BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 8, Number 4, December 1980

ON SUPPORTS AND ADMISSIBLE TRANSLATES OF STABLE MEASURES ON HILBERT SPACES

BY

C. H. LIN AND JOHN YUAN

Abstract. We study the support $S(\mu)$ and the set $A(\mu)$ of admissible translates of a stable measure μ of index $\alpha \in [1, 2[$ on a real separable Hilbert space H. We prove that for each $y \in S(\mu)$, $A(\mu)^- = S(\mu) - y = S(M)^{\perp\perp}$ where M is the associated Levy measure of μ and $S(M)^{\perp\perp}$ is the closed subspace of H generated by the support $S(M)$ of M.

1. **Introduction.** Let μ be a probability measure (p. m.) on a real separable Hilbert space H. The support of μ is the set $S(\mu) = \{x \in H : \mu(U) > 0$ for each open neighborhood U of $x\}$. An element $a \in H$ is called an admissible translate of μ if $\mu^*\delta_a$ is absolutely continuous with respect to μ $(\mu^*\delta_a \ll \mu)$ where $*$ and δ_a denote convolution and the point-mass at a respectively. It is clear that $A(\mu)$ is a semigroup containing 0.

For an infinitely divisible p. m. μ without Gaussian component on H, there exist a unique $x_0 \in H$ and a unique measure M which is called the Levy measure of μ such that

$$\hat{\mu}(y) = \exp\left\{i(y, x_0) + \int\left(e^{i(y,x_0)} - 1 - \frac{i(y, x)}{1 + \|x\|^2}\right) dM(x)\right\} \tag{1.1}$$

where $\hat{\mu}(y) = \int e^{i(y,x)}\, d\mu(x)$ [12, Theorem 4.10, p. 181]. For convenience, we shall denote it by $\mu \sim [x_0; M]$. Recall that M is a measure on $H\backslash\{0\}$ satisfying $M(H\backslash U) < \infty$ for each Borel neighborhood U of 0 and $\int \|x\|^2/(1 + \|x\|^2)\, dM(x) < \infty$. We shall always denote $G(M) = (\bigcup_{k\geq 0}(k)S(M))^-$, where $(0)S(M) = \{0\}$, $(1)S(M) = S(M)$ and $(k+1)S(M) = (k)S(M) + S(M)$ the vectorial

Received by the editors July 6, 1978.
A. M. S. 1970 Classification: Primary 60E05, 52A05, Secondary 60B99, 46C05.
Key words and Phrases: Supports, stable measures on Hilbert spaces, Convex sets, cones, locally interior points, strong law of large number, admissible translates.

sums for all $k \geq 1$. Clearly, $G(M)$ is a closed semigroup containing 0.

A stable measure of index $\alpha \in]0, 2[$ is an infinitely divisible p.m. $\mu \sim [x_0; M]$ satisfies

$$M(A) = \int_B \int_0^\infty I_A(rs) \frac{dr}{r^{1+\alpha}} d\sigma(s) \tag{1.2}$$

where I_A is the indicator of A and σ is a finite measures on the unit ball $B = \{x \in H : \|x\| = 1\}$ satisfying

$$\sigma(W) = \alpha M\left(\left\{x \in H : \|x\| > 1, \frac{x}{\|x\|} \in W\right\}\right) ([5];\ [8]). \tag{1.3}$$

The work of this paper is to study $S(\mu)$ and $A(\mu)$ of a stable measure μ on H. For the index $\alpha = 2$, μ is purely a Gaussian measure which has been adequately discussed ([9]; [14]). For $\alpha < 1$, Brockett ([2]) has proved that $S(\mu) = a + G(M)$ for some $a \in H$. For $\alpha > 1$, De Acosta [1] has shown that $S(\mu) = m(\mu) + T$ where $m(\mu)$ is the mean of μ and T is the closed truncated cone (i.e. T is a convex semigroup satisfying $aT \subseteq T$ for all $a \geq 1$). For $\alpha > 1$, Gikhman and Skorokhod [5] have indicated a sufficient condition for $a \in A(\mu)$. However, we shall prove that for $\alpha \geq 1$, $A(\mu)^- = S(\mu) - y = S(M)^{\perp\perp} = (G(M) - G(M))^-$ for each $y \in S(\mu)$, where $S(M)^{\perp\perp}$ is the closed subspace generated by $S(M)$.

2. **Supports of stable measures.** The main result of this section is Theorem 1 which indicates that for $\mu \sim [y; M]$ of index $\alpha \in [1, 2[$, $S(\mu) = y + (G(M) - G(M))^-$.

DEFINITION. A set $T \subseteq H$ is called convex if for any $x, y \in T$, $ax + (1-a)y \in T$ for all $0 \leq a \leq 1$. A convex set T is called a cone if $aT \subseteq T$ for all $a > 0$.

DEFINITION. For a set $T \subseteq H$, the point $t \in T$ is called a locally interior point of T if 0 is a interior point of $T - t$ in the relative topology of the closed subspace V generated by $T - t$. In other words, there is an $\varepsilon > 0$ such that $t + U_\varepsilon \subseteq T$ where $U_\varepsilon = \{y \in V : \|y\| < \varepsilon\}$. For convenience, we will write int T and loc-int T the totalities of interior and locally interior points of T respectively.

REMARK. Moreover, if $0 \in T$, then T and $T - t$ for all $t \in T$ generate the same closed subspace V. In this case the set of all locally interior points of T is the interior of T in the relative topology of V.

DEFINITION. Let μ be a $p. m.$ on H. If $\int |(x, y)| \, d\mu(x) < \infty$ for all $y \in H$, and there exists a unique $x_0 \in H$ such that $(x_0, y) = \int (x, y) d\mu(x)$, then x_0 is called the mean of μ which is denoted by $x_0 = m(\mu) = \int x \, d\mu(x)$. Let X be an H-valued random variable (r. v.) whose distribution $L(X) = \mu$. Define $E(X) = m(\mu)$ provided $m(\mu)$ exists.

It is well-known that if T is a convex subset of R^n and X is an n-dimensional r. v. for which $P(X \in T) = 1$ and $E(X)$ exists, then $E(X) \in T$ (See [4, p. 74]). Our first lemma will improve and extend this result to H (even to Banach spaces).

LEMMA 1. *Let T be a closed convex set and let μ be a $p. m.$ on H with $S(\mu) \subseteq T$. Suppose $\int \|x\| d\mu(x) < \infty$. Then $m(\mu) \in T$, and $m(\mu) \in$ loc-int T provided loc-int $T \neq \emptyset$.*

Proof. It is obvious that $m(\mu)$ exists [12, p. 168]. We first prove $m(\mu) \in T$. Let X_n, $n = 1, 2, \cdots$ be independent identically distributed r. v.'s with $L(X_1) = \mu$. By assumption, $E(\|X_1\|) = \int \|x\| d\mu(x) < \infty$ and hence the strong law of large number holds for X_n (See [11]). Let $Z_n = (X_1 + X_2 + \cdots + X_n)/n$ for all $n \geq 1$. Then $P(\lim_{n\to\infty} Z_n = m(\mu)) = 1$. Since T is a closed convex set and $S(\mu) \subseteq T$, then $P(Z_n \in T) = 1$ and $P(\lim_{n\to\infty} Z_n \in T) = 1$. This implies $m(\mu) \in T$. Now, we prove that $m(\mu) \in$ loc-int T. Let V be the closed subspace generated by $T - m(\mu)$. It suffices to show that $0 \in$ int $(T - m(\mu))$ in the relative topology of V. Suppose not, then there exists a vector $v \in V$ such that $(v, y) \geq 0$ for all $y \in T - m(\mu)$ ([3, 21.17, p. 35], [10, p. 133]). Then either $\mu(\{x \in T : (v, x - m(\mu) > 0\}) > 0$ or $\mu(\{x \in T ; \ (v, x - m(\mu)) = 0\}) = 1$. Suppose the latter is true, then $T - m(\mu) \subseteq \{y \in V : (v, y) = 0\}$ (Note that $x \mapsto (v, x - m(\mu))$ is a continuous map), contradicting the fact that V is generated by $T - m(\mu)$. Therefore, $\int (v, \ x - m(\mu)) \, d\mu(x) > 0$, i.e.

$(v, m(\mu)) < \int (v, x)\, d\mu(x)$ which is impossible by the definition of mean of μ. Hence 0 is an interior point of $T - m(\mu)$ in the relative topology of V.

LEMMA 2. *Let M be a Levy measure satisfying $\int_{\|x\|\le 1} \|x\|\, dM(x) < \infty$. Suppose $G(M)$ is a cone. Then $\int x/(1 + \|x\|)\, dM(x) \in G(M)$, and $\int x/(1 + \|x\|^2)\, dM(x) \in$* loc-int *$G(M)$ provided* loc-int $G(M) \neq \emptyset$.

Proof. Define the p. m. ν as follows;

$$\nu(A) = \int_A \frac{\|x\|}{1 + \|x\|^2}\, dM(x) \Big/ \int_H \frac{\|x\|}{1 + \|x\|^2}\, dM(x)\,.$$

Then $S(\nu) \subseteq G(M)$. Let X and Y be r. v.'s such that $L(X) = \nu$ and $Y = X/\|X\|$. Therefore, $E\|Y\| = 1$ and hence $EY \in G(M)$. Since $\int_{\|x\|\le 1} \|x\|\, dM(x) < \infty$, it follows that $0 < \int \|x\|/(1 + \|x\|^2)\, dM(x) < \infty$. Hence $\int x/(1 + \|x\|^2)\, dM(x) = EY \int \|x\|/(1+\|x\|^2)\, dM(x) \in G(M)$ as $G(M)$ is a cone.

LEMMA 3. *Suppose $\mu \sim [y; M]$ is an infinitely divisible p. m. on H with $G(M)$ a cone. Define the Levy measure M_n by*

$$\frac{dM_n}{dM}(x) = \begin{cases} 1 & \text{if } \|x\| \ge 1/n\,, \\ \|x\| & \text{otherwise.} \end{cases}$$

Then $S(\mu) = y + (\bigcup_{n\ge 1} G(M) - a_n)^-$ where $a_n = \int x/(1+\|x\|^2)\, dM_n(x) \in G(M)$.

Proof. This follows from [15, Lemma 1.3] (See also [2]).

LEMMA 4. *Let $p : H \to H$ be a continuous projection and let M be a Levy measure. Then $p(M)$ defined via*

$$p(M)(B) = M(p^{-1}(B))$$

is a Levy measure on pH so that $S(p(M)) = p(S(M))^-$ and $G(p(M)) = p(G(M))^-$.

Moreover, if M satisfies

$$M(A) = \int_B \int_0^\infty I_A(rs)\, \frac{dr}{r^{1+\alpha}}\, d\sigma(s) \tag{1.2}$$

and

(1.3) $\sigma(W) = \alpha M(\{x \in H : \|x\| > 1,\ x/\|x\| \in W\})$,

then $p(M)$ *also satisfies*

(1.2) $$p(M)(\dot{A}) = \int_{pB}\int_0^\infty I_{\dot{A}}(r\dot{s})\,\frac{dr}{r^{1+\alpha}}\,dp(\sigma)(\dot{s})$$

and

(1.3) $$p(\sigma)(\dot{W}) = \alpha\, p(M)(\{\dot{x} \in pH : \|\dot{x}\| > 1,\ \dot{x}/\|\dot{x}\| \in \dot{W}\}).$$

Proof. Straightforward.

LEMMA 5. *Let* M *be a Levy measure satisfying* (1.2) *and* (1.3). *Then*

(1) $G(M)$ *is a closed cone,*

(2) $a_0 = \int_B s d\sigma(s) \in G(M)$,

(3) $p(a_0) = \int_{pB} \dot{s}\, dp(\sigma)(\dot{s}) \in$ loc-int $G(p(M))$ *for each projection* p *of finite rank.*

Proof. (1) (See [2]).

(2) As $S(\sigma) \subseteq S(M) \subseteq G(M)$ by (1.2) and (1.3), we have by Lemma 1 that $a_0 \in G(M)$.

(3) For the equality, we refer to [6, Theorem. 3.7.12, p. 83]. Without losing generality, we any assume that $H = (G(M) - G(M))^-$ and $pH \cong R^n$. Then $pH = pG(M) - pG(M) = R^n$, whence loc-int $pG(M) \neq \emptyset$. By Lemma 1, $p(a_0) \in$ loc-int $pG(M)$.

REMARK. Let $\mu \sim [y; M]$ be a stable measure. Then $S(M)^{\perp\perp} = (G(M) - G(M))^-$ where $S(M)^\perp = \{z \in H : (z, x) = 0$ for all $x \in S(M)\}$.

Now we need the following technique result to prove Theorem 1;

LEMMA 5. *Let* G *be a closed cone whose* loc-int $G \neq \emptyset$. *Assume* $a_0 \in$ loc-int G *and* $t_n \uparrow \infty$. *Then* $(\bigcup_{n\geq 1} G - t_n a_0)^- = (G-G)^- = G - G$ *and is a closed subspace of* H.

Proof. Without losing generality, we may assume that $H = (G - G)^-$, whence int $G \neq \emptyset$ and so $H = G - G$. Now, as $a_0 \in$ int G, there exists an $\varepsilon > 0$ so that $a_0 + U_\varepsilon \subseteq G$ where $U_\varepsilon = \{x \in H : \|x\| < \varepsilon\}$, whence $U_{t\varepsilon} \subseteq G - ta_0$ for all $t > 0$. Hence $G - G \subseteq \bigcup_{n\geq 1} U_{t_n a_0} \subseteq \bigcup_{n\geq 1} (G - t_n a_0) \subseteq G - G$.

THEOREM 1. *Let* $\mu \sim [y; M]$ *be a stable measure of index* $\alpha \in [1, 2[$ *on* H. *Then* $S(\mu) = V + y$ *where* V *is the closed subspace generated by* $G(M)$, *i.e.* $V = (G(M) - G(M))^{-} = S(M)^{\perp\perp}$. *Moreover,* $S(\mu) = V + x$ *for all* $x \in S(\mu)$.

Proof. Without losing generality, we may assume that $y = 0$. By Lemma 3, $S(\mu) = (\bigcup_{n\geq 1} G(M) - a_n)^{-}$ where $a_n = \int x/(1 + \|x\|^2)\, dM_n(x) \in G(M)$. Since μ is stable, there exists a finite measure σ on the unit ball B satisfying (1.2). Then

$$\begin{aligned} a_n &= \int_{\|x\|<1/n} \frac{\|x\|\ x}{1 + \|x\|^2}\, dM(x) + \int_{\|x\|\geq 1/n} \frac{x}{1 + \|x\|^2}\, dM(x) \\ &= \left(\int_0^{1/n} \frac{r^{1-\alpha}}{1 + r}\, dr + \int_{1/n}^{\infty} \frac{1}{(1 + r^2) r^{\alpha}}\, dr\right) \int_B s\, d\sigma(s) \\ &= t_n a_0 \text{ where } a_0 = \int_B s\, d\sigma(s) \text{ and } t_n \uparrow \infty. \end{aligned}$$

By Lemma 2 or Lemma 3, $a_0 \in G(M)$. Now, we let V be the closed subspace generated by $S(\mu)$.

Claim 1. $\overline{pS(\mu)} = pV \cong R^n$ for any projection p of rank n; By the linear property of p, Lemma 5 (3) and Lemma 6, we have that

$$\begin{aligned} \overline{pS(\mu)} &= p\left(\overline{\bigcup_{n\geq 1} G(M) - t_n a_0}\right) \\ &= \left(\bigcup_{n\geq 1} G(p(M)) - t_n p(a_0)\right)^{-} \\ &= (G(p(M)) - G(p(M)))^{-} \\ &= (pG(M) - pG(M))^{-} = pV \cong R^n. \end{aligned}$$

Claim 2. $S(\mu) = V$; As $S(\mu)$ is closed and convex, we have by [3, 21. 14, p. 32] that $S(\mu) = \bigcap H_i$ where H_i are closed half spaces of V. It remains to show that $H_i = V$ for all i. Suppose that $V \backslash H_i \neq \emptyset$ for some i. Then there exists an $h \in V$ and an $a \in R$ so that $H_i = \{x \in V\colon (x, h) \geq a\}$ or $\{x \in V : (x, h) \leq a\}$. Without losing generality, we may assume that $a = 0$ and $H = \{x \in V : (x, h) \geq 0\}$, whence $h/\|h\| \in H_i$. Let $g = h/\|h\|$. Then $p : H \to H$ defined via

$$p(y) = (y, g)g$$

is a projection of rank 1 so that $pH \cong [0, \infty[$ which is a contradiction as by Claim 1 that $\overline{pH_i} \supseteq \overline{pS(\mu)} \cong R$.

REMARK. Let $\mu \sim [y; M]$ be a stable measure of index α. Then for every $r < \alpha$, $\int \|x\|^r d\mu(x) < \infty$ [1, Theorem 3.2]. Then for $\alpha > 1$, $m(\mu)$ always exists and $m(\mu)$ is a locally interior point of $S(\mu)$. In particular, if μ is strictly stable [1], then $m(\mu) = 0$ and so $S(\mu) = S(\mu) + 0 = (G(M) - G(M))^- = S(M)^{\perp\perp}$.

3. **Admissible translates of stable measures.** Let $\mu \sim [y; M]$ be a stable measure of index $\alpha < 2$ and let σ be a finite measure on the unit ball B satisfying (1.2) and (1.3). In the case $\alpha > 1$, Gikhmann and Skorokhod have proved a sufficient condition for $a \in A(\mu)$ [5, Theorem 6.2] as follows;

(*) Suppose that for a given $a \in H$ there exists a sequence of non-negative measurable functions g_n so that

$$a = \lim_{n\to\infty} \int g_n(x) x \, d\sigma(x) ,$$

Then $a \in A(\mu)$.

We note that the proposition (*) is also true for the case $\alpha = 1$ (since in precisely the same manner as in the proof of Theorem 6.2 in [5], we can take ε_n and $\delta_n(x)$ such that $\delta_n(x) < \varepsilon_n$ and $\int_{\delta_n(x)}^{\varepsilon_n} dr/r = g_n(x)$ for all $x \in B$). We shall use these facts to show the following result;

THEOREM 2. *Suppose $\mu \sim [y; M]$ is a stable measure of index $\alpha \in [1, 2[$ on H. Then $A(\mu) \supseteq G(M) - G(M)$. In particluar, $A(\mu)^- = (G(M) - G(M))^- = S(\mu) - y$.*

Proof. Let $a \in S(\sigma)$ where σ satisfies (1.2) and (1.3). Then $\sigma(B \cap (U_\varepsilon + a)) > 0$ for all $\varepsilon > 0$ where $U_\varepsilon = \{x \in H : \|x\| < \varepsilon\}$. Given an $\varepsilon > 0$, let $g_\varepsilon : H \to R$ be defined via

$$g_\varepsilon(x) = \begin{cases} 1/\sigma(B \cap (U_\varepsilon + a)) & \text{if } x \in U_\varepsilon + a, \\ 0 & \text{otherwise}. \end{cases}$$

By Lemma 1, $\int_B g_\varepsilon(x) x \, d\sigma(x) \in U_\varepsilon + a$ for all $\varepsilon > 0$, whence $a \in A(\mu)$ by(*) and so $tS(\sigma) \subseteq A(\mu)$ for all $t \geq 0$. It follows from (1.2) and (1.3) that $S(M) \subseteq A(\mu)$ (note that for $a \in S(M)$, $a \neq 0$ we have $a/\|a\| \in S(\sigma)$). As $A(\mu)$ is a semigroup, $\bigcup_{n\geq 1} (n)\, S(M) \subseteq A(\mu)$. Next, we want to prove $G(M) \subseteq A(\mu)$ in the following steps;

Claim 1. Given an $a \in S(M)$ and given $n \geq 1$, there exists a sequence of non-negative measurable functions f_k such that

$$\left\|\int f_k(x) x \, d\sigma(x) - a\right\| < 1/nk .$$

Proof of the Claim 1: This follows from the fact that $a/\|a\| \in S(\sigma)$.

Claim 2. For each $n \geq 1$ and each $a \in (n) S(M)$, there exists a sequence of non-negative measurable functions f_k such that

$$\left\|\int f_k(x) x \, d\sigma(x) - a\right\| < 1/k .$$

Proof of the Claim 2: This follows from Claim 1.

Claim 3. Let $a_n \rightarrow a$, $a_n \in \bigcup_{k\geq 1}(k) S(M)$. Then $a \in A(\mu)$.

Proof of the Claim 3: By Claim 2, we can choose a sequence g_k^n, $k, n \geq 1$ such that $\|g_k^n(x) x \, d\sigma(x) - a_n\| < 1/k$ for each $k, n \geq 1$. Let $f_n = g_n^n$. Then

$$\left\|\int f_n(x) x \, d\sigma(x) - a\right\| < \left\|\int g_n^n(x) x \, d\sigma(x) - a_n\right\| + \|a_n - a\|$$
$$< 1/n + \|a_n - a\| .$$

Then $a \in A(\mu)$, i.e. $G(M) \subseteq A(\mu)$. By the Remark 6.2 in [5], $G(M) - G(M) \subseteq A(\mu)$. Now, let $\nu = \delta_{-y} * \mu$. Then $A(\nu) = A(\mu)$ and $S(\nu) = (G(M) - G(M))^-$, whence $A(\mu)^- = S(\mu) - y$.

REFERENCES

1. A. De Acosta, *Stable measures and seminorms*, Ann. Probability **3** (1975), 865-875.
2. P. L. Brockett, *Supports of infinitely divisible measures on Hilbert spaces*, Ann. Probability **5** (1977), No. 6, 1012-1018.
3. G. Choquet, *Lectures on Analysis* II, W. A. Benjamin, Inc., New York, 1969.
4. T. S. Ferguson, *Mathematical Statistic*, Academic Press, New York, 1967.
5. I. I. Gikhmann and A. V. Skorokhod, *On the densities of probability measures in function spaces*, Russian Mathematical Surveys **21** (1966), 83-156.
6. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Colloq. Publ. Amer. Math. Soc. Providence, R. I., 1957.
7. W. N. Hudson and H. G. Tucker, *On admissible translates of infinitely divisible distributions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **31** (1975), 67-72.
8. J. Kuelbs, *A representation theorem for symmetric stable processes and stable measures on H*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **26** (1973), 256-271.
9. ———, *Some results for probability measures on linear topological vector space with an application to Strassen's* log log *law*, J. Functional Analysis **14** (1973), 28-43.
10. D. G. Luenberger, *Optimization by vector space methods*, John Wiley & Sons, Inc., New York, 1969.
11. E. Mourier, *Elements aleatoires dans un espaces de Banach*, Ann. Inst. H. Poincaré **13** (1953), 161-244.

12. K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York and London, 1967.

13. A. V. Skorokhods, *On admissible translations of measures in Hilbert spaces*, Theory of Prob. and Appl. **15** (1970), 557-580.

14. G. N. Sytaya, *On admissible displacements of suspended Gaussian measures*, Theory of Prob. and Appl. **14** (1969), 506-509.

15. J. Yuan and T. C. Liang, *On the supports and absolute continuity of infinitely divisible probability measures*, Semigroup Forum **12** (1976), 34-44.

DEPARTMENT OF STATISTICS, FENGCHIA COLLEGE OF ENGINEERING AND BUSSINESS, TAICHUNG, TAIWAN 4-0.

INSTITUTE OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU, TAIWAN 300.