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OSCILLATORY STRUCTURE OF THE SOLUTION SPACE OF A PAIR OF SECOND ORDER DIFFERENTIAL EQUATIONS

BY

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Abstract. This paper is a continuation of two earlier studies by the author [2, 3], and is concerned with the oscillatory structure of the solution space of a pair of second order differential equations of the form: $x'' = A(t)x + B(t)y$, $y'' = C(t)x + D(t)y$.

1. **Introduction.** This paper is a continuation of two earlier studies by the author [2, 3] concerning a pair of second order differential equations of the form

$$
\begin{aligned}
x'' &= A(t)x + B(t)y, \\
y'' &= C(t)x + D(t)y
\end{aligned}
\tag{1.1}
$$

where $A \geq 0$, $B > 0$, $C > 0$, and $D \geq 0$ are continuous in $(0, \infty)$. The above-mentioned studies are mainly concerned with oscillatory behavior of solutions of (1.1). Here, however, we are more interested in the "oscillatory structure" of the solution space of (1.1).

The oscillatory structure of the solution space of a linear differential equation has been the subject of interest in a number of recent studies. Of these studies, we mention the works by Ahmad [1], Dolan [4], Dolan and Klaasen [5, 6], Etgen and Taylor [7, 8], Gustafson [9], Gustafson and Sedziwy [10], Jones and Rankin [11] and Keener [12]. Although the questions raised in these studies are not yet in their final form, a common concern seems to be the problem of decomposing the solution space into the direct

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sum of subspaces of different (oscillatory) categories. A result of Gustsfson and Sedziwy [10] says, for instance, that the solution space of a general nth order linear scalar differential equation can be decomposed into the direct sum of two subspaces M_1 and M_2 such that (1) each solution in $M_1\backslash\{0\}$ does not vanish eventually (nonoscillatory); and (2) each solution in M_2 has infinitely many zeros (oscillatory). Such results, if proved by constructive means and/or complemented by additional analysis, are often desirable because they usually yield information about the number of linearly independent oscillatory solutions.

In this paper, we shall derive some decomposition theorems for the solution space of (1.1). The technique used here is similar to that of Dolan [4]. We first introduce a dual for the solution space of (1.1) in 2. We then study their orthogonal subspaces in 3 and show in 4 that they serve as stepping stones for decomposing the solution space of (1.1). In the final section, we shall derive some representation theorems as applications.

The reader is assumed to be familiar with [2, 3]. However, some of the important terminologies and facts developed there will be recalled in the following section. Here we briefly mention the notations which we shall adopt in the sequel. The letter J will denote the matrix

$$J=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

The transpose of a column vector z will be denoted by $\tilde{z}$. If M is any subset of a vector space V, Sp M will denote the linear span of M, and $-M$ will denote the set $\{-z|z\in M\}$. The direct sum of two spaces M, N of V will be denoted by $M\oplus N$. Finally, if $W=(z_1,\cdots,z_i,\cdots,z_n)$ is a matrix, then the matrix obtained from W by changing z_i to z will be denoted by $(z_1,\cdots,z_i\to z,\cdots,z_n)$.

2. **Preliminaries.** A nontrivial solution $z(t)=\{x(t),y(t)\}$ of (1.1) is said to be oscillatory if each of its components is oscillatory and nonoscillatory otherwise. If K is a nonempty subset of the plane, then a vector function $z(t)$ defined in $(0,\infty)$ is said to be

K-nonoscillatory if $a > 0$ exists such that $z(t)$ belongs to K for $t \geq a$. It is shown in [2, Theorem 2.9] that $z(t)$ is a nonoscillatory solution of (1.1) if and only if it is K_i-nonoscillatory for some integer i, $1 \leq i \leq 4$. Here and in the sequel, K_i denotes the ith open quadrant of the plane.

The following system [2]

$$(1.1^*) \qquad \begin{aligned} x'' &= D(t)x + B(t)y, \\ y'' &= C(t)x + A(t)y \end{aligned}$$

is called the weak adjoint system of (1.1). It is obtained from (1.1) by interchanging the coefficients $A(t)$ and $D(t)$. There are some important relationships between (1.1) and (1.1*), and the interested reader is referred to [2, 3].

Denote by S and S^* the solution spaces of (1.1) and (1.1*) respectively. Several subsets of S and S^* will be of interest in the sequel. We list those of S below, while the corresponding ones of S^* will be indicated by the superscript '*':

(i) $O = \{z \in S \mid z \text{ is oscillatory}\}$;

(ii) $N = \{z \in S \mid z \text{ is nonoscillatory}\}$;

(iii) $MD = \{z \in S \mid z \in K_1,\ z' \in K_3 \text{ on } (0, \infty)\}$; and

(iv) $MI = \{z \in S \mid z, z' \in K_1 \text{ on } [a, \infty) \text{ for some } a > 0\}$.

There are certain relationships between these subsets, the most important one being the following [2, 3]:

THEOREM 2.1. *O is nonempty if and only if* $N \subset MI \cup MD \cup (-MI) \cup (-MD)$.

THEOREM 2.2. (i) *O is nonempty if and only if O^* is nonempty; and* (ii) *MD, MI, MD^*, and MI^* are nonempty.*

Since linear combinations of oscillatory or nonoscillatory solutions need not be oscillatory respectively, the subsets O and N need not be subspaces of S. Similarly, MI and MD are not subspaces in general. Instead, if argumented by the trivial solution, they become convex cones [6, 13].

We now introduce a bilinear functional $(\cdot, \cdot)$ in $S \times S^*$. In so doing, S and S^* will form a dual pair of vector spaces [14, p. 88]. For $z = \{x, y\} \in S$ and $w = \{u, v\} \in S^*$, define

$$(2.1) \qquad (z, w) = \tilde{z}(Jw)' - \tilde{z}'(Jw) = xv' + yu' - x'v - y'u.$$

It is clear that $(\cdot, \cdot)$ is bilinear. Furthermore, by direct calculation, we see that $(z, w)' = 0$. Hence (z, w) is a constant. In case $(z, w) = 0$, we say that they are orthogonal. Since $(z, w)(t) = (z, w)(a)$, it is easy to construct orthogonal solutions $z \in S$ and $w \in S^*$. On the other hand, note that $(z, w) < 0$ for any $z \in MI$ and $w \in MD^*$. Since MI and MD^* are nonempty, there are solutions $z \in S$ and $w \in S^*$ which are not orthogonal.

3. **Orthogonal subspaces.** Let F be a subset of S. Let $F_\perp = \{w \in S^* \mid (z, w) = 0 \text{ for } z \in F\}$. Similarly, for $G \subset S^*$, let $G_\perp = \{z \in S^* \mid (z, w) = 0 \text{ for } w \in G\}$. Clearly, $F_\perp$ and $G_\perp$ are subspaces of S^* and S respectively. These subspaces will play special roles in analyzing the structures of S and S^*. For this reason, we shall discuss some of their properties in this section.

For each interger n, $1 \le n \le 4$, let $w_n = \{u_n, v_n\} \in S^*$. Let x_i and y_i, $1 \le i \le 4$, be respectively the cofactors of v_i' and u_i' in the following matrix

$$W^* = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1' & w_2' & w_3' & w_4' \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ u_1' & u_2' & u_3' & u_4' \\ v_1' & v_2' & v_3' & v_4' \end{bmatrix}.$$

Calculation of x_i', y_i', x_i'', y_i'' shows that the following hold:

(i) for each i, $z_i = \{x_i, y_i\}$ is a solution of (1.1);

(ii) if $\det W^*$ is a nonzero constant and if W denotes the matrix obtained from W^* by changing u_i, v_i, u_i', v_i', respectively, to x_i, y_i, x_i', y_i' then $\det W$ is also a nonzero constant; and

(iii)

$$\det \begin{bmatrix} w_1, \cdots, & w_i \to w, & \cdots, & w_4 \\ w_1', \cdots, & w_i' \to w', & \cdots, & w_4' \end{bmatrix} = (z_i, w). \tag{3.1}$$

As a consequence of (ii), if $\{w_1, w_2, w_3, w_4\}$ is a basis for S^*, then $\{z_1, z_2, z_3, z_4\}$ is a basis for S. The following can now be proved easily (cf. [11, Lemma 4]).

LEMMA 3.1. *Let $\{h_1, h_2, h_3, h_4\}$ be a basis for S, then there exist a basis $\{w_1, w_2, w_3, w_4\}$ for S^* and nonzero constants k_1, k_2, k_3, k_4 such that $z_i = k_i h_i$, $1 \le i \le 4$, where w_i and z_i are defined above.*

THEOREM 3.2. *Suppose* F *is a subspace of* S, *then* $\dim F_\perp = \dim S - \dim F$.

Proof. Let $\dim F = m$, where $0 \le m \le 4$. Choose a basis $\{h_1, \cdots, h_m\}$ for F. By Lemma 3.1, there exist a basis $\{w_1, w_2, w_3, w_4\}$ and nonzero constants k_1, k_2, k_3, k_4 such that $z_i = k_i h_i$, $1 \le i \le m$. It is easily verified that $F_\perp = \{h_1, \cdots, h_m\}_\perp$. Hence $F_\perp = \{w \in S^* \mid (h_i, w) = k_i(z_i, w) = 0 \text{ for } 1 \le i \le m\}$. The system

$$(z_i, w) = 0, \quad 1 \le i \le m \tag{3.2}$$

clearly has $w_{m+1}, \cdots, w_4$ as solutions. We assert that any other solution is a linear combination of them. To see this, let $w^\circ = c_1 w_1 + \cdots + c_4 w_4$ be a solution of (3.2). Substituting w° into $(z_1, w) = 0$, we get from (3.1) that

$$c_1 \det \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1' & w_2' & w_3' & w_4' \end{bmatrix} = 0 .$$

Hence $c_1 = 0$. Similarly, $c_2 = \cdots = c_m = 0$. Thus $w^\circ = c_{m+1} w_{m+1} + \cdots + c_4 w_4$ as required. Q. E. D.

There is a dual result for Theorem 3.2. Namely, if G is a subspace of S^*, then $\dim G_\perp = \dim S - \dim G$. Theorem 3.2 and its dual have several important consequences.

COROLLARY 3.3. *Suppose* F *is a subspace of* S, *then* $F_{\perp\perp} = F$.

Proof. Clearly. $F \subset F_{\perp\perp}$. Furthermore, the equalities $\dim F_{\perp\perp} = \dim S - \dim F_\perp$ and $\dim F_\perp = \dim S - \dim F$ imply $\dim F = \dim F_{\perp\perp}$. Hence $F_{\perp\perp} = F$.

COROLLARY 3.4. *If* $w_1, w_2 \in S^*$ *are linearly independent, then* $\{w_1\}_\perp \neq \{w_2\}_\perp$.

Proof. Since $\{w_i\}_\perp = \text{Sp}\ \{w_i\}_\perp$, $i = 1, 2$. If $\{w_1\}_\perp = \{w_2\}_\perp$, then

$$\text{Sp}\ \{w_1\} = (\text{Sp}\ \{w_1\}_\perp)_\perp = (\text{Sp}\ \{w_2\}_\perp)_\perp = \text{Sp}\ \{w_2\} ,$$

which is a contradiction.

COROLLARY 3.5. *Suppose* $0 \neq z \in S$. *If* $\{w_1, w_2, w_3\}$ *is a basis for* $\{z\}_\perp$, *then* $z^\circ = \{x^\circ, y^\circ\}$ *is a nonzero multiple of* z, *where*

$$x^\circ = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ u_1' & u_2' & u_3' \end{vmatrix}, \quad y^\circ = -\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1' & v_2' & v_3' \end{vmatrix} \quad \textit{and } w_i = \{u_i, v_i\}.$$

Proof. First we note that z° is the previously defined $z_4 \in S$. In view of (3.1), $(z^\circ, w_i)=0$ for $1 \leq i \leq 3$. In other words $\{w_1, w_2, w_3\}$ is a basis for $\{z^\circ\}_\perp$. The proof now follows from Corollary 3.4.

4. **Solution space decompositions.** We begin with two lemmas. The technique employed in proving these results is well known ([1, Theorem 1], [12, Theorem 3.1]), hence their proofs will only be sketched.

LEMMA 4.1. *If* $z \in O$, *then* $\{z\}_\perp$ *has an element belonging to* MD^*.

Proof. Let $z = \{x, y\}$. Let $w_i = \{u_i, v_i\}$, $1 \leq i \leq 3$, be linearly independent solutions in $\{z\}_\perp$. By Corollary 3.5, $x^\circ = kx$ where $k \neq 0$. Since $x(t)$ is oscillatory, there is an increasing sequence $\{t_i\}$ of zeros of $x(t)$ which is unbounded. Since $x^\circ(t_i) = 0$, for each i, it is easy to see that constants c_{1i}, c_{2i} and c_{3i} exist and satisfy

$$\begin{aligned} &c_{1i}\, u_1(t_i) + c_{2i}\, u_2(t_i) + c_{3i}\, u_3(t_i) = 0, \\ &c_{1i}\, v_1(t_i) + c_{2i}\, v_2(t_i) + c_{3i}\, v_3(t_i) = 0, \\ &c_{1i}\, u_1'(t_i) + c_{2i}\, u_2'(t_i) + c_{3i}\, u_3'(t_i) = 0, \\ &c_{1i}\, v_1'(t_i) + c_{2i}\, v_2'(t_i) + c_{3i}\, v_3'(t_i) < 0, \\ &c_{1i}^2 + c_{2i}^2 + c_{3i}^2 = 1. \end{aligned}$$

Let $w_i = c_{1i} w_1 + c_{2i} w_2 + c_{3i} w_3$. Then by Theorem 2.2 in [2], $w_i(t) \in K_1$ and $w_i'(t) \in K_3$ for $0 < t < t_i$. Because the sequences $\{c_{ji}\}$, $j = 1, 2, 3$, are bounded, we may assume, without loss of generality, that $\lim_{j\to\infty} c_{ji} = c_i$ and $c_1^2 + c_2^2 + c_3^2 = 1$. The function $w = c_1 u_1 + c_2 u_2 + c_3 u_3$ will then belong to MD^*.

LEMMA 4.2. *For any* $w \in MD^*$, $\{w\}_\perp$ *has an element in* MD.

Proof. For each positive integer n, we can easily choose solution h_n of (1.1) such that $h_n(n) \in K_1$, $h_n'(n) \in K_3$ and $(h_n, w) = 0$. By Theorem 2.2 in [2], $h_n(t) \in K_1$ and $h_n'(t) \in K_3$ for $0 < t < n$. Let $\{f_1, \cdots, f_4\}$ be a basis for S. There then exist constants $c_{n1}, \cdots, c_{n4}$ (not all zero) such that $h_n = \sum_{i=1}^4 c_{ni} f_i$ and $\sum_{i=1}^4 c_{ni}^2 = 1$.

Assume without loss of generality that $\lim_{n\to\infty} c_{ni} = c_i$. Let $h = \sum_{i=1}^{4} c_i f_i$. Then h will belong to MD and $(h, w) = 0$ as required.

We remark that in case $S = S^*$, then $w \in \{w\}_\perp$ as can be seen from $(w, w) = 0$.

We have stated in Theorem 2.2 that MD^* is not empty. For any $w \in MD^*$, since dim $\{w\}_\perp = 3$, and since $(f, w) < 0$ for any $f \in MI$, we have

$$S = \mathrm{Sp}\ \{f\} \oplus \{w\}_\perp .$$

The subspace $\{w\}_\perp$ can be decomposed further, if we assume that O is not empty.

THEOREM 4.3. *Suppose O is not empty. For any w in MD^*, if $z \in \{w\}_\perp$, then either $z \in MD \cup (-MD)$ or $z \in O$.*

Proof. Suppose $z \in N$, then in view of Theorem 2.1, $z \in MI \cup (-MI) \cup MD \cup (-MD)$. Since $z \in \{w\}_\perp$, $z \notin MI \cup (-MI)$. Thus $z \in MD \cup (-MD)$.

COROLLARY 4.4. *Suppose O is nonempty. Let $w \in MD^*$. For any $a > 0$ and $z = \{x, y\} \in \{w\}_\perp$, if one of the numbers $x(a)$, $y(a)$, $x'(a)$ and $y'(a)$ is zero, then z is oscillatory.*

As a consequence of Corollary 4.4, we can now construct a basis for $\{w\}_\perp$ with all oscillatory elements. To see this, note that we can easily choose three solutions $z_i = \{x_i, y_i\}$, $1 \le i \le 3$, such that $(z_i, w)(a) = 0$ and

$$x_1(a) = 0,\ y_1(a) \neq 0,\ x_1'(a) \neq 0,\ y_1'(a) \neq 0$$

$$x_2(a) \neq 0,\ y_2(a) = 0,\ x_2'(a) \neq 0,\ y_2'(a) \neq 0$$

$$x_3(a) \neq 0,\ y_3(a) \neq 0,\ x_3'(a) = 0,\ y_3'(a) \neq 0 .$$

Clearly these solutions are linearly independent. Furthermore, they are oscillatory by Corollary 4.4. We have thus shown the following.

THEOREM 4.5. *If O is nonempty, then S has three linearly independent oscillatory solutions.*

This result generalizes those of Ahmad [1, Theorem 4] and Etgen and Taylor [7, Theorem 4.4]. Similarly, we can show the following.

THEOREM 4.6. *Suppose O is nonempty. For any $w \in MD^*$ and any $a > 0$, there exist two linearly independent oscillatory solutions $z_1 = \{x_1, y_1\}$ and $z_2 = \{x_2, y_2\}$ in $\{w\}_\perp$ such that $x_1(a) = x_2(a) = x_2'(a) = 0$, $x_1'(a) \neq 0$ and $y_2(a) \neq 0$.*

Note that every nontrivial linear combination of the solutions z_1 and z_2 in Theorem 4.6 is oscillatory because its first component vanishes at a. This fact, together with Lemma 4.2, implies the following decomposition theorem.

THEOREM 4.7. *Suppose O is nonempty. Then S has a basis $\{z_1, z_2, g, f\}$ such that $z_1, z_2 \in O$, $g \in MD$ and $f \in MI$. Furthermore, any nontrivial linear combination of z_1 and z_2 is oscillatory.*

Accordingly, $S = \text{Sp}\ \{z_1, z_2\} \oplus \text{Sp}\ \{g, f\}$. Since every solution in Sp $\{g, f\}$ is nonoscillatory, this result illustrates the Theorem of Gustafson and Sedziwy mentioned in the Introduction.

We now turn to some immediate consequences of the above Theorems.

THEOREM 4.8. *Suppose O is nonempty. Let $g \in MD^*$. If $z = \{x, y\} \in S$ but $z \notin \{g\}_\perp$, then x and y are unbounded.*

Proof. Since $S = \text{Sp}\ \{f\} \oplus \{g\}_\perp$ where $f \in MI$, $z = z_1 + z_2$ where $z_1 \in \text{Sp}\ \{f\}$ and $z_2 \in \{g\}_\perp$. If $z \notin \{g\}_\perp$, then $z_1 \neq 0$. Furthermore, since $f \in MI$ and z_1 is a nonzero multiple of f, the components of z_1 are unbounded. This, together with the fact that $z_2 \in MD \cup (-MD) \cup O$ (by Theorem 4.3), implies x and y are unbounded. Q. E. D.

THEOREM 4.9. *Suppose O is nonempty. If z_1, z_2 and z_3 are any three linearly independent solutions of* (1.1), *then some linear combination of z_1, z_2 and z_3 is nonoscillatory.*

The proof of the above Theorem follows easily from Theorem 4.7, and is similar to the proof of Theorem 6 in [1].

THEOREM 4.10. *Suppose O is nonempty. Then MD^* has two linearly independent elements if and only if S has a basis with all oscillatory elements.*

Proof. Suppose $w_1, w_2 \in MD^*$ are linearly independent. Recall that $\{w_1\}_\perp$ has a basis $\{z_1, z_2, z_3\}$ with all oscillatory elements.

Furthermore, by Corollary 3.4, $\{w_2\}_\perp$ has an element z which does not belong to $\{w_1\}_\perp$. In view of Theorem 4.8, z is unbounded and hence oscillatory by Theorem 4.3. Clearly, $\{z, z_1, z_2, z_3\}$ is a basis for S. To see the converse, assume that $\{h_1,\cdots, h_4\}$ is a basis for S such that $h_i(1 \leq i \leq 4)$ is oscillatory. By Lemma 4.1, for each i, $1 \leq i \leq 4$, $\{h_i\}_\perp$ has a solution $w_i \in MD^*$. At least two of w_1, w_2, w_3 and w_4 are linearly independent. Otherwise, $\{w_1\}_\perp = \mathrm{Sp}\ \{h_1,\cdots, h_4\}$ which contradicts the fact that $\dim\{w_1\}_\perp = 3$. Q. E. D.

5. **Applications.** As further applications of our previous development, we shall extend some of the results of Keener [12]. We shall assume throughout this section that O is nonempty and that if $z=\{x, y\} \in O$, then x is unbounded.

The following theorem provides a representation for any oscillatory solution of (1.1) [12, Theorem 4.7].

THEOREM 5.1. *If $z = \{x, y\}$ is an oscillatory solution of* (1.1), *then it is a linear combination of solutions $z_1 = \{x_1, y_1\}$, $z_2 = \{x_2, y_2\}$ and $z_3 = \{x_3, y_3\}$ of* (1.1) *such that $z_3 \in MD$, $x_1(a) = x_2(a) = x_2'(a) = 0$, $x_1'(a) \neq 0$ and $y_2(a) \neq 0$.*

Proof. By Theorems 4.6 and 4.7, $z = c_1 z + \cdots + c_4 z_4$ where $z_4 = \{x_4, y_4\} \in MI$. We assert that $c_4 = 0$. To see this, note that $x - c_3 x_3 - c_1 x_1 - c_2 x_2 = c_4 x_4$. Since z and $c_1 z_1 + c_2 z_2$ are oscillatory, $x - c_1 x_1 - c_2 x_2$ is bounded by assumption. Since $z_3 \in MD$, $c_3 x_3$ is bounded. Thus the left hand side of the above equality is bounded, which contradicts the fact that x_4 is unbounded unless $c_4 = 0$. Q. E. D.

COROLLARY 5.2. *Every oscillatory solution $z = \{x, y\}$ of* (1. 1) *satisfying $x(a) = 0$ is a linear combination of z_1 and z_2 where z_1 and z_2 have been mentioned in Theorem* 5.1.

COROLLARY 5.3. *If $h_1 = \{f_1, g_1\}$ and $h_2 = \{f_2, g_2\}$ are two linearly independent oscillatory solutions of* (1.1) *such that $f_1(a) = f_2(a) = 0$ for some $a > 0$, then every linear combination of h_1 and h_2 is oscillatory or identically zero.*

COROLLARY 5.4. *If $h_1 = \{f_1, g_1\}$ and $h_2 = \{f_2, g_2\}$ are oscillatory* *solutions of* (1.1) *such that* f_1 *and* f_2 *have two zeros* (*coincident, or distinct*) *in common, then they are linearly dependent.*

Proof. Let z_1 and z_2 be as in Theorem 5.1. If $f_1(a) = f_1'(a) = f_2(a) = f_2'(a) = 0$, then by Corollary 5.2, $h_1 = cz_2$ and $h_2 = dz_2$ for some constants c and d. If $f_1(a) = f_2(a) = f_1(b) = f_2(b) = 0$ for $0 < a < b$. Then by Theorem 2.4 in [2], $f_1'(b) f_2'(b) \neq 0$. If h_1 and h_2 are linearly independent, then by Corollary 5.3, the linear combination $\{x, y\} = f_2'(b)h_1 - f_1'(b)h_2$ is oscillatory or identically zero. Note that $x(a) = x(b) = x'(b) = 0$, hence by Theorem 2.4 in [2], $\{x, y\}$ is nonoscillatory. Accordingly, $\{x, y\} \equiv 0$ so that h_1 and h_2 are linearly dependent. Q. E. D.

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