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ERROR ESTIMATES IN POLYNOMIAL INTERPOLATION

BY

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Abstract. In this paper, we have obtained some inequalities for the k-th derivative $(k=0, 1, \cdots, n-1)$ of the function $x(t)$ satisfying n conditions, in terms of the nth derivative of $x(t)$ over the compact interval $[a, b]$.

1. **Introduction.** In polynomial interpolation theory, the following inequalities are well known

THEOREM 1.1. *Let* $x(t) \in C^{(n)}[a, b]$ *satisfying*

$$x(a_i) = x'(a_i) = \cdots = x^{(k_i)}(a_i) = 0; \ 1 \le i \le r$$

$$a \le a_1 < a_2 < \cdots < a_r \le b, \ 0 \le k_i \tag{1.1}$$

$$\sum_{i=1}^{r} k_i + r = n.$$

Then,

$$|x^{(k)}(t)| \le C_{n,k}\, \mu (b-a)^{n-k}, \qquad k = 0, 1, \cdots, n-1 \tag{1.2}$$

where $\mu = \max_{a \le t \le b} |x^{(n)}(t)|$, *and*

$$C_{n,k} = \frac{1}{(n-k)!}. \tag{1.3}$$

The proof follows from osculatory interpolation formula

$$x(t) = \frac{1}{n!} P(t) x^{(n)}(p)$$

where

$$P(t) = \prod_{i=1}^{r} (t-a_i)^{k_i+1}$$

and p is in (a, b), and the observation that the k-th derivative of $x(t)$ has at least $n-k$ zeros in (a, b). The constants $C_{n,k}$ in (1.2) are obviously the best possible.

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In the case of non-zero conditions in (1.1), one can consider $x(t)=y(t)-l(t)$, where $l(t)$ is the $n-1$-th degree polynomial satisfying the conditions (1.1), then the inequalities (1.2) provide error estimates for the polynomial interpolation and also the polynomial differentiation only in terms of $\max_{a\le t\le b}|y^{(n)}(t)|$.

If we consider only the segment $[a_1, a_r]$, which corresponds to interpolation in the exact sense of the word, then the inequalities (1.2) can be improved.

THEOREM 1.2. *Let* $x(t)\in C^{(n)}[a_1, a_r]$ *satisfying* (1.1). *Then*

$$|x^{(k)}(t)|\le C^*_{n,k}\, m(a_r-a_1)^{n-k} \qquad k=0,1,\cdots,n-1 \tag{1.4}$$

where $m=\max_{a_1\le t\le a_r}|x^{(n)}(t)|$, *and*

$$C^*_{n,o}=\frac{1}{n!}\ \frac{(n-1)^{n-1}}{n^n}$$
$$C^*_{n,k}=\frac{k}{n(n-k)!} \qquad k=1,2,\cdots,n-1. \tag{1.5}$$

This theorem has been proved in two different ways one using a theorem due to Krein and Milman concerning extremal points and the second using a suitable integral representation of $x(t)$. Hukuhara [1] indicates that Tumura [2] proved this result, this result has also been mentioned in [3]-[6]. The constants $C^*_{n,k}(k=0,1,\cdots,n-1)$ are the best possible, as they are exact for the functions

$$x_1(t)=(t-a_1)^{n-1}(a_r-t)$$
$$x_2(t)=(t-a_1)(a_r-t)^{n-1}$$

and only for these functions, up to a constant factor.

Naturally the constants $C^*_{n,k}$ are free from any nature of multiplicity at the points a_i, $1\le i\le r$. In this paper we shall assume $\alpha=\min(k_1, k_r)$ and obtain similar type of inequalities. It is shown that the result is best possible for $k=0$ and for $k\neq 0$ it is still an open problem, but the inequalities we have obtained are much sharper than (1.4) if $\alpha\neq 0$, and if $\alpha=0$ it reduces to theorem 1.2. Results for several other type of conditions have been also discussed.

In the second part of this paper entitled 'Boundary value problems for higher order differential equations' we have used these inequalities to prove the existence and uniqueness for the non-linear boundary value problems of higher orders, some lower estimates for the iterative scheme quasilinearization to converge are also given.

In the third part of this paper entitled 'Boundary value problems for differential equations with deviating arguments' we have used these inequalities to find sufficient conditions to prove the existence and uniqueness for the nonlinear boundary value problems of higher order with deviating arguments.

In the fourth part of this paper entitled 'Best possible length estimates for nonlinear boundary value problems' we have used shooting type of methods.

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2. **Some inequalities.**

THEOREM 2.1. *Let* $x(t) \in C^{(n)}[a_1, a_r]$ *satisfying* (1.1). *Then*

$$|x^{(k)}(t)| \le C^{**}_{n,k}\, m(a_r - a_1)^{n-k} \qquad k = 0, 1, \cdots, n-1 \tag{2.1}$$

where $m = \max_{a_1 \le t \le a_r} |x^{(n)}(t)|$, *and*

$$\begin{aligned} C^{**}_{n,k} &= \frac{1}{(n-k)!}\,\frac{(n-\alpha-1)^{n-\alpha-1}}{(n-k)^{n-k}}\,(\alpha-k+1)^{\alpha-k+1} & k &= 0, 1, \cdots, \alpha \\ C^{**}_{n,\alpha+k} &= \frac{k}{(n-\alpha)(n-\alpha-k)!} & k &= 1, 2, \cdots, n-\alpha-1. \end{aligned} \tag{2.2}$$

The constants $C^{**}_{n,k}$ *are smaller than* $C^{*}_{n,k}$.

To, prove this theorem we shall need the following:

LEMMA 2.2. *Let* $g(t, s)$ *be the Green's function for the boundary value problem*

$$x^{(n)}(t) = 0 \tag{2.3}$$

Conditions (1.1).

Then, the following identity follows

$$\int_{a_1}^{a_r} |g(t, s)|\, ds = \frac{1}{n!} \prod_{i=1}^{r} |t - a_i|^{k_i+1}. \tag{2.4}$$

Proof. The explicit form of the Green's function for the boundary value problem (2.3) is given in [7], in his (1.57). In [8]

Beesack has given a recurrence relation to obtain this, where as in [9] Das and Vatsala for $r=n$ have put this in two different forms which yield conclusions as to the sign of $g(t, s)$. But the sign of $g(t, s)$ for (2.3) is known, for this Levin [10] and Pokornyi [11] independently proved that $g(t, s)/P(t)>0$ for $a_1<s<a_r$, $a_1\le t\le a_r$, where $P(t)$ is defined earlier. For the proof one can refer to Coppel [12, p. 105–109] also.

Now, the identity (2.4) follows immediately by the simple observation

$$\int_{a_1}^{a_r} g(t, s)\, ds = \frac{1}{n!} P(t) \tag{2.5}$$

where the right side of (2.5) is the unique solution of $x^{(n)}(t)=1$ satisfying the boundary conditions (1.1).

The equality (2.4) for a particular case $r=n$ has been proved earlier in [9]. In [13] one can find several applications of (2.4).

Proof of the Theorem 2.1. First, we shall prove for $k=0,1,\cdots,\alpha$. Since k_1+1 and k_r+1 is the multiplicity of zeros at a_1 and a_r respectively we find that $x^{(k)}(t)$ will have at least $(n-k_1-k_r+k-2)$ zeros (counting with multiplicity) in (a_1, a_r) and (k_1-k+1) at a_1 and (k_r-k+1) at a_r. Define, $h(t)=x^{(k)}(t)$, then we have

$$\begin{aligned} & h(a_1)=h'(a_1)=\cdots=h^{(k_1-k)}(a_1)=0 \\ & h(a_{k,i})=0,\ i=1,2,\cdots, n-k_1-k_r+k-2=N \text{ (say)} \\ & h(a_r)=h'(a_r)=\cdots=h^{(k_r-k)}(a_r)=0 \end{aligned} \tag{2.6}$$

where $a_{k,i}$ denotes the point where $x^{(k)}(t)$ vanishes in (a_1, a_r), for two different i, $a_{k,i}$ may be same. Now, using lemma 2.2 we find

$$h(t)=\int_{a_1}^{a_r} g_k(t, s)\, h^{(n-k)}(s)\, ds$$

or

$$\begin{aligned} |h(t)| &\le \max_{a_1\le t\le a_r} |h^{(n-k)}(t)| \int_{a_1}^{a_r} |g_k(t, s)|\, ds \\ &= m\frac{1}{(n-k)!} Q(t) \end{aligned}$$

where

$$Q(t)=(t-a_1)^{k_1-k+1}(a_r-t)^{k_r-k+1}\prod_{i=1}^{N}|t-a_{k,i}|,\ k=0, 1,\cdots, \alpha$$

and $g_k(t, s)$ is the Green's function for the boundary value problem $h^{(n-k)}(t) = 0$, satisfying (2.6).

Now to prove (2.2), suppose $a_{k,j} \leq t \leq a_{k,j+1}$, $0 \leq j \leq N$, where $a_{k,0} = a_1$ and $a_{k,N+1} = a_r$, then we obtain

$$Q(t) \leq (t - a_1)^{k_1-k+j+1}(a_r - t)^{n-k_1-j-1}$$
$$\leq \begin{cases} (t - a_1)^{n-\alpha-1}(a_r - t)^{\alpha-k+1} = \phi(t), & t - a_1 \geq a_r - t \\ (t - a_1)^{\alpha-k+1}(a_r - t)^{n-\alpha-1} = \psi(t), & t - a_1 \leq a_r - t \end{cases}$$

where in obtaining $\phi(t)$, we have used $n - k_1 - j - \alpha + k - 2 \geq 0$ (since $j \leq n - k_1 - k_r - 2 + k$ and $k_r \geq \alpha$) and $\psi(t)$ follows from $k_1 - \alpha + j \geq 0$. Now an absolute maximum of $\phi(t)$ is at

$$t = a_1 + \frac{n - \alpha - 1}{n - k}(a_r - a_1)$$

and of $\psi(t)$ is at

$$t = a_1 + \frac{\alpha - k + 1}{n - k}(a_r - a_1)$$

also, an absolute maximum value of $\phi(t)$ and $\psi(t)$ is same which is

$$\frac{(n - \alpha - 1)^{n-\alpha-1}}{(n - k)^{n-k}}(\alpha - k + 1)^{\alpha-k+1} \cdot (a_r - a_1)^{n-k}$$

this proves (2.2).

To, show $C^{**}_{n,k}$ are smaller than $C^{*}_{n,k}$, we note that

$$\phi(t) \leq \frac{n - k}{n}(t - a_1)^{n-k-1}(a_r - t) + \frac{k}{n}(t - a_1)^{n-k} = \phi^*(t)$$

$$\psi(t) \leq \frac{k}{n}(a_r - t)^{n-k} + \frac{n - k}{n}(t - a_1)(a_r - t)^{n-k-1} = \psi^*(t)$$

and for $k = 0$, $\phi^*(t)$ has an absolute maximum at $t = ((n-1)a_r + a_1)/n$ and $\psi^*(t)$ has an absolute maximum at $t = ((n - 1)a_1 + a_r)/n$, also in both the cases the absolute maximum value in $((n - 1)^{n-1}/n^n)(a_r - a_1)^n$; this proves for $k = 0$. For $\alpha \geq k \geq 1$, $\phi^*(t)$ has an absolute maximum at $t = a_r$ and $\psi^*(t)$ has an absolute maximum at $t = a_1$, also in both the cases the absolute maximum value is $(k/n)(a_r - a_1)^{n-k}$.

Now, to prove for $k = \alpha + 1, \alpha + 2, \cdots, n - 1$; we observe that $x^{(\alpha)}(t)$ has one zero at a_1(or a_r), and at a_r(or a_1) may be more than one zero, also at least $n - k_1 - k_r + \alpha - 2$ in (a_1, a_r), counting with multiplicity. Now, we define $h(t)=x^{(\alpha)}(t)$, then on using theorem 1.2, we obtain

$$|h^{(k)}(t)|\le\frac{k}{(n-\alpha)(n-\alpha-k)!}m(a_r-a_1)^{n-\alpha-k},\qquad k=1,2,\cdots,n-\alpha-1$$

which proves (2.2). Also, it is easy to see that $C^*_{n,\alpha+k}>C^{**}_{n,\alpha+k}$. This completes the proof of the theorem.

The constant $C^{**}_{n,0}$ is the best possible, as this is exact for the functions

$$x_1(t)=(t-a_1)^{n-\alpha-1}(a_r-t)^{\alpha+1}$$

$$x_2(t)=(t-a_1)^{\alpha+1}(a_r-t)^{n-\alpha-1}$$

and only for these functions, up to a constant factor.

For a particular case $n=4$, $r=2$, $\alpha=1$ i.e. $x(a_1)=x'(a_1)=x(a_2)=x'(a_2)=0$, we find the following comparison between $C^*_{n,k}$ and $C^{**}_{n,k}$

	$C^*_{n,k}$	$C^{**}_{n,k}$
$k=0$	$\frac{9}{2048}$	$\frac{1}{384}$
$k=1$	$\frac{1}{24}$	$\frac{2}{81}$
$k=2$	$\frac{1}{4}$	$\frac{1}{6}$
$k=3$	$\frac{3}{4}$	$\frac{2}{3}$.

LEMMA 2.3. *The Green's function of the boundary value problem*

$$-x^{(n)}(t)=0 \tag{2.7}$$

$$x^{(i)}(a_1)=0,\quad i=0,1,\cdots,n-2;\qquad x^{(p)}(a_2)=0\quad(0\le p\le n-1) \tag{2.8}$$

and all its derivatives with respect to t up to order p are non-negative.

Proof. It can easily be verified that the Green's function of the problem (2.7), (2.8) is

$$(2.9)\quad g_1(t,s)=\frac{1}{(n-1)!}\begin{cases}(t-a_1)^{n-1}\left(\dfrac{a_2-s}{a_2-a_1}\right)^{n-p-1}-(t-s)^{n-1}, & a_1\le s\le t\le a_2\\ (t-a_1)^{n-1}\left(\dfrac{a_2-s}{a_2-a_1}\right)^{n-p-1}, & a_1\le t\le s\le a_2.\end{cases}$$

Hence $\partial^k g_1(t,s)/\partial t^k \ge 0$, $k\le p$, provided $(t-a_1)^{n-k-1}[(a_2-s)/(a_2-a_1)]^{n-p-1}\ge (t-s)^{n-k-1}$ when $a_1\le s\le t\le a_2$. Since it is true if $t=s$, we consider only $a_1\le s<t\le a_2$. Because

$$\frac{t-a_1}{t-s}\ge 1,\ \frac{a_2-a_1}{a_2-s}\ge 1$$

and

$$\frac{t-a_1}{t-s}\ge\frac{a_2-a_1}{a_2-s}$$

we have

$$\left(\frac{t-a_1}{t-s}\right)^{n-k-1}\ge\left(\frac{a_2-a_1}{a_2-s}\right)^{n-p-1}$$

and hence the result.

LEMMA 2.4. *The Green's function of the boundary value problem*

$$\begin{aligned}&-x^{(n)}(t)=0\\ &x^{(p)}(a_1)=0\qquad (0\le p\le n-1)\\ (2.10)\qquad &x^{(i)}(a_2)=0,\qquad i=0,1,\cdots,n-2\end{aligned}$$

and its kth derivative $(0\le k\le p)$ *with respect to t is non-negative if* $n+k$ *is even and non-positive if* $n+k$ *is odd.*

Proof. The Green's function of the boundary value problem (2.10) is

$$(2.11)\quad g_2(t,s)=\frac{(-1)^n}{(n-1)!}\begin{cases}(a_2-t)^{n-1}\left(\dfrac{s-a_1}{a_2-a_1}\right)^{n-p-1}; & a_1\le s\le t\le a_2\\ (a_2-t)^{n-1}\left(\dfrac{s-a_1}{a_2-a_1}\right)^{n-p-1}-(s-t)^{n-1}; & a_1\le t\le s\le a_2.\end{cases}$$

The proof is same as in Lemma 2.3.

For several applications of these lemmas see [14].

THEOREM 2.5. *Let* $x(t)\in C^{(n)}[a_1, a_2]$ *satisfying* (2.8) *or* (2.10). *Then*

$$|x^{(k)}(t)|\le \alpha_{n,k}\, m(a_2-a_1)^{n-k}, \qquad k=0,1,\cdots,n-1 \tag{2.12}$$

where $m=\max_{a_1\le t\le a_2}|x^{(n)}(t)|$, *and*

$$\alpha_{n,k}=\begin{cases}\dfrac{(n-k-1)^{n-k-1}}{(n-k)!(n-p)^{n-k}} & \text{if } n-1\ge p=k\\[2mm] \dfrac{(p-k)}{(n-p)(n-k)!} & \text{if } n-1\ge p\ge k+1\end{cases} \tag{2.13}$$

$$\alpha_{n,p+k}=\frac{(p-k)}{(n-p)(n-p-k)!} \qquad \text{if } k=1,2,\cdots,n-p-1.$$

In (2.13), *we follow* $0^0=1$.

Proof. First, we shall prove for $0\le k\le p$. The function $x(t)$ satisfying (2.8) can be written as

$$x(t)=-\int_{a_1}^{a_2} g_1(t,s)\; x^{(n)}(s)\,ds$$

where $g_1(t,s)$ is defined in (2.9), and hence using lemma 2.3, we find

$$\begin{aligned}|x^{(k)}(t)| &\le \max_{a_1\le t\le a_2}|x^{(n)}(t)|\int_{a_1}^{a_2}\frac{\partial^k g_1(t,s)}{\partial t^k}\,ds\\ &= m\,\frac{1}{(n-k-1)!}\,(t-a_1)^{n-k-1}\left[\frac{a_2-a_1}{n-p}-\frac{t-a_1}{n-k}\right]\\ &= m\,\phi_k(t)\ \text{(say)}.\end{aligned}$$

Similarly, for the function $x(t)$ satisfying (2.10), we obtain on using lemma 2.4

$$\begin{aligned}|x^{(k)}(t)| &\le m\,\frac{1}{(n-k-1)!}\,(a_2-t)^{n-k-1}\left[\frac{a_2-a_1}{n-p}-\frac{a_2-t}{n-k}\right]\\ &= m\psi_k(t)\ \text{(say)}.\end{aligned}$$

Now, the result follows from the observation that $\phi_k(t)$ attains an absolute maximum at $t=a_1+[(n-k-1)/(n-p)](a_2-a_1)$ if $k=p\le n-1$ and at $t=a_2$, if $k+1\le p\le n-1$. Also, $\psi_k(t)$ attains an absolute maximum at $t=a_2-[(n-k-1)/(n-p)](a_2-a_1)$ if $k=p\le n-1$ and at $t=a_1$ if $k+1\le p\le n-1$.

To prove for $k = p+1, p+2, \cdots, n-1$ we note that for the function $x(t)$ satisfying (2.8), $x^{(p)}(t)$ will have $n-p-1$ zeros at a_1 and 1 zero at a_2 (counting with multiplicity). Hence, if we define $h(t) = x^{(p)}(t)$, then on using theorem 1.2, we obtain

$$|h^{(k)}(t)| \leq \frac{k}{(n-p)(n-p-k)!} m(a_2 - a_1)^{n-p-k}$$

which proves the result. For the function $x(t)$ satisfying (2.10) the result follows analogously. This completes the proof.

The constants $\alpha_{n,k} (k = 0, 1, \cdots, n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (t - a_1)^{n-1}\left[\frac{a_2 - a_1}{n-p} - \frac{t - a_1}{n}\right]$$

$$x_2(t) = (a_2 - t)^{n-1}\left[\frac{a_2 - a_1}{n-p} - \frac{a_2 - t}{n}\right]$$

and only for these functions, up to a constant factor.

THEOREM 2.6. *Let* $x(t) \in C^{(n)}[a_1, a_3]$ *satisfying*

$$x^{(p)}(a_1) = 0,\ x^{(i)}(a_2) = 0,\ i = 0, 1, \cdots, n-3,\ x^{(p)}(a_3) = 0. \qquad (0 \leq p \leq n-2) \tag{2.14}$$

Then

$$|x^{(k)}(t)| \leq \alpha_{n,k}\, m(a_3 - a_1)^{n-k}, \qquad k = 0, 1, \cdots, n-1 \tag{2.15}$$

where $m = \max_{a_1 \leq t \leq a_3} |x^{(n)}(t)|$ *and* $\alpha_{n,k}$ *are same as defined in* (2.13).

Proof. For $k = p+1, p+2, \cdots, n-1$ the proof is same as in theorem 2.5 hence we shall prove only for $k = 0, 1, \cdots, p$. Following the proof of the theorem 2.1, we have

$$|x^{(p)}(t)| \leq m \frac{1}{(n-p)!} (t - a_1)|t - a_2|^{n-p-2}(a_3 - t).$$

Hence, if $t \geq a_2$, we find

$$|x^{(p)}(t)| \leq m \frac{1}{(n-p)!} [(t - a_1)^{n-p-1}(a_3 - a_1) - (t - a_1)^{n-p}].$$

Now, on using $x^{(i)}(a_2) = 0$, $i = 0, 1, \cdots, n-2$ we obtain

$$|x^{(k)}(t)| \le \Big|\underbrace{\int_{a_2}^{t}\cdots\int_{a_2}^{t}}_{p-k\text{ times}} x^{(p)}(s)\,ds\Big| \le \underbrace{\int_{a}^{t}\cdots\int_{a_2}^{t}}_{p-k\text{ times}} |x^{(p)}(s)|\,ds$$

$$\le m\frac{1}{(n-k-1)!}(t-a_1)^{n-k-1}\left[\frac{a_3-a_1}{n-p}-\frac{t-a_1}{n-k}\right]$$

and from this the result follows, for $t \ge a_2$. A similar argument can be used for $t \le a_2$ to prove this theorem completely.

THEOREM 2.7. *Let* $x(t) \in C^{(n)}[a_i, a_{i+1}](i=1, 2)$ *satisfying*

(2.16) $x(a_1)=0,\ x^{(j)}(a_2)=0,\ j=0, 1,\cdots, n-3,\ x^{(n-1)}(a_2)=0$

or

(2.17) $x^{(j)}(a_2)=0,\ j=0, 1,\cdots, n-3,\ x^{(n-1)}(a_2)=0,\ x(a_3)=0.$

Then

$$|x^{(k)}(t)| \le \beta_{n,k}\, m_i(a_{i+1}-a_i)^{n-k} \tag{2.18}$$

where $m_i=\max_{a_i\le t\le a_{i+1}}|x^{(n)}(t)|$, *and*

$$\beta_{n,0}=\frac{1}{n!}\,\frac{2}{n}\left(\frac{n-2}{n}\right)^{(n-2)/2}$$

$$\beta_{n,k}=\frac{k}{(n-1)(n-k-1)!}, \qquad k=1, 2,\cdots, n-3 \tag{2.19}$$

$$\beta_{n,n-2}=\frac{1}{2}-\frac{1}{n(n-1)},\ \beta_{n,n-1}=1.$$

Proof. We shall prove for (2.17) and for (2.16) it will follow similarly. The function $x(t)$ satisfying (2.17) can be written as

$$x(t)=\frac{1}{(n-1)!}\left[\int_{a_2}^{t}\left\{(t-s)^{n-1}-\frac{(t-a_2)^{n-2}(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}}\right\}x^{(n)}(s)\,ds\right.$$
$$\left.+\int_{t}^{a_3}-\frac{(t-a_2)^{n-2}(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}}\,x^{(n)}(s)\,ds\right].$$

Thus, we find that

$$|x(t)| \le m_2\frac{1}{n!}[(t-a_2)^{n-2}(a_3-a_2)^2-(t-a_2)^n]$$
$$\le m_2\frac{1}{n!}\,\frac{2}{n}\left(\frac{n-2}{n}\right)^{(n-2)/2}(a_3-a_2)^n.$$

Hence, the result follows for $k=0$, also for $k=n-1$ the result is immediate, so now we shall prove for $k=n-2$. In fact, we have from the above integral representation

$$|x^{(n-2)}(t)| \leq m_2\left[\frac{1}{n(n-1)}(a_3-a_2)^2 - \frac{1}{2}(t-a_2)^2\right]$$
$$\leq m_2\left[\frac{1}{2} - \frac{1}{n(n-1)}\right](a_3-a_2)^2$$

for $t \geq a_2 + (a_3-a_2)/(n-1)$ and the result follows. Also, if $t \leq a_2 + (a_3-a_2)/(n-1)$, then we find

$$|x^{(n-2)}(t)| \leq m_2\left[\frac{2}{n}(t-t_1)(a_3-t_1) - \frac{1}{n(n-1)}(a_3-a_2)^2 - (t-t_1)^2 + \frac{1}{2}(t-a_2)^2\right]$$
$$= m_2 f(t,t_1(t))(\text{say})$$

where

$$\frac{(a_3-t)^{n-1}}{(n-1)(a_3-a_2)^{n-2}} = (t-t_1)\,.$$

Now, it is easy to verify that the maximum value of $f(t, t_1(t))$ attains at $t = a_3$ and then $t_1 = a_3$, also the maximum value is $\beta_{n,n-2}(a_3-a_2)^2$.

To, prove for $k = 1, 2, \cdots, n-3$ we note that $x(t)$ satisfies $x^{(j)}(a_2) = 0$, $j = 0, 1, \cdots, n-3$; $x(a_3) = 0$, thus on using theorem 1.2. we find

$$|x^{(k)}(t)| \leq \frac{k}{(n-1)(n-k-1)!}(a_3-a_2)^{n-k-1}\max_{a_2\leq t\leq a_3}|x^{(n-1)}(t)|.$$

Since $|x^{(n-1)}(t)| \leq (a_3-a_2)m_2$, the result follows.

The constants $\beta_{n,k}(k = 0, n-2, n-1)$ are the best possible as they are exact for the functions

$$x_1(t) = (a_2-t)^{n-2}\,[(a_2-a_1)^2 - (a_2-t)^2]$$
$$x_2(t) = (a_2-t)^{n-2}[(a_3-a_2)^2 - (a_2-t)^2\,].$$

THEOREM 2.8. *Let* $x(t) \in C^{(n)}[a_i, a_{i+1}](i = 1, 2)$ *satisfying*

(2.20) $x'(a_1) = 0$, $x^{(j)}(a_2) = 0$, $j = 0, 1, \cdots, n-3$; $x^{(n-1)}(a_2) = 0$

or

(2.21) $x^{(j)}(a_2) = 0$, $j = 0,1, \cdots, n-3$, $x^{(n-1)}(a_2) = 0$, $x'(a_3) = 0$.

Then

$$|x^{(k)}(t)| \leq \gamma_{n,k} m_i (a_{i+1} - a_i)^{n-k} \tag{2.22}$$

where $m_i = \max_{a_i \leq t \leq a_{i+1}} |x^{(n)}(t)|$, *and*

$$\gamma_{n,0} = \frac{2}{(n-2)n!}, \quad \gamma_{n,1} = \frac{(n-3)^{n-3}}{(n-2)!(n-2)^{n-2}}$$

$$\gamma_{n,k} = \frac{k}{(n-2)(n-k-2)!}, \quad k = 2, 3, \cdots, n-3 \tag{2.23}$$

$$\gamma_{n,n-2} = \frac{1}{2} - \frac{1}{(n-1)(n-2)} \text{ if } n > 3 \text{ and } \frac{1}{2} \text{ if } n = 3$$

$$\gamma_{n,n-1} = 1.$$

The proof of this theorem is same as of theorem 2.7. The constants $\gamma_{n,k}$ $(k = 0, n-2, n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (a_2 - t)^{n-2}\left[(a_2 - t)^2 - \frac{n}{(n-2)}(a_2 - a_1)^2\right]$$

$$x_2(t) = (t - a_2)^{n-2}\left[(t - a_2)^2 - \frac{n}{(n-2)}(a_3 - a_2)^2\right].$$

THEOREM 2.9. *Let* $x(t) \in C^{(n)}[a_i, a_{i+1}]$ $(i = 1, 2)$ *satisfying*

$$x(a_1) = 0, \; x^{(n-2)}(a_2) = 0, \; x^{(n-1)}(a_2) = 0 \tag{2.24}$$

$$x^{(j)}(a_2) = 0 \; (j = 0, 1, \cdots, n-4) \text{ when } n > 3$$

or

$$x^{(n-1)}(a_2) = 0, \; x^{(n-2)}(a_2) = 0, \; x(a_3) = 0$$

$$x^{(j)}(a_2) = 0 \; (j = 0, 1, \cdots, n-4) \text{ when } n > 3. \tag{2.25}$$

Then

$$|x^{(k)}(t)| \leq \delta_{n,k} m_i (a_{i+1} - a_i)^{n-k} \tag{2.26}$$

where $m_i = \max_{a_i \leq t \leq a_{i+1}} |x^{(n)}(t)|$, *and*

$$\delta_{n,0} = \frac{3}{n \cdot n!}\left(\frac{n-3}{n}\right)^{(n-3)/3}$$

$$\delta_{n,k} = \frac{k}{2(n-2)(n-k-2)!}, \quad k = 1, 2, \cdots, n-3 \tag{2.27}$$

$$\delta_{n,n-2} = \frac{1}{2}, \quad \delta_{n,n-1} = 1.$$

The proof of this theorem is same as of Theorem 2.7. The constants $\delta_{n,k}(k=0,\ n-2,\ n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (a_2-t)^{n-3}[(a_2-a_1)^3-(a_2-t)^3]$$

$$x_2(t) = (t-a_2)^{n-3}[(a_3-a_2)^3-(t-a_2)^3].$$

THEOREM 2.10. *Let* $x(t) \in C^{(3)}[a_i,\ a_{i+1}](i=1,\ 2)$ *satisfying*

$$x(a_1)=0,\ x'(t_1)=0,\ x''(a_2)=0,\ t_1 \in (a_1,\ a_2] \tag{2.28}$$

or

$$x''(a_2)=0,\ x'(t_1)=0,\ x(a_3)=0,\ t_1 \in [a_2,\ a_3). \tag{2.29}$$

Then

$$|x(t)| \le \frac{1}{6} m_i(a_{i+1}-a_i)^3, |x'(t)| \le \frac{1}{2} m_i(a_{i+1}-a_i)^2, \quad |x''(t)| \le m_i(a_{i+1}-a_i) \tag{2.30}$$

where $m_i = \max_{a_i \le t \le a_{i+1}} |x'''(t)|$.

THEOREM 2.11. *Let* $x(t) \in C^{(3)}[a_i,\ a_{i+1}](i=1,\ 2)$ *satisfying*

$$x'(a_1)=0,\ x''(t_1)=0,\ x(a_2)=0,\ t_1 \in (a_1,\ a_2] \tag{2.31}$$

or

$$x(a_2)=0,\ x''(t_1)=0,\ x'(a_3)=0,\ t_1 \in [a_2,\ a_3). \tag{2.32}$$

Then

$$|x(t)| \le \frac{1}{3} m_i(a_{i+1}-a_i)^3,\ |x'(t)| \le \frac{1}{2} m_i(a_{i+1}-a_i)^2 \quad |x''(t)| \le m_i(a_{i+1}-a_i). \tag{2.33}$$

THEOREM 2.12. *Let* $x(t) \in C^{(3)}[a_i,\ a_{i+1}](i=1,\ 2)$ *satisfying*

$$x(a_1)=0,\ x''(t_1)=0,\ x(a_2)=0,\ t_1 \in (a_1,\ a_2] \tag{2.34}$$

or

$$x(a_2)=0,\ x''(t_1)=0,\ x(a_3)=0,\ t_2 \in [a_2,\ a_3). \tag{2.35}$$

Then

$$|x(t)| \le \frac{\sqrt{3}}{27} m_i(a_{i+1}-a_i)^3,\ |x'(t)| \le \frac{1}{3} m_i(a_{i+1}-a_i)^2, \quad |x''(t)| \le m_i(a_{i+1}-a_i). \tag{2.36}$$

The proof of these theorems 2.10 — 2.12 is similar as of theorem 2.7.

It is possible to find several other inequalities for the function satisfying other different conditions. Here we have given only those results which will be required in our subsequent papers on nonlinear boundary value problems which are mentioned in the introduction. We shall refer to a particular equation (α, β) of this paper in the subsequent ones as $(1\cdot\alpha\cdot\beta)$.

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