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INDEX OF A SIMPLE ADJOINT GROUP

BY

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Abstract. Let G be an algebraic group over an algebracially closed field k. For $g \in G$, define ind(g) to be the smallest positive integer q for which g^q is contained in some connected abelian algebraic subgroup of G.

If G is a connected simple adjoint group over k, $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$ the highest root of G with respect to a maximal torus T expressed in terms of a simple root system $\{\alpha_1, \cdots, \alpha_l\}$. We prove that ind($g$) is a factor of some m_j for any $g \in G$. Furthermore, in case char $k = 0$, we prove that $\{\text{ind}(g), g \in G\} = \{q | q$ is a factor some m_j $(1 \leq j \leq l)\}$.

Let G be an algebraic group over an algebraically closed field k. As proved by Goto in [1], there exists a positive integer q, such that, for any $g \in G$, g^q is contained in some connected abelian algebraic subgroup of G.

Following Goto [2], for each $g \in G$, we define the algebraic index ind*(g) of g to be the smallest positive integer $q(g)$ for which $g^{q(g)}$ lies in some connected abelian algebraic subgroup, and the algebraic index ind*(G) of G to be the least common multiple of all ind*(g) ($g \in G$). The result in [1] implies that ind*$(G) < \infty$. One question arises: For a certain G, can we find $\{\text{ind}^*(g); g \in G\}$ explicitly?

When $k = C$ the field of complex numbers, G has a Lie structure. Goto in [2] defined another concept of index: Let G be a Lie group with $\mathfrak{g}$ as its Lie algebra, exp: $\mathfrak{g} \to G$ the exponential map. For any $g \in G$, the index (of the exponential map) ind(g) of g is the smallest positive integer q for which $g^q \in \exp \mathfrak{g}$. It is easy to see that, for an algebraic group over C, the two indices coincide: ind$(g) =$ ind*(g).

On the other hand, in [4] the author proved the following: Let G be a connected complex simple adjoint group, $\mathfrak{g}$ its Lie algebra. If $m_1\alpha_1 + \cdots + m_l\alpha_l$ is the highest root of $\mathfrak{g}$ with respect

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to a Cartan subalgebra $\mathfrak{h}$ expressed in terms of a simple root system $\{\alpha_1, \cdots, \alpha_l\}$, then $\{\mathrm{ind}(g);\ g \in G\} = \{1, m_1, \cdots, m_l\}$.

In this note, we shall generalize the above result to a connected simple adjoint group G over an algebraically closed field of characteristic zero. Since we are interested only in the algebraic index, we simply call it the index, and denote it by $\mathrm{ind}(g)$ $(g \in G)$ and $\mathrm{ind}(G)$.

Throughout the note, k denotes an algebraically closed field, G a connected simple adjoint group over k with Lie algebra $\mathfrak{g}$, T a (fixed) maximal torus of G, Δ the root system of G with respect to T, $\{\alpha_1, \cdots, \alpha_l\}$ a simple root system for Δ, and $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$ the highest root in Δ. For each $\alpha \in \Delta$, let $X_\alpha = \{x_\alpha(c_\alpha);\ c_\alpha \in k\}$ be the one dimensional unipotent subgroup determined by α, $U = \Pi_{\alpha \in \Delta^+} X_\alpha$ a maximal connected unipotent subgroup, and $B = TU$ a Borel subgroup of G. For an algebraic group H, we denote by H° the connected component of H containing the identity element e. For $a \in H$ (respectively, $X \in \mathfrak{h}$, the Lie algebra of H), $Z_H(a) = \{x \in H;\ ax = xa\}$ (respectively, $Z_H(X) = \{g \in H;\ \mathrm{Ad}\ g \cdot X = X\}$) denotes the centralizer of a (respectively, X) in H, and $z_{\mathfrak{h}}(a) = \{Y \in \mathfrak{h};\ \mathrm{Ad}\ a \cdot Y = Y\}$ (respectively, $z_{\mathfrak{h}}(X) = \{Y \in \mathfrak{h};\ [X, Y] = 0\}$) the centralizer of a (respectively, X) in the Lie algebra $\mathfrak{h}$.

THEOREM 1. *Let G be a connected simple adjoint group over k. If* char$k$ *is either* 0 *or a "good" prime, then with the above notation, for any $g \in G$,* $\mathrm{ind}(g)$ *is a factor of some m_i* $(1 \le i \le l)$.

Recall that $\mathrm{char}k = p$ is a good prime if p is not a factor of any m_i. Precisely,

if G is of type A, p can be arbitrary;
if G is of type B, C, or D, $p \neq 2$;
if G is of type E_6, E_7 or G_2, $p \neq 2, 3$;
if G is of type E_8, $p \neq 2, 3, 5$.

Proof of Theorem 1. Any element $g \in G$ has a Jordan decomposition $g = su$ into semisimple part s and unipotent part u, satisfying $su = us$.

Consider $G_1 = Z_G(s)^\circ$, which is a reductive subgroup with the same rank as G. Since char k is either 0 or a good prime, we know that $u \in G_1$ (Springer-Steinberg [6, III 3.15, p. 230]). Since u is unipotent, we can find a connected abelian subgroup V of G_1 containing u.

Let C be the center of G_1, $s \in C$. As proved in [6, III 3.17 & 3.18, p. 231], there is some i, so that $s^{m_i} \in C^\circ$.

Since C° is central in G_1, $A = C^\circ V$ is a connected abelian subgroup of G_1, and s^{m_i}, $u \in A$. Hence $g^{m_i} = s^{m_i} u^{m_i} \in A$. Q.E.D

Notice that if $B = TU$ is a Borel subgroup of G containing $g = su$, so that $s \in T$, $u \in U$. Then, in the above discussion, $C^\circ \subset T$, and V can be chosen to be contained in U, so that $A \subset B$.

COROLLARY. *Let B be a Borel subgroup of G (G as in Theorem 1). For any $g \in B$, denote by* $\mathrm{ind}_B(g)$ *the index of g regarded as an element of B. Then* $\mathrm{ind}_B(g)$ *is also a factor of m_i for some i.*

***The case* char $k = 0$.** In this case, for each $j = 1, \cdots, l$, we can find an element with index exactly equal to m_j.

We first prove a lemma. Recall that an element $x \in G$ (respectively, $X \in \mathfrak{g}$) is said to be regular if its centralizer in G has minimal dimension (which equals to rank G).

LEMMA. *Let G_1 be a connected semisimple subgroup of G. Assume that* rank G_1 = rank G. *If u is a unipotent element regular in G_1, then $Z_G(u)^\circ$ consists only of unipotent elements.*

Proof. Any algebraic group has a faithful linear representation, so we may assume that G is a closed subgroup of some $GL(n, k)$. In such a case, for $g \in G$, the linear map Ad g: $\mathfrak{g} \to \mathfrak{g}$ is nothing but the conjugation by g: Ad $g(X) = gXg^{-1} (X \in \mathfrak{g})$ if we regard G and $\mathfrak{g}$ as subsets of $\mathfrak{gl}(n, k)$ (Humphreys [3, Proposition III. 10. 3, p. 73]).

Since we are assuming char$k = 0$, the maps exp and log set up a bijective correspondence between the totality of nilpotent elements in $\mathfrak{g}$ and the totality of unipotent elements in G:

$$\exp N = I + N + \frac{N^2}{2!} + \cdots + \frac{N^{n-1}}{(n-1)!}$$

if N is nilpotent,

$$\log u = (u-I) - \frac{(u-I)^2}{2} + \cdots + (-1)^n \frac{(u-I)^{n-1}}{n-1}$$

if u is unipotent, and $\exp(\log u) = u$, $\log(\exp N) = N$.

Given any unipotent element u in G, let $N = \log u$. Consider the centralizers $Z_G(u) = \{g \in G;\ gug^{-1} = u\}$ and $Z_G(N) = \{g \in G;\ \mathrm{Ad}\, g \cdot N = N\} = \{g \in G;\ gNg^{-1} = N \text{ in } gl(n, k)\}$. Since N can be expressed as a polynomial in u, and vice versa, it follows that $Z_G(u) = Z_G(N)$. In particular, u is regular in G if and only if N is regular in $\mathfrak{k}$.

Now, let u be a unipotent element which is regular in G_1, so that $N = \log u$ is regular in $\mathfrak{g}_1$. To prove that $Z_G(u)^\circ = Z_G(N)^\circ$ consists only of unipotent elements, it suffices to prove that its Lie algebra is composed of nilpotent elements.

The Lie algebra of $Z_G(N)$ is easily seen to be $z_{\mathfrak{g}}(N) = \{X \in \mathfrak{g};\ [X, N] = 0\}$, the centralizer of N in $\mathfrak{g}$. Since N is regular in $\mathfrak{g}_1$, and $\operatorname{rank} \mathfrak{g}_1 = \operatorname{rank} \mathfrak{g}$, Lemma C in [5] (which is also true in case $\operatorname{char} k = 0$) shows that $z_{\mathfrak{g}}(N)$ consists only of nilpotent elements. Therefore $Z_G(u)^\circ = Z_G(N)^\circ$ consists only of unipotent elements. Q.E.D.

REMARK. The conclusion of the lemma (and its proof) holds for any connected semisimple algebraic group over an algebraically closed field k of characteristic 0.

Now, fix j for which $m_j > 1$. Let ω be a primitive m_jth root of 1 in k.

Let $x_0 \in T$ be chosen so that $\alpha_i(x_0) = 1$ $(1 \le i \le l,\ i \ne j)$ and $\alpha_j(x_0) = \omega$. Consider $G_1 = Z_G(x_0)^\circ$, which is generated by T together with those X_α's with α satisfying $\alpha(x_0) = 1$. Then $X_{\alpha_i} \subset G_1$ $(1 \le i \le l,\ i \ne j)$ and $X_{\alpha_0} \subset G_1$. Hence G_1 is a semisimple subgroup of G with rank l $(= \operatorname{rank} G)$, and x_0 is a central element in G_1, the order of x_0 is exactly m_j (because $\alpha_j(x_0) = \omega$ and ω is a primitive m_jth root of 1). Let u be a regular unipotent element of G_1 (which exists, see e.g. Springer-Steinberg [6, III 1.8, p. 217]), $g = x_0 u$. Since $g^{m_j} = x_0^{m_j} u^{m_j} = u^{m_j}$, $\operatorname{ind}(g)$ is clearly a factor of m_j. We want to show that $\operatorname{ind}(g) = m_j$.

Let q be a positive integer. Assume that $g^q = x_0^q u^q$ lies in some connected abelian subgroup A of G. Then $A \subset Z_G(g^q) \subset Z_G(u^q)$, so that $A \subset Z_G(u^q)^\circ = Z_G(qN)^\circ = Z_G(N)^\circ = Z_G(u)^\circ$, where $N = \log u$. (Since $\mathrm{Ad}\, x \cdot (qN) = q(\mathrm{Ad}\, x \cdot N)$, and $\mathrm{char} k = 0$, so $x \in Z_G(qN)$ if and only if $x \in Z_G(N)$). But u is a unipotent element regular in G_1, by the lemma preceding Theorem 2, $Z_G(u)^\circ$ consists only of unipotent elements. This implies that A consists only of unipotent elements. Since $x_0^q \in A$ is semisimple, we must have $x_0^q = e$. This can happen only when q is a multiple of m_j. Thus $\mathrm{ind}(g) = m_j$.

So we have proved: For each j $(1 \leq j \leq l)$, there is an element $g \in G$ with $\mathrm{ind}(g)$ exactly equal to m_j.

This together with Theorem 1 imply the following:

THEOREM 2. *Let G be a connected simple adjoint group over k. Retain the notation used in Theorem 1. Then* $\{\mathrm{ind}(g);\ g \in G\} = \{1, m_1, \cdots, m_l\}$. *In particular,* $\mathrm{ind}(G) = \mathrm{l.\,c.\,m.}\ \{m_i;\ 1 \leq i \leq l\}$.

Let $B = TU$ be a Borel subgroup of G as before. For any element $g \in B$, we denote by $\mathrm{ind}_G(g)$, $\mathrm{ind}_B(g)$ the indices of g when regarded as an element of G and B respectively.

In the above discussion, we may choose u from $U \cap G_1$, so that $g \in B_1 = U \cap G_1$. Then we have $\mathrm{ind}_B(g) = m_j = \mathrm{ind}_G(g)$.

COROLLARY. *Let G be as above, B a Borel subgroup of G. Then* $\{\mathrm{ind}_B(g);\ g \in B\} = \{\mathrm{ind}_G(g);\ g \in B\} = \{\mathrm{ind}_G(g);\ g \in G\}$. *In particular,* $\mathrm{ind}(B) = \mathrm{ind}(G)$.

REMARK. If $\mathrm{char} k = p > 0$ is a good prime. A primitive m_jth root ω of 1 still exists in k. Choose $x_0 \in T$ and $u \in Z_G(x_0)^\circ = G_1$ as in the proof of Theorem 2. A similar argument can show that $\mathrm{ind}(g) > 1$ (which is clearly a factor of m_j). Otherwise, assume that g lies in some connected abelian subgroup A of G. Then $A \subset Z_G(g) \subset Z_G(x_0)$, so that $A \subset G_1$. This implies that $A \subset G_1 \cap Z_G(u) = Z_{G_1}(u)$ (because $Z_G(g) \subset Z_G(u)$), and $A \subset Z_{G_1}(u)^\circ$. But u is a regular unipotent element of G_1, $Z_{G_1}(u)^\circ$ consists only of unipotent elements (Steinberg [7, p. 112]). Since $x_0 \in A$ is semisimple, we must have $x_0 = e$. This contradiction shows that $\mathrm{ind}(g) > 1$.

So, in case G is of classical type or of type E_6, G_2 (m_j are all prime in these cases), we can also conclude that $\mathrm{ind}(g) = m_j$, and a statement similar to Theorem 2 holds.

But in case G is of type E_7, E_8 or F_4, m_j not necessarily prime. The above proof is no longer applicable. It is probable the same result holds, but we need some other method to prove the lemma.

BIBLIOGRAPHY

1. M. Goto, *On an integer associated with an algebraic group*, J. Math. Soc. Japan **29** (1977), 161–163.

2. ______, *Index of exponential map of a semi-algebraic group*, Amer. J. Math. **100** (1978), 837–843.

3. J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York 1975.

4. H. L. Lai, *Index of the exponential map of a center-free complex simple Lie group*, Osaka J. Math. **15** (1978), 553–560.

5. ______, *Corrections and supplements to the above*, Osaka J. Math. **17** (1980), 525–530.

6. T. A. Springer and R. Steinberg, *Conjugacy classes*, *Seminar on Algebraic Groups and Related Finite Groups*, ed. A. Borel, Lecture Notes in Math. #131, pp. 167–266, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

7. R. Steinberg, *Conjugacy Classes in Algebraic Groups*, Lecture Notes in Math. #366, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

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