 0, 0, \cdots, 0)|$ for $a_1 \leq t \leq a_r$ or merely

$$M(a_r - a_1)^n\, C^{**}_{n,0} \leq N \tag{2.25}$$

if $M = \max_D |f(t, u_0, u_1, \cdots, u_q)|$, θ is defined in (2.10). *Then, the boundary value problem* (1.1), (2.1) *has one and only one solution $x(t) \in D$.*

Proof. Let the space S consist of q times continuously differentiable functions on $[a_1, a_r]$ with the norm

$$\|x\| = \max_{0 \le j \le q} \left\{ \frac{C^{**}_{n,0} (a_r - a_1)^j}{C^{**}_{n,j}} \max_{a_1 \le t \le a_r} |x^{(j)}(t)| \right\}.$$

We shall show that the mapping $T : S \to C^{(n)} [a_1, a_r]$ satisfies the conditions of lemma 2.8, where

$$(Tx)(t) = \int_{a_1}^{a_r} g_{(2.1)}(t, s) \, f(s, x(s), \; x'(s), \cdots, x^{(q)}(s)) \, ds. \quad (2.26)$$

Let $x_0(t) \equiv 0$ and B be the ball $\{w \in S : \|w\| \le N\}$. Then, if $x_1(t), x_2(t) \in B$, we have on using theorem 2.1 [1]

$$\begin{aligned}
&|(Tx_1)^{(j)}(t) - (Tx_2)^{(j)}(t)| \\
&\le C^{**}_{n,j} (a_r - a_1)^{n-j} \times \max_{a_1 \le t \le a_r} |f(t, x_1(t), x_1'(t), \cdots, x_1^{(q)}(t)) \\
&\quad - f(t, x_2(t), x_2'(t), \cdots, x_2^{(q)}(t))| \\
&\le C^{**}_{n,j} (a_r - a_1)^{n-j} \max_{a_1 \le t \le a_r} \sum_{i=0}^{q} L_i |x_1^{(i)}(t) - x_2^{(i)}(t)| \\
&\le C^{**}_{n,j} (a_r - a_1)^{n-j} \sum_{i=0}^{q} L_i \frac{C^{**}_{n,i}}{C^{**}_{n,0} (a_r - a_1)^i} \|x_1 - x_2\|
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{C^{**}_{n,0} (a_r - a_1)^j}{C^{**}_{n,j}} |(Tx_1)^{(j)} - (Tx_2)^{(j)}(t)| \\
&\le \sum_{i=0}^{q} L_i C^{**}_{n,i} (a_r - a_1)^{n-i} \|x_1 - x_2\|, \qquad j = 0, 1, \cdots, q
\end{aligned}$$

from which it follows that

$$\|Tx_1 - Tx_2\| \le \theta \|x_1 - x_2\|.$$

To apply lemma 2.8, we need to show that (2.22) holds. Let (2.24) hold, then we have

$$\begin{aligned}
|(Tx_0)^{(j)}(t)| &\le C^{**}_{n,j} (a_r - a_1)^{n-j} \max_{a_1 \le t \le a_r} |f(t, 0, 0, \cdots, 0)| \\
&\le C^{**}_{n,j} (a_r - a_1)^{n-j} m
\end{aligned}$$

or

$$\frac{C^{**}_{n,0} (a_r - a_1)^j}{C^{**}_{n,j}} |(Tx_0)^{(j)}(t)| \le (a_r - a_1)^n m C^{**}_{n,0}, \qquad j = 0, 1, \cdots, q.$$

Hence, we have

$$\|Tx_0 - x_0\| \leq N(1-\theta)\,.$$

Next, let (2.25) hold, then for any $x(t) \in B$, we have

$$|x^{(j)}(t)| \leq \frac{C^{**}_{nj,}}{C^{**}_{n,0}\,(a_r - a_1)^j}\,N$$

hence, by the hypothesis $M = \max_D |f(t, x(t), \cdots, x^{(q)}(t))|$ and it follows that

$$|(Tx)^{(j)}(t)| \leq C^{**}_{n,j}\,(a_r - a_1)^{n-j}\,M$$

or

$$\frac{C^{**}_{n,0}\,(a_r - a_1)^j}{C^{**}_{n,j}}\,|(Tx)^{(j)}(t)| \leq (a_r - a_1)^n M\; C^{**}_{n,0}, \qquad j = 0, 1, \cdots, q\,.$$

Thus

$$\|\,Tx\,\| \leq N.$$

This completes the proof of theorem 2.9.

For several particular cases of theorem 2.9 and its analogous see [2], [7], [9].

3. **Quasilinarization.** This is a practical method to construct the solution of the nonlinear problems in an iterative way, the nonlinear problem is being reduced to solving a sequence of linear problems. This method has attracted considerable attention in recent years, for example see Bellman *et. al.* [13], Lee [14]. Bernfeld *et. al.* [4], Kalaba [15], also for the systems and component-wise analysis see Agarwal [16], [17].

Here, for the equation (2.1), we define an iterative scheme as follows

$$\begin{aligned} x^{(n)}_{m+1}(t) &= f(t, x_m(t), \cdots, x^{(q)}_m(t)) \\ &\quad + C\sum_{j=0}^{q} (x^{(j)}_{m+1}(t) - x^{(j)}_m(t))\,p_j(t), \qquad (3.1) \\ &\qquad m = 0, 1, \cdots \end{aligned}$$

where

$$p_j(t) = \frac{\partial f(t, x_m(t), \cdots, x^{(q)}_m(t))}{\partial x^{(j)}_m(t)}$$

and C is any constant.

In (3.1), $x_0(t)$ is any function at least q times continuously differentiable and satisfy the boundary conditions (2.1) (we shall consider only this and for other boundary conditions it follows analogously). For each m the equation (3.1) is solved with the conditions that $x_{m+1}(t)$ satisfy (2.1). Thus, the problem (1.1), (2.1) is being reduced to solving the sequence of problems (3.1), satisfying (2.1), we shall denote this as $\{x_m(t)\}$.

It has been proved that the sequence $\{x_m(t)\}$ (even for more general boundary conditions) under certain conditions on f will actually exist and converge in a suitable space provided the length of the interval $a_r - a_1$ is sufficiently small. Here we shall give some lower bound on $a_r - a_1$.

We shall denote B as the Banach space $C^{(q)}[a_1, a_r]$ with the norm

$$\|x\| = \sum_{i=0}^{q} L_i \max_{a_1 \leq t \leq a_r} |x^{(i)}(t)| \tag{3.2}$$

and consider the closed, bounded subset B_1 of B such that $\|x - \ell_{(2.1)}\| \leq 1$. In (3.2) the constants L_j $(j = 0, 1, \cdots, q)$ are defined as follows: for $t \in [a_1, a_r]$, $x(t) \in B_1$

$$\left| \frac{\partial f(t, x(t), \cdots, x^{(q)}(t))}{\partial x^{(j)}(t)} \right| \leq L_j \qquad j = 0, 1, \cdots, q. \tag{3.3}$$

The maximum of $|f(t, x(t), \cdots, x^{(q)}(t))|$ over $[a_1, a_r] \times R^{q+1}$ we shall denote as L, also we shall define $L^* = \max(L, 1)$.

THEOREM 3.1. *Let us assume*

(i) $f(t, x(t), \cdots, x^{(q)}(t))$ *be continuous over* $[a_1, a_r] \times B_1$ *and hence bounded by* L.

(ii) $(\partial f / \partial x^{(j)}(t))(t, x(t), \cdots, x^{(q)}(t))$ *exist and are continuous for all* $j = 0, 1, \cdots, q$ *on* $[a_1, a_r] \times B_1$ *and hence bounded by* L_j. *Then, if* $k = (L^* + C)\theta/(1 - C\theta) < 1$, $(1 > C\theta)$ *where* θ *is defined in* (2.10) *the sequence generated by* (3.1) *with* $\|x_0 - \ell_{(2.1)}\| \leq 1$ *converges uniformly in* t *to the unique solution of* (1.1), (2.1) *say* $x(t)$.

A bound on the error is given by

$$\|x_m - x\| \leq k^m (1 - k)^{-1} \|x_1 - x_0\|.$$

Proof. First we shall show that the sequence $\{x_m(t)\}$ exists in B_1. We define an implicit operator T

$$Tx(t) = \ell_{(2.1)}(t) + \int_{a_1}^{a_r} g_{(2.1)}(t, s)\Big[f(s, x(s), \cdots, x^{(q)}(s)) + C\sum_{j=0}^{q} ((Tx)^{(j)}(s) - x^{(j)}(s))\, p_j(s)\Big] ds \tag{3.4}$$

whose form is patterned on the integral equation representation of (3.1).

Since $x_0(t) \in B_1$, it is sufficient to show that, if $x(t) \in B_1$ then $Tx(t) \in B_1$, i.e. T is a mapping of B_1 into B_1. From (3.4) and theorem 2.1 [1], we have

$$\begin{aligned} |(Tx)^{(k)}(t) - \ell^{(k)}_{(2.1)}(t)| &\le C^{**}_{n,k}(a_r - a_1)^{n-k} \\ &\quad \times \max_{a_1 \le t \le a_r}\Big[|f(t, x(t), \cdots, x^{(q)}(t))| \\ &\quad + C\sum_{j=0}^{q} |(Tx)^{(j)}(t) - x^{(j)}(t)|\,|p_j(t)|\Big] \\ &\le C^{**}_{n,k}(a_r - a_1)^{n-k}\,[L^* + C\{\|Tx - \ell_{(2.1)}\| \\ &\quad + \|x - \ell_{(2.1)}\|\}], \qquad k = 0, 1, \cdots, q. \end{aligned} \tag{3.5}$$

Multiplying (3.5) by L_k and summing over from $k=0$ to $k=q$, we obtain

$$\|Tx - \ell_{(2.1)}\| \le \theta\,[L^* + C + C\,\|Tx - \ell_{(2.1)}\|]$$

and hence

$$\|Tx - \ell_{(2.1)}\| \le \frac{(L^* + C)\,\theta}{1 - C\theta} = k.$$

Thus, the sequence $\{x_m(t)\}$ exists in B_1, provided $k \le 1$.

Now, we shall show that $\{x_m(t)\}$ converges: For this, we have

$$\begin{aligned} x_{m+1}(t) - x_m(t) = \int_{a_1}^{a_r} g_{(2.1)}(t, s)\Big[& f(s, x_m(s), \cdots, x_m^{(q)}(s)) \\ &+ C\sum_{j=0}^{q} (x_{m+1}^{(j)}(s) - x_m^{(j)}(s))\, p_j(t) \\ &- f(s, x_{m-1}(s), \cdots, x_{m-1}^{(q)}(s)) \\ &- C\sum_{j=0}^{q} (x_m^{(j)}(s) - x_{m-1}^{(j)}(s))\, q_j(s)\Big] ds \end{aligned}$$

where $q_j(s)$ is same as $p_j(s)$ replacing m by $m-1$.

Thus using theorem 2.1 [1], we obtain

$$\begin{aligned}|x_{m+1}^{(k)}(t)-x_m^{(k)}(t)| \leq C_{n,k}^{**}(a_r-a_1)^{n-k} \max_{a_1\leq t\leq a_r}\Big[&|f(t, x_m(t),\cdots, x_m^{(q)}(t))-f(t, x_{m-1}(t),\cdots, x_{m-1}^{(q)}(t))| \\ &+C\sum_{j=0}^{q}\{|x_{m+1}^{(j)}(t)-x_m^{(j)}(t)|\,|p_j(t)| \\ &+|x_m^{(j)}(t)-x_{m-1}^{(j)}(t)|\,|q_j(t)|\}\Big] \\ \leq C_{n,k}^{**}(a_r-a_1)^{n-k}\,[\|x_m-x_{m-1}\|&+C\|x_{m+1}-x_m\| \\ &+C\|x_m-x_{m-1}\|]\,.\end{aligned} \tag{3.6}$$

Multiplying (3.6) by L_k and summing over from $k=0$ to $k=q$ we obtain

$$\|x_{m+1}-x_m\| \leq \theta\,[\|x_m-x_{m-1}\|(L^*+C)+C\|x_{m+1}-x_m\|]\,.$$

Hence, we find

$$\|x_{m+1}-x_m\| \leq k\,\|x_m-x_{m-1}\|$$

or

$$\|x_{m+1}-x_m\| \leq k^m\,\|x_1-x_0\|\,.$$

Since, $k<1$ the sequence $\{x_m(t)\}$ converges to the solution of (1.1), (2.1), say $x(t)$.

The error bound follows from the following triangular inequality

$$\begin{aligned}\|x_{m+p}-x_m\| &\leq \|x_{m+p}-x_{m+p-1}\|+\|x_{m+p-1}-x_{m+p-2}\|+\cdots \\ &\quad+\|x_{m+1}-x_m\| \\ &\leq (k^{m+p-1}+k^{m+p-2}+\cdots+k^m)\|x_1-x_0\| \\ &\leq k^m(1-k)^{-1}\|x_1-x_0\|\end{aligned}$$

and taking $p\to\infty$.

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