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FIXED POINT THEOREMS FOR POINT-TO-SET MAPPINGS IN LOCALLY CONVEX SPACES AND CHARACTERIZATIONS OF COMPLETE METRIC SPACES

BY

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Abstract. Let $\mathscr{R}$ be a family of continuous seminorms determining the topology of a locally convex space X. Conditions for the existence of fixed points of upper semi-continuous point-to-set $\mathscr{R}$-Kannan maps are determined in terms of compactness and weakly compact convexity of X. Applications include characterizations of the completeness of metric spaces.

Introduction. Fixed point theorems for point-to-set mappings in Fan [1] and Su and Sehgal [8] have some form of continuity conditions. Recently many works on single valued mappings without the assumption of continuity have been done on the type of maps initiated by Kannan [4, 5]. Motivated by these ideas and a work of Ko [6], we have two fixed point theorems in §1 for point-to-set mappings on locally convex spaces.

As we all know, the completeness of a metric space is essential to ensure the existence of a fixed point under a contraction map or a Kannan map on it. (Here by a Kannan map we mean a map f on a metric space (X, d) to itself such that $d(f(x), f(y)) \leqq \alpha\{d(x, f(x)) + d(y, f(y))\}$ for all x, y in X and some constant $0 \leqq \alpha < \frac{1}{2}$). One might expect that the converse is true. That is, one thinks a metric space will be complete if every contraction map or every Kannan map on it has a fixed point. It turns out that this is not true for the case of a contraction map, but is true for a Kannan map. In §2 we shall deal with these problems.

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1. Fixed Point Theorems. Throughout this section we assume X to be a Hausdorff locally convex topological linear space (or simply locally convex space), $\mathcal{L}$ to be a local base at the origin consisting of absolutely convex neighborhoods, (we follow the terminologies used in Robertson [7]), and $\mathcal{R}$ to be the family of gauges p_U for $U \in \mathcal{L}$. Then each p in $\mathcal{R}$ is a continuous seminorm on X, and the topology on X is determined by $\mathcal{R}$. Denote the family of all nonempty closed subsets of X by 2^X.

DEFINITION 1. A map f on X to 2^X is said to be ***upper semi-continuous*** (abbreviated by u.s.c.) at a point x_0 in X if for any open set G containing $f(x_0)$, there is a neighborhood V of x_0 such that $f(y) \subset G$ for all $y \in V$. f is upper semicontinuous on X if it is u.s.c. at every x in X.

Let $f : X \to 2^X$. For each $p \in \mathcal{R}$, define a real valued function g_p on X by

$$g_p(x) = \inf \{p(x-y) : y \in f(x)\}.$$

g_p can be interpreted as the p-distance of x from $f(x)$. Recall that the function g_p is lower semicontinuous on X if $\liminf_{x \to x_0} g_p(x) \geqq g_p(x_0)$ for each x_0 in X, or, equivalently, if the set $\{x \in X : g_p(x) > r\}$ is open for each real number r.

LEMMA 1. *If $f : X \to 2^X$ is u.s.c. on X, then for each $p \in \mathcal{R}$, the function g_p is lower semicontinuous on X.*

The proof is exactly the same as that of Proposition 2 in Ko [6], with the distance $d(x, y)$ replaced by $p(x-y)$.

DEFINITION 2. Let K be a subset of X; $f : K \to 2^K$ is said to be a ***$\mathcal{R}$-Kannan map*** if for each $p \in \mathcal{R}$, there exists $0 \leqq \lambda_p < \frac{1}{2}$ such that for any x, y in K and $x_1 \in f(x)$, there exists $y_1 \in f(y)$ for which the inequality

$$p(x_1 - y_1) \leqq \lambda_p\{g_p(x) + g_p(y)\}$$

holds.

Note that if the set K is compact, then each set A in 2^K is compact and hence bounded under each p in $\mathcal{R}$. Thus the Hausdorff metric D_p [2] induced from the seminorm p is well defined on 2^K. It should be noted that D_p is a metric on 2^K although p is a seminorm on K. Hence if K is compact, Definition 2 is equivalent to the following:

DEFINITION 2'. Let K be a compact subset of X. A map $f: K \to 2^K$ is said to be a $\mathcal{R}$-Kannan map if for each $p \in \mathcal{R}$, there exists $0 \leqq \lambda_p < \frac{1}{2}$ such that

$$D_p(f(x), f(y)) \leqq \lambda_p\{g_p(x) + g_p(y)\}.$$

LEMMA 2. *Let K be a nonempty subset of X, $f: K \to 2^K$ be a u.s.c. $\mathcal{R}$-Kannan map on K. Then for each $p \in \mathcal{R}$, $\inf\{g_p(x): x \in K\} = 0$. Further if K is compact, then there is x_0 in K such that $g_p(x_0) = 0$.*

Proof. Let $x \in K$, $x_1 \in f(x)$. As f is a $\mathcal{R}$-Kannan map, for each p there is $0 \leqq \lambda_p < \frac{1}{2}$ such that for x and x_1 there is $x_2 \in f(x_1)$ with

$$p(x_1 - x_2) \leqq \lambda_p\{g_p(x) + g_p(x_1)\}.$$

Then

$$p(x_1 - x_2) \leqq \lambda_p\{g_p(x) + g_p(x_1)\} \leqq \lambda_p\{p(x - x_1) + p(x_1 - x_2)\}.$$

Hence

$$p(x_1 - x_2) \leqq [\lambda_p/(1 - \lambda_p)]\, p(x - x_1). \tag{1}$$

Similarly, as $x_2 \in f(x_1)$, there is $x_3 \in f(x_2)$ such that

$$p(x_2 - x_3) \leqq \lambda_p\{g_p(x_1) + g_p(x_2)\};$$

therefore we get

$$p(x_2 - x_3) \leqq [\lambda_p/(1 - \lambda_p)]\, p(x_1 - x_2). \tag{2}$$

From (1) and (2), we have

$$p(x_2 - x_3) \leqq [\lambda_p/(1 - \lambda_p)]^2\, p(x - x_1).$$

Continuing this process, we get a sequence $x_n \in K$ with $x_{n+1} \in f(x_n)$ and

$$p(x_n - x_{n+1}) \leqq [\lambda_p/(1 - \lambda_p)]^n\, p(x - x_1).$$

Then

$$g_p(x_n) \leqq p(x_n - x_{n+1}) \leqq [\lambda_p/(1 - \lambda_p)]^n\, p(x - x_1). \tag{3}$$

Now $0 \leqq \lambda_p < \frac{1}{2}$ implies $\lambda_p/(1-\lambda_p) < 1$. Hence (3) implies $\lim_{n\to\infty} g_p(x_n) = 0$. Therefore $\inf\{g_p(x) : x \in K\} = 0$.

If K is compact, then there is a convergent subnet $\{x_{n(\alpha)}\}$ of the sequence $\{x_n\}$ converging to, say, $x_0 \in K$. We claim that $g_p(x_0) = 0$. Since g_p is lower semicontinuous at x_0, from Lemma 1, we have

$$0 \leqq g_p(x_0) \leqq \liminf_{\alpha} g_p(x_{n(\alpha)}) = \lim_{\alpha} g_p(x_{n(\alpha)}) = 0 .$$

Hence $g_p(x_0) = 0$.

THEOREM 1. *Let K be a compact subset of X and $f : K \to 2^K$ be a u.s.c. $\mathscr{R}$-Kannan map on K. Then f has a fixed point on K.*

Proof. For each $p \in \mathscr{R}$, let $A_p = \{x \in K : g_p(x) = 0\}$. Then, from Lemma 2, A_p is nonempty. We claim:

(a) A_p is closed. To prove it, let $\{x_\alpha\}$ be a net in A_p converging to $x_0 \in K$. We prove that $g_p(x_0) = 0$. Again, using the lower semicontinuity of g_p at x_0 we have

$$0 \leqq g_p(x_0) \leqq \liminf_{\alpha} g_p(x_\alpha) = 0 ,$$

since $g_p(x_\alpha) = 0$ for each α. Hence $g_p(x_0) = 0$; therefore $x_0 \in A_p$, i.e. A_p is closed.

(b) The family $\{A_p : p \in \mathscr{R}\}$ has the finite intersection property. To prove it, let $p_i \in \mathscr{R}$, and let U_i in the local base $\mathscr{L}$ be such that p_i is the gauge of U_i, $i = 1, \cdots, n$. Let $U = \bigcap_{i=1}^n U_i$. Then there is a neighborhood V in $\mathscr{L}$ such that $V \subset U$. Let p_V and p_U be the gauges of V and U respectively. Since $V \subset U \subset U_i$, $i = 1, \cdots, n$, we have $p_V \geqq p_U \geqq p_i$, $i = 1, \cdots, n$. By Lemma 2, there is $y \in K$ such that $p_V(y) = 0$. Hence

$$0 = p_V(y) \geqq p_U(y) \geqq p_i(y) \geqq 0 , \qquad i = 1, \cdots, n .$$

Therefore $p_i(y) = 0$, $i = 1, \cdots, n$; i.e. $y \in \bigcap_{i=1}^n A_{p_i}$. By the compactness of K we have $\bigcap_{p \in \mathscr{P}} A_p \neq \emptyset$. Let $x_0 \in \bigcap_{p \in \mathscr{P}} A_p$. Then $g_p(x_0) = 0$ for all $p \in \mathscr{R}$. This means that x_0 is in the closure of $f(x_0)$, and hence $x_0 \in f(x_0)$ as $f(x_0)$ is closed. Thus x_0 is a fixed point of f.

REMARK. The fixed point theorem in Fan [1] does not apply to a map f without the condition that each set $f(x)$ is convex.

In Theorem 1 we did not assume that the set K or the sets $f(x)$ are convex. It is because of the absence of convexity that the condition of "$\mathscr{R}$-Kannan map" steps in.

In the following we replace the compactness of K by the weak compactness and add one condition to the map f.

THEOREM 2. *Let K be a weakly compact convex subset of X. Assume that $f : K \to 2^K$ is a u.s.c. $\mathscr{R}$-Kannan map on K. If f satisfies the following condition*:

(*) *for any x, y in K and $m = \mu x + (1-\mu) y$, $0 \leqq \mu \leqq 1$, we have $g_p(m) \leqq \max\{g_p(x), g_p(y)\}$ for each $p \in \mathscr{R}$,*

then f has a fixed point in K.

Proof. For each $p \in \mathscr{R}$, Lemma 2 implies that

$$\inf\{g_p(x) : x \in K\} = 0 . \tag{4}$$

For each $r > 0$, let $H_r = \{x \in K : g_p(x) \leqq r\}$. Then from (4), H_r is nonempty. The condition (*) and Lemma 1 imply that H_r is convex and closed. Hence H_r is weakly closed. It is evident that the family $\{H_r : r > 0\}$ has the finite intersection property. Now by weak compactness of K, the set $A_p = \bigcap_{r>0} H_r$ is nonempty. We see that $x \in A_p$ if and only if $g_p(x) = 0$. Therefore the set A_p is exactly what we defined in the proof of Theorem 1 and thus the family $\{A_p : p \in \mathscr{R}\}$ has the finite intersection property. By the weak compactness of K, we have

$$\bigcap_{p \in \mathscr{R}} A_p \neq \emptyset ,$$

and every point in $\bigcap_{p \in \mathscr{R}} A_p$ is a fixed point of f.

REMARK. Condition (*) in Theorem 2 was termed semiconvex as of the map $I - f$, where I is the identity map on K.

2. **Characterizations of complete metric spaces by means of fixed point theorems.** Let (X, d) be a metric space and α be a positive number. We denote by $C(X, \alpha)$ the family of all maps $f : X \to X$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all x, y in X, and by $K(X, \alpha)$ the family of all maps $f : X \to X$ such that

$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))]$ for all x, y in X. For simplicity, we introduce the following definition.

DEFINITION 3. Let X be a set and F be a family of maps from X into itself. Then X is said to have the fixed point property relative to F if each function in F has a fixed point in X.

Now we are able to characterize a complete metric space by the relative fixed point property.

THEOREM 3. *Let* (X, d) *be a metric space. If there is a positive number* α *with* $0 < \alpha < \frac{1}{2}$ *such that* X *has the fixed point property relative to* $K(X, \alpha)$, *then* (X, d) *is complete.*

Proof. Suppose that (X, d) is not complete, then there is a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ such that $\{x_n\}_{n=1}^{\infty}$ is not convergent, and $x_n \neq x_m$ whenever $n \neq m$. Let N be the set of all natural numbers. For each x in X, let

$$a_x = \inf\{d(x, x_n) : n \in N \text{ and } x_n \neq x\}. \tag{5}$$

Then $a_x > 0$ since $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence. Now $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, thus for each $x \in X$, there is a positive integer $N(x)$ such that

$$d(x_n, x_m) < \alpha a_x \quad \text{for all } n, m \geq N(x). \tag{6}$$

We claim that

$$N(x_n) > n \quad \text{for all } n \in N. \tag{7}$$

If not, then $N(x_n) \leq n$ for some $n \in N$. Then from (6) we have $d(x_n, x_m) < \alpha a_n < a_n$ for all $m \geq n$. But from (5) we have $d(x_n, x_m) \geq a_n$ for all $m > n$. This is a contradiction. Hence (7) holds.

Define $f : X \to X$ by $f(x) = x_{N(x)}$. Then f has no fixed point; for if $x = x_n$ for some n, then from (7) we have $N(x_n) > n$, hence $f(x_n) = x_{N(x_n)} \neq x_n$, and if $x \neq x_n$ for all n, then $f(x) = x_{N(x)} \neq x$. Now

$$\begin{aligned} d(f(x), f(y)) &= d(x_{N(x)}, x_{N(y)}) \\ &\leq d(x_{N(x)}, x_{N(x)+N(y)}) + d(x_{N(x)+N(y)}, x_{N(y)}) \\ &\leq \alpha(a_x + a_y), \end{aligned}$$

$$d(x, f(x)) + d(y, f(y)) = d(x, x_{N(x)}) + d(y, x_{N(y)}) \geq a_x + a_y .$$

Hence

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))] \quad \text{for all } x, y \text{ in } X .$$

Therefore $f \in K(X, \alpha)$ and f has no fixed point. This is a contradiction. Hence (X, d) is complete.

For contraction mappings, the following theorem is a variation of a result of T. K. Hu [3]. We include it here because our proof seems to be more transparant, and because the result is quite different from that of Theorem 3 (cf. the counter-example which follows its proof) in spite of the similarity of their statements.

THEOREM 4. *Let (X, d) be a metric space. If for every closed subset Y of X, there exists an α with $0<\alpha<1$ such that Y has the fixed point property relative to $C(Y, \alpha)$, then (X, d) is complete.*

Proof. Suppose that (X, d) is not complete, then there is a nonconvergent Cauchy sequence $\{x(n)\}$ in X with distinct points $x(n)$. Then $Y = \{x(n) : n \in N\}$ is a closed subset of X. For this set Y, there is a positive number α with $0<\alpha<1$ such that Y has the fixed point property relative to $C(Y, \alpha)$.

Let f be the function defined in the proof of Theorem 3, where the sequence $\{N(x(N))\}$ is assumed to be non-decreasing. Now let g be the restriction of f to the subspace Y. Then g has no fixed point in Y since f has none.

Now we claim that $g \in C(Y, \alpha)$. Let $m>n$. Then $N(x(m)) \geq N(x(n))$. It follows from (6) that

$$d(g(x(n)), g(x(m))) = d(x(N(x(n))), x(N(x(m)))) < \alpha\, a_{x(n)} .$$

From (5) we have $d(x(n), x(m)) \geq a_{x(n)}$ for all $m>n$. Thus we have

$$d(g(x(n)), g(x(m))) < \alpha\, a_{x(n)} \leq \alpha d(x(n), x(m)) \quad \text{for } m>n .$$

This shows that $g \in C(Y, \alpha)$. But g has no fixed point. This contradicts the assumption that Y has the fixed point property relative to $C(Y, \alpha)$. Hence the metric space (X, d) must be complete.

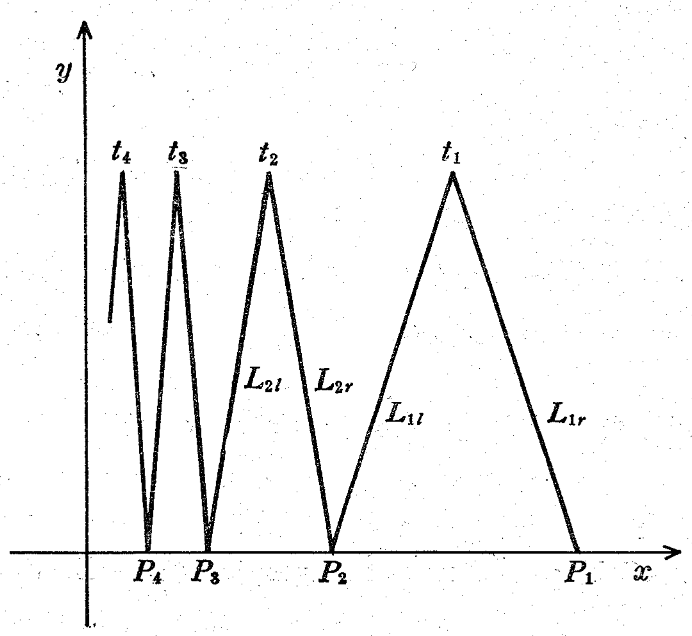
REMARK. Here we will call the attention of the reader that X has the fixed point property relative to $C(X, \alpha)$ for some α in $(0, 1)$ if and only if it has the fixed point property relative to $C(X, \alpha)$ for all α in $(0, 1)$. This follows from the following two facts: (i) if $f \in C(X, \alpha)$, then $f^n \in C(X, \alpha^n)$ for every positive integer n; (ii) if some iterate f^n of f has a unique fixed point x_0, then x_0 is also a fixed point of f.

Now a question arises: Is it true that (X, d) is complete if X has the fixed point property relative to $C(X, \alpha)$ for some α in $(0, 1)$? The answer is negative, as the following counterexample shows:

EXAMPLE. In the Euclidean plane R^2 consider the points $p_n = (2^{-(n-1)}, 0)$ and $t_n = (3 \times 2^{-(n+1)}, 1)$. Let L_{nr} denote the line segment joining p_n to t_n and L_{nl} the line segment joining t_n to p_{n+1}. Set

$$X = \left(\bigcup_{n=1}^{\infty} L_{nr}\right) \cup \left(\bigcup_{n=1}^{\infty} L_{nl}\right)$$

and equip it with the usual metric d as a subspace of the Euclidean plane R^2. Then (X, d) is connected but not complete. We claim that (X, d) has the fixed point property relative to $C(X, \frac{1}{2})$. Let



$f \in C(X, \frac{1}{2})$. Denote the image set of X under f by Y. Then (1) Y is connected, since X is connected; (2) the diameter of Y is at most $\sqrt{2}/2$, since $f \in C(X, \frac{1}{2})$. From (2) we know that Y does not contain the whole line segment L_{nr} or L_{nl} for any positive integer n. Therefore Y must be a connected subset of one (and only one) of the following sets: (a) L_{nr}, (b) L_{nl}, (c) $L_{nr} \cup L_{nl}$, (d) $L_{nl} \cup L_{n+1,r}$ for some positive integer n. Now consider the sequence $\{f^n(p_1)\}$. As $f \in C(X, \frac{1}{2})$, the sequence $\{f^n(p_1)\}$ is a Cauchy sequence, and hence it converges to some point x_0 of the closure $X \cup \{(0, x) : x \in [0, 1]\}$ of X in R^2. But as $f^n(p_1) \in Y$ for all $n \geq 1$, and Y is a subset of one of the sets (a), (b), (c) and (d), the limit point x_0 of the sequence $\{f^n(p_1)\}$ cannot fall on the y-axis. Hence $x_0 \in X$. Now

$$d(f(x_0), x_0) = \lim_{n\to\infty} d(f(x_0), f^n(p_1)) \leq \lim \frac{1}{2} d(x_0, f^{n-1}(p_1))$$
$$= \frac{1}{2} d(x_0, x_0) = 0 .$$

Hence $d(f(x_0), x_0) = 0$ and x_0 is the fixed point of f on X.

At first glance, there seems to be no relation between contraction mappings and Kannan maps. But if we take a second look, we can find that there are some interesting relations between them, such as

(a) If $f \in C(X, \alpha)$ for some α in $(0, \frac{1}{3})$, then f is a Kannan map. Thus if f is a contraction mapping, select n such that $\alpha^n < \frac{1}{3}$. Then f^n is a Kannan map. Therefore the iterations of every contraction mapping are eventually in the class of all Kannan maps.

(b) Let (X, d) be a metric space and $f : X \to X$ be a contraction mapping or a Kannan map. Then there is a metric ρ on X such that f is both a contraction map and a Kannan map on (X, ρ). Indeed, we may define ρ by $\rho(x, x) = 0$ and $\rho(x, y) = d(x, f(x)) + d(y, f(y))$ for all $x, y \in X$ such that $x \neq y$. Then $\rho(x, y) = 0 \Leftrightarrow x = y$, since f, being a contraction map or a Kannan map, cannot have more than one fixed point. It is easy to show that ρ is a metric on X. For such ρ, we have

(i) If $f \in C((X, d), \alpha)$, then

$$f \in C((X, \rho), \alpha) \cap K((X, \rho), \alpha/(1+\alpha)).$$

(ii) If $f \in K((X, d), \alpha)$ with $0 < \alpha < 1$, then

$$f \in K((X, \rho), \alpha) \cap C((X, \rho), \alpha/(1-\alpha)).$$

(iii) The space (X, ρ) has at most one limit point (i.e. not an isolated point), and the limit point is the fixed point of f.

(iv) The space (X, ρ) is complete if f has a fixed point, and hence it is complete if the original metric space (X, d) is complete.

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