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ORDINARY DIFFERENTIAL OPERATORS AND THEIR CONVOLUTION ADJOINTS

BY

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Abstract. The concept of reflection in connection with a formal linear differential operator L is formalized in this paper in terms of the convolution adjoint L°. Intimate relations between the Lagrange adjoint L^*, L and L° are studied. These relations indicate that the convolution adjoint is important in analyzing symmetric boundary value problems as well as initial value problems.

1. Introduction. The reader is undoubtedly familiar with the fact that for any solution $u(t)$ of the equation

$$u'' + q(t)\,u = 0\,, \qquad q \in C[a, b]\,, \tag{1.1}$$

$v(t) = u(a+b-t)$ is a solution of the equation

$$v'' + q(a+b-t)\,v = 0\,. \tag{1.2}$$

This fact may be regarded as elementary; however, by asking ourselves whether it is a specific manifestation of more general phenomenon, we come up with some interesting results. Some of these are presented in the following sections. Here we briefly describe the route we take to arrive at them.

We consider, as a general extension of (1.1), the equation $Lu = 0$ where L is a formal linear differential operator of order n on an interval $I = [a, b]$,

$$L = \sum_{j=0}^{n} p_j(t)(d/dt)^j \qquad (p_j \in C^j[a, b],\ p_n(t) \neq 0)\,, \tag{1.3}$$

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and we take, after some guess work, as an ad hoc extension of (1.2) the equation $L^\circ v = 0$ where

$$L^\circ = \sum_{j=0}^{n} (d/dt)^j p_j(a+b-t). \tag{1.4}$$

We discover many intimate relations between the equations $Lu = 0$ and $L^\circ v = 0$. Furthermore, we find that these relations are similar to those that exist between the equation $Lu = 0$ and its (Lagrange) adjoint $L^* v = 0$ where

$$L^* = \sum_{j=0}^{n} (-1)^j (d/dt)^j p_j(t).$$

As is well known, the relations between L and L^* are intimately related to the inner product defined in the space $C[a, b]$. These similarities thus prompt us to introduce a bilinear functional in the space $C[a, b]$ and study linear differential operators defined there. It turns out that $C[a, b]$, endowed with the bilinear functional, is an indefinite inner product space and L° is a formal adjoint of L (for definitions see [1]). In this setting our discoveries become natural and allow further augmentation. Even though the theory of indefinite inner product space has been developed to some extent, we believe that the results presented below are new and can be readily applied to concrete problems. Moreover, they are quite accessible to readers with moderate background in differential equations.

We begin by introducing some terminologies and notations. Matrices will be represented by capital, or boldfaced capital letters, in particular, the identity matrix will be represented by J. A vector is always meant to be a column vector, and is represented by a boldfaced letter. The transpose of a matrix A is denoted by $\bar{A}$. If f is any function with $n-1$ derivatives, the vector

$$\boldsymbol{k}[f] = \text{col}\,(f, f', \cdots, f^{(n-1)})$$

is called the Wronskian vector of f. If $f_1, f_2, \cdots, f_n$ is any set of functions, each having $n-1$ derivatives, and if $\boldsymbol{f}$ is the vector with these functions as components, then the matrix

$$\boldsymbol{K}[\boldsymbol{f}] = (\boldsymbol{k}[f_1], \boldsymbol{k}[f_2], \cdots, \boldsymbol{k}[f_n])$$

is called the Wronskian matrix of f. A fundamental vector of a homogeneous equation $Lu=0$ is a vector whose components form a fundamental system of solutions of the equation. Finally, a function f defined on an interval $I=[a, b]$ is said to be symmetric or even if $f(t)=f(a+b-t)$, and is said to be odd if $f(t)=-f(a+b-t)$.

It is hoped that the results, to be developed below, would reach a wider range of audience; hence, we will neither present the theory in its most general form nor include highly specialized information. Instead, we will restrict ourselves to real functions, real parameters, etc., and to linear differential operators which are ordinary. Furthermore, we will only discuss those topics which are found in most elementary and intermediate texts such as [2, 3, 5, 6, 7].

2. **Convolution product and the convolution adjoint operator.** If f and g are real continuous function defined on the interval $I=[a, b]$, then the convolution product of f and g is defined to be the integral

$$(f, g)_I = \int_a^b f(a+b-t)\, g(t)\, dt\,.$$

For convenience, we shall call $(f, g)_I$ the C-product of f and g on the interval I. It is easily seen that the C-product is commutative, associative and linear in each variable. Furthermore, it is separating, that is, if $(f, g)_I=0$ for every f which belongs to a dense subset of $C[a, b]$, then $g\equiv 0$ in I. However, the C-product is not positive definite, that is, it is quite possible that $f\not\equiv 0$ and yet $(f, f)_I\leq 0$. The vector space $C[a, b]$ endowed with the C-product is an example of indefinite inner product spaces (in fact, it is a Krein space [1]) but knowledge of these spaces will not be required in the sequel.

Let $u(t)$, $v(t)\in C^{(j)}[a, b]$, then the following formula is easily verified:

$$vu^{(j)} = (-1)^j uv^{(j)} + (d/dt)\sum_{i=0}^{j-1}(-1)^i v^{(i)} u^{(j-i-1)}\,. \tag{2.1}$$

Let the formal linear differential operator L be given by (1.3).

If we replace $v(t)$ by $p_j(t)\,v(a+b-t)$ in (2.1), then for any s in $[a, (a+b)/2]$, we readily see that

$$(2.2)\qquad \begin{aligned}(v, Lu)_I &= \sum_{j=0}^{n} (v, p_j u^{(j)})_I \\ &= \sum_{j=0}^{n} (-1)^j \int_s^{a+b-s} u(t)[p_j(t)\,v(a+b-t)]^{(j)}\,dt \\ &\quad + B(u, v)\big|_s^{a+b-s}\end{aligned}$$

where $I = [s, a+b-s]$ and

$$B(u, v) = \sum_{j=0}^{n} \sum_{i=0}^{j-1} (-1)^i u^{(j-i-1)}(t)[p_j(t)\,v(a+b-t)]^{(i)}\,.$$

Note that by the change of variable $a+b-t=x$,

$$\begin{aligned}&\int_s^{a+b-s} u(t)[p_j(t)\,v(a+b-t)]^{(j)}\,dt \\ &\qquad = (-1)^j \int_s^{a+b-s} u(a+b-x)[p_j(a+b-x)\,v(x)]^{(j)}\,dx\,.\end{aligned}$$

Hence if we let the operator L° be defined by (1.4), then (2.2) can be written in the form

$$(2.3)\qquad (v, Lu)_I - (u, L^\circ v)_I = B(u, v)\big|_s^{a+b-s}\,, \qquad I = [s, a+b-s]\,.$$

The operator L° shall be called the convolution adjoint, or in short, C-adjoint of L. Clearly, equation (2.3) relates L and L°. We shall see that it plays roughly the same role as the Green's formula that relates L and L^*. The bilinear form $B(u, v)$ will play an important role part in subsequent discussions, hence it is useful to examine its structure. This is achieved by observing that $B(u, v)$ is identical to the Lagrange bilinear concomitant $P(u, v)$ except that $v(t)$ is replaced by $v(a+b-t)$. It is well known [3, p. 61] that

$$(2.4)\qquad P(u, v) = \bar{k}[v](t)\,P(t)\,k[u](t)\,,$$

where $P(t)$ is the concomitant matrix. Furthermore, if we replace $v(t)$ by $v(a+b-t)$ in $\bar{k}[v](t)$, we obtain

$$\begin{aligned}&(v(a+b-t), [v(a+b-t)]', \cdots, [v(a+b-t)]^{(n-1)}) \\ &\quad = (v(a+b-t), -v'(a+b-t), \\ &\qquad \cdots, (-1)^{n-1} v^{(n-1)}(a+b-t)) \\ &\quad = \bar{k}[v](a+b-t)\,M\end{aligned}$$

where $M = \operatorname{diag}[1, -1, \cdots, (-1)^{n-1}]$. In other words,

$$B(u, v) = \bar{k}[v](a + b - t)\, B(t)\, k[u](t),$$

where $B(t) = MP(t)$ is the matrix obtained from $P(t)$ by changing the signs of all the elements on the even rows. Recall [3, p. 63] that every component of the concomitant matrix below the secondary diagonal is zero and the components on the secondary diagonal are alterantively p_n and $-p_n$, with p_n in the upper right hand corner. Hence B is a matrix with zeros below the secondary diagonal and functions p_n on the secondary diagonal. In the sequel, the matrix B will be called the C-concomitant matrix of L.

Numerous properties of the C-adjoint formally duplicate those of the Lagrange adjoint. Below and in the following sections we list some of them in a more or less logical order. The proofs can be obtained either by imitating the corresponding arguments for L^*, or by applying the properties of L^* combined with identity (2.3).

THEOREM 2.1. *If L and M are differential operators of the form* (1.3), *then for any real number c,*

$$(L + M)^\circ = L^\circ + M^\circ \quad \text{and} \quad (cL)^\circ = cL^\circ. \tag{2.5}$$

THEOREM 2.2. $(L^\circ)^\circ = L$ *and* $(LM)^\circ = M^\circ L^\circ$.

THEOREM 2.3. *Let B and D be the C-concomitant matrix of L and L° respectively, then* $B(a + b - t) = \bar{D}(t)$.

So far we have been concerned with properties of L° which are similar to those of L^*. But as we shall see, things start to look somewhat different when we restrict our attention to operators which satisfy the condition $L = L^\circ$.

DEFINITION 2.4. If $L = L^\circ$ we say that the operator L is C-selfadjoint and also that the equation $Lu = 0$ is C-selfadjoint.

Note that in view of (2.5), a sum of C-selfadjoint differential operators is also C-selfadjoint, and so is the product of a C-selfadjoint operator with a real number. The following theorem is concerned with the general form of all C-selfadjoint operators and its proof is similar to an argument in [7, p. 7].

THEOREM 2.5. *Any C-selfadjoint differential operator is a sum of linear differential operators defined by*

$$M_{2j}u = (pu^{(j)})^{(j)}$$

and

$$M_{2j-1}u = [(pu^{(j-1)})^{(j)} + (pu^{(j)})^{(j-1)}]/2,$$

where $p(t)$ *is an even function in* $[a, b]$.

We remark that any selfadjoint operator L (i.e. an operator L satisfying $L = L^*$) with real coefficients is necessarily of even order and has the form

$$L = \sum_{j=0}^{n} (d/dt)^j p(t) (d/dt)^j,$$

where $p_0, \cdots, p_n$ are real functions. In view of Theorem 2.5 a C-selfadjoint operator is, in general, nonselfadjoint. Note also that, if L is C-selfadjoint, then $B = D$ in Theorem 2.1, and hence $B(a+b-t) = \overline{B}(t)$.

Next we consider some examples.

EXAMPLE 1. All operators of the form (1.3) with constant coefficients are C-selfadjoint.

EXAMPLE 2. The following equation

$$Lu = [p(t)u' + q(t)u]' + [q(t)u' + r(t)]u = 0, \quad (p(t) \neq 0) \tag{2.6}$$

where $p \in C^{(1)}$, $q \in C^{(1)}$ and $r \in C$ are even functions in $[a, b]$, is the most general C-selfadjoint equation of the second order. The C-concomitant matrix, in this case, is easily shown to be

$$\begin{pmatrix} 2q(t) & p(t) \\ p(t) & 0 \end{pmatrix}.$$

EXAMPLE 3. The following is an analogue of the Sturm's classical comparison theorem: If u is a solution of (2.6), $u \neq 0$ for $t \in (a, b)$, $u(a) = u(b) = 0$ and if $R(t)$ is an even function on $[a, b]$ satisfying $r(t) \leq R(t)$, as well as $r(t) \not\equiv R(t)$, then every solution of the following equation

$$Mv = [p(t)v' + q(t)v]' + [q(t)v' + R(t)v] = 0$$

has at least one zero in (a, b). The proof essentially duplicates that of the Sturm's theorem (see for example [3, p. 223]), and is thus omitted.

3. **Relations between the one sided Green's functions of C-adjoint operators.** Let us first recall the definition of the one sided Green's function or Cauchy function for the linear differential operator L given by (1.3). Let $\{u_j(t) \mid 1 \leq j \leq n\}$ be a fundamental set of solutions of $Lu = 0$. Denote the column vector $\operatorname{col}(u_1, u_2, \cdots, u_n)$ by $\boldsymbol{u}$. Then the one sided Green's function for L is defined to be the function [6, p. 33]

$$\begin{aligned} (3.1) \qquad & H(t, s) \\ & = (-1)^{n-1} \det(\boldsymbol{u}(t), \boldsymbol{u}(s), \boldsymbol{u}'(s), \cdots, \boldsymbol{u}^{(n-2)}(s))/p_n(s) W(s) \end{aligned}$$

where $W(s)$ is the Wronskian of the functions $u_j(s)$. Note that $H(t, s)$ can also be written in the form

$$H(t, s) = \sum_{i=1}^{n} u_i(t) u_i^*(s) \tag{3.2}$$

where

$$u_i^*(s) = \frac{1}{p_n(s) W(s)} (\partial/\partial u_i^{(n-1)}(s)) W(s) .$$

Using the properties [6, pp. 29–42] of the Cauchy function and identity (2.3), the following theorem can easily be proved.

THEOREM 3.1. *Let the one sided Green's functions of L and L° be denoted by $H(t, s)$ and $K(t, s)$ respectively, then*

$$K(t, s) = H(a + b - s, a + b - t) . \tag{3.3}$$

THEOREM 3.2. *Let $H(t, s)$ be the one sided Green's function given by (3.2). Then the functions $v_i(t) = u_i^*(a + b - t)$, $1 \leq i \leq n$, form a fundamental system of solutions of $L^\circ v = 0$.*

Proof. By Theorem 3.1, if $K(t, s)$ denotes the one sided Green's function for L°, then

$$K(t, s) = \sum_{i=0}^{n} u_i(a + b - s) u_i^*(a + b - t) .$$

Furthermore $K(t, s)$, as a function of t, satisfies (3.3). Hence

$$L_t K(t, s) = \sum_{i=0}^{n} u_i(a+b-s)\, Lu_i^*(a+b-t) = 0.$$

But since $\{u_i \mid 1 \le i \le n\}$ is a fundamental system of solutions of $Lu = 0$, $Lu_i^*(a+b-t) = 0$ for all i, $1 \le i \le n$. Next we show that the functions $u_i^*(a+b-t) = v_i(t)$, $1 \le i \le n$, are independent. A well known property of the one sided Green's function is that, for any $t_0 \in [a, b]$, the set of functions

$$(3.4)\quad \begin{aligned} z_{j+1}(t) &= (\partial/\partial s)^j K(t, t_0) \\ &= \sum_{i=1}^{n} (-1)^j u_i^{(j)}(a+b-t_0)\, u_i^*(a+b-t) \\ &= \sum_{i=1}^{n} (-1)^j u_i^{(j)}(a+b-t_0)\, v_i(t) \qquad 0 \le j \le n-1 \end{aligned}$$

is a fundamental system of solutions of $L^\circ v = 0$. Let $z = \mathrm{col}(z_1, z_2, \cdots, z_n)$, $v = \mathrm{col}(v_1, \cdots, v_n)$ and $u = \mathrm{col}(u_1, \cdots, u_n)$, then (3.4) can be rewritten in the following form

$$z(t) = MK[u](a+b-t_0)\, v(t)$$

where $M = \mathrm{diag}[1, -1, \cdots, \pm 1]$. Since the Wronskian $\det K[u]$ does not vanish at any point of $[a, b]$ and since $\det M = \pm 1$, $v(t) = \{MK[u](a+b-t_0)\}^{-1} z(t)$ is thus a fundamental vector for $Lu = 0$. Q.E.D.

It follows from the above theorem that if a fundamental system of solutions of $Lu = 0$ is known, then we can construct a fundamental system of solution of $L^\circ v = 0$, and vice versa.

COROLLARY 3.3. *If* $u_1, u_2, \cdots, u_{n-1}$ *are linearly independent solutions of*

$$Nu = p_n(t)\, u^{(n)} + p_{n-2}(t)\, u^{(n-2)} + p_{n-3}(t) + \cdots + p_0(t)\, u = 0$$

where $p_j \in \mathbf{C}[a, b]$ *and* $p_n(t) \neq 0$, *then* $v(a+b-t)$, *where*

$$v = p_n^{-1} \begin{vmatrix} u_1 & u_2 & \cdots & u_{n-1} \\ u_1' & u_2' & \cdots & u_{n-1}' \\ \cdots & \cdots & \cdots & \cdots \\ u_1^{(n-2)} & u_2^{(n-2)} & & u_{n-1}^{(n-2)} \end{vmatrix}$$

is a solution of $N^\circ v = 0$.

In particular, if $u(t)$ is a solution of (1.1), then $u(a+b-t)$ is a solution of (1.2) as observed in the Introduction.

4. **Two point boundary problems.** We begin with homogeneous two point boundary value problems. Consider the homogeneous equation

$$Lu = 0 \tag{4.1.a}$$

together with the auxiliary condition

$$Ek[u](a) + Fk[u](b) = 0 \tag{4.1.b}$$

where E, F are m by n constant matrices. We will assume throughout this section that the m by $2n$ augmented matrix $W = (E : F)$ has rank m. From standard theory we recall that the problem (4.1) is $(n-p)$-ply compatible (i.e. (4.1) has $n-p$ independent solutions) if and only if the characteristic matrix

$$D = EK[z](a) + FK[z](b)\,, \tag{4.2}$$

where z is any fundamental vector for $Lu = 0$, has rank p.

We are concerned with the problem of defining an "adjoint boundary problem" of (4.1) in face of the existing definition of the C-adjoint differential operator L°. Recall first that the C-concomitant matrix $B(t)$ of L is equal to $MP(t)$ where $M = \operatorname{diag}[1, -1, \cdots, (-1)^{n-1}]$. Since $P(t)$ is a triangular matrix with nonzero elements on the secondary diagonal, the block matrix

$$B = \begin{pmatrix} 0 & B(b) \\ -B(a) & 0 \end{pmatrix} \tag{4.3}$$

is clearly nonsingular and its inverse is given by

$$B^{-1} = \begin{pmatrix} 0 & -B^{-1}(a) \\ B^{-1}(b) & 0 \end{pmatrix}. \tag{4.4}$$

DEFINITION 4.1. Let M be any matrix whose rows, considered as vectors, form a basis for the orthogonal complement of the row space of the matrix WB^{-1}; and let M be written in the partitioned form $M = (P : Q)$ where each of P and Q is a $(2n-m) \times n$ matrix. The boundary problem

$$L^\circ v = 0\,, \tag{4.5.a}$$

(4.5.b) $$Pk[v](a)+Qk[v](b)=0$$

is called the C-adjoint boundary problem of (4.1) and the boundary condition (4.5.b) is called the C-adjoint boundary condition of (4.1.b).

It may be noted that the relation $WB^{-1}\overline{M}=0$ follows directly from the definition of M. Furthermore, if N is any nonsingular $(2n-m)\times(2n-m)$ matrix, the replacement of M by NM merely effects a change of basis for the orthogonal complement of the row space of the matrix WB^{-1}, and replaces relation (4.5.b) by an equivalent condition

$$N\{Pk[v](a)+Qk[v](b)\}=0.$$

Clearly if P' and Q' are $(2n-m)$ by n matrices such that $M'=(P':Q')$ has rank $2n-m$, then the condition

$$P'k[v](a)+Q'k[v](b)=0$$

is the C-adjoint boundary condition of (4.1.b) if and only if the matrix M' satisfies $WB^{-1}\overline{M}'=0$.

We remark that our Definition 4.1 does ensure a reciprocal relationship between a given boundary problem of the form (4.1) and its C-adjoint, so that the C-adjoint of the boundary problem (4.5) is the original boundary problem. To see this, it suffices to show that

(4.6) $$(P:Q)\begin{pmatrix}0 & -D^{-1}(a)\\ D^{-1}(b) & 0\end{pmatrix}\begin{pmatrix}\bar{E}\\ \bar{F}\end{pmatrix}=0,$$

where D is the C-concomitant matrix of L°. We first recall from Theorem 2.3 that $B(b)=\bar{D}(a)$ and $B(a)=\bar{D}(b)$. Transposing the equality $WB^{-1}\overline{M}=0$ and substituting $\bar{B}^{-1}(b)=D^{-1}(a)$ and $\bar{B}^{-1}(a)=D^{-1}(b)$ into the resulting equation then yield (4.6) as required.

In view of the above discussions, we may now make the following definition: Suppose $m=n$, the boundary condition (4.1.b) is said to be C-selfadjoint if the condition $WB^{-1}\overline{W}=0$ is satisfied; furthermore, the boundary problem (4.1) is said to be C-selfadjoint if $L=L^{\circ}$ and (4.1.b) is C-selfadjoint.

EXAMPLE 4. Consider the boundary conditions

$$u(a) = u'(a) = \cdots = u^{(j)}(a) = 0\,,$$
$$u(b) = u'(b) = \cdots = u^{(m-j)}(b) = 0 \qquad 1 \leq j < m \leq n$$

or, in matrix notations,

$$\begin{pmatrix} J_{j\times j} & 0 \\ 0 & 0 \end{pmatrix} k[u](a) + \begin{pmatrix} 0 & 0 \\ J_{(m-j)\times(m-j)} & 0 \end{pmatrix} k[u](b) = 0 \tag{4.7}$$

where the symbol J is used to denote an identity matrix. Let W and M be respectively the augmented matrices

$$\begin{pmatrix} J_{j\times j} & 0 & 0 & 0 \\ 0 & 0 & J_{(m-j)\times(m-j)} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} J_{(n-m+j)\times(n-m+j)} & 0 & 0 & 0 \\ 0 & 0 & J_{(n-j)\times(n-j)} & 0 \end{pmatrix}.$$

Then upon direct matrix multiplications, it is not difficult to see that $WB^{-1}\overline{M} = 0$, so that the C-adjoint boundary condition of (4.7) is

$$\begin{cases} J_{(n-m+j)\times(n-m+j)}\, k[v](a) = 0\,, \\ J_{(n-j)\times(n-j)}\, k[v](b) = 0\,. \end{cases}$$

In particular, if $n = m = 2j$, then (4.7) is C-selfadjoint.

EXAMPLE 5. Consider the general C-selfadjoint second order equation (2.6) together with the boundary condition

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} u(a) \\ u'(b) \end{pmatrix} + \begin{pmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{pmatrix} \begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = 0\,. \tag{4.8}$$

We will assume that at least one of the six determinants $\delta_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$ contained in the matrix

$$W = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

is not zero, so that W has rank 2. We want to determine the condition such that the boundary problem is C-selfadjoint. The C-concomitant matrix is given in Example 2 and its inverse is

$$B^{-1}(t) = \begin{pmatrix} 0 & p^{-1}(t) \\ p^{-1}(t) & -2q(t)\,p^{-2}(t) \end{pmatrix}.$$

Furthermore, since p and q are even, $B(a)=B(b)$ and $B^{-1}(a) = B^{-1}(b)$. By definition, (4.8) is C-selfadjoint if and only if

$$W\begin{pmatrix} 0 & -B^{-1}(a) \\ B^{-1}(a) & 0 \end{pmatrix}\overline{W} = 0. \tag{4.9}$$

Upon straightforward matrix multiplication, we see that (4.9) holds if and only if $2q(a)\,\delta_{24} - p(a)(\delta_{14} + \delta_{23}) = 0$, which is the desired condition (cf. Ince [5, pp. 215–217]). In particular, assuming $q(a) = 0$, then the separated boundary condition

$$\begin{cases} \alpha_1 u(a) + \alpha_2 u'(a) = 0, \\ \beta_3 u(b) + \beta_4 u'(b) = 0, \end{cases}$$

is C-selfadjoint if and only if $\alpha_1\beta_4 = \alpha_2\beta_3$.

Clearly, the above examples show that the two sets of self-adjoint and C-selfadjoint boundary problems overlap but they do not conincide with each other.

EXAMPLE 6. Consider the boundary condition

$$J_{n\times n}k[u](a) + 0 \cdot k[u](b) = 0, \tag{4.10}$$

which is equivalent to the one point boundary condition $k[u](a)=0$. Note that

$$(J_{n\times n} : 0)\begin{pmatrix} 0 & -B^{-1}(a) \\ B^{-1}(b) & 0 \end{pmatrix}\begin{pmatrix} J_{n\times n} \\ 0 \end{pmatrix} = 0,$$

hence (4.10) is C-selfadjoint! We remark that initial value problems are never selfadjoint. This example, however, shows that initial value problems of the form

$$\begin{cases} Lu = 0, \\ k[u](a) = 0, \qquad (L = L^{\circ},\ a \le t \le b) \end{cases}$$

are C-selfadjoint and suggests the possibility of constructing variational principles for initial value problems.

Indeed, the reader may care to verify the following:

EXAMPLE 7. Let $I = [a, b]$, let $f \in C[a, b]$ and let

$$D(L) = \{u \in C^{(n)}[a, b] \mid k[u](a) = 0\}.$$

Then $u(t) \in D(L)$ is a solution of $Lu = f$ if and only if it is a critical point of the functional

$$F[u]=(Lu,\ u)_I-2(u,\ f)_I.$$

For variational formulation of other initial value problems, the reader is referred to [8] and the bibliograph contained therein.

We now turn to the relations between the boundary problem (4.1) and its C-adjoint (4.5).

THEOREM 4.2. *Suppose u, v are arbitrary functions in $C^{(n)}[a, b]$ such that u satisfies* (4.1.b), *then* $(v, Lu)_I=(u, L^\circ v)_I$, *where* $I=[a, b]$, *if and only if* v *satisfies* (4.5.b).

Proof. If we let $I=[a, b]$ and U be the matrix

$$(k[v](a),\ k[v](b))\begin{pmatrix}0 & B(b)\\ -B(b) & 0\end{pmatrix},$$

then identity (2.3) becomes

$$\begin{aligned}&(v,\ Lu)_I-(u,\ L^\circ v)_I\\ \tag{4.11} &\qquad=k[v](a)\,B(b)\,k[u](b)-k[v](b)\,B(a)\,k[u](a)\\ &\qquad=U\begin{pmatrix}k[u](a)\\ k[u](b)\end{pmatrix}.\end{aligned}$$

Note that (4.1.b) can be written in the form

$$W\begin{pmatrix}k[u](a)\\ k[u](b)\end{pmatrix}=0,$$

hence the left hand side of (4.11) is zero if and only if the matrix U belongs to the row space of the matrix W, that is, $U=dW$ for some $m\times 1$ column vector d. But $U=dW$ is equivalent to

$$(k[v](a),\ k[v](b))=dW\begin{pmatrix}0 & -B^{-1}(a)\\ B^{-1}(b) & 0\end{pmatrix},$$

which holds if and only if $(k[v](a),\ k[v](b))\,\overline{M}=0$. Q. E. D.

There exists a certain duality between the number of nontrivial solutions of the boundary problem (4.1) and problem (4,5), as shown by the following theorem. The proof of this theorem is similar to that of Theorem 6.6.3 in [3, p. 160] and is omitted.

THEOREM 4.3. *If the boundary problem* (4.1) *has exactly* p

linearly independent solutions, then the C-adjoint boundary problem (4.5) *has exactly* $m + p - n$ *linearly independent solutions.*

In particular, if $m = n$, then the boundary problems (4.1) and (4.5) have the same number of independent solutions.

Next we investigate the relation between the Green's functions of the boundary problem (4.1) and its C-adjoint (4.5). Recall that if $m = n$ and if the boundary problem is incompatible, then the Green's function $G(t, s)$ for (4.1) exists. Furthermore, since (4.5) is also incompatible by Theorem 4.3, the Green's function $G^{\circ}(t, s)$ for (4.5) also exists.

THEOREM 4.4. *Suppose* $m = n$ *and* (4.1) *is incompatible, suppose further that* $G(t, s)$ *and* $G^{\circ}(t, s)$ *denote the Green's function for* (4.1) *and* (4.5) *respectively, then for all* $t, s \in [a, b]$, $G^{\circ}(t, s) = G(a + b - s, a + b - t)$.

We will sketch the proof as follows. For $f, g \in C[a, b]$, let $u(t) = \int_a^b G(t, s) f(s)\, ds$ and $v(t) = \int_a^b G^{\circ}(t, s)\, g(s)\, ds$. Then it is well known that $Lu = f$, $L^{\circ} v = g$ and u, v satisfies (4.1.b) and (4.5.b) respectively. By Theorem 4.2,

$$(v, Lu)_I = (v, f)_I = (u, g)_I = (u, L^{\circ} v)_I .$$

By the change of order of integration and the change of variables in the C-product $(u, g)_I$, we infer from the above equality that

$$0 = \int_a^b f(a + b - t) \cdot \int_a^b [G(a + b - s, a + b - t) - G^{\circ}(t, s)]\, g(s)\, ds\, dt .$$

The proof now follows from the separating property of the C-product.

By the uniqueness of the Green's function, it is not difficult to show the following [2, p. 193]

COROLLARY 4.5. *Suppose the boundary problem* (4.1) *is incompatible, then it is C-selfadjoint if and only if for all* $t, s \in [a, b]$, $G(t, s) = G^{\circ}(a + b - s, a + b - t)$.

COROLLARY 4.6. *If the boundary problem* (4.1) *is selfadjoint,* *C-selfadjoint and incompatible, then for all* $t, s \in [a, b]$, $G(t, s) = G(a+b-s, a+b-t) = G(a+b-t, a+b-s) = G(s, t)$ *and* $G(a+b-t, s) = G(t, a+b-s)$.

EXAMPLE 8. The boundary problem

$$-u'' = 0\,, \qquad u(a) = u(b) = 0$$

is selfadjoint, C-selfadjoint and incompatible. Its Green's function is given by

$$G(t, s) = \begin{cases} (b-t)(s-a)/(b-a)\,, & \text{for } a \le s \le t\,, \\ (t-a)(b-s)/(b-a)\,, & \text{for } t \le s \le b\,. \end{cases}$$

Clearly, the conclusions of Theorem 4.5 and Corollary 4.6 hold.

In view of the importance of the Green's function, we will interpret Theorem 4.5 and Corollary 4.6 geometrically. The Green's function $G(t, s)$ is a function defined on the rectangle $[a, b] \times [a, b]$ lying in the (t, s)-plane. If (4.1) is selfadjoint, $G(t, s) = G(s, t)$, that is, G is symmetric with respect to the line joining (a, a) and (b, b). On the other hand, if (4.1) is C-selfadjoint, then Theorem 4.5 says that G is symmetric with respect to the line joining (a, b) and (b, a). Clearly then, if (4.1) is both selfadjoint and C-selfadjoint, then the Green's function takes on the same value at the corners of any parallelogram lying in the rectangle $[a, b] \times [a, b]$ and having sides parallel to the diagonals of the rectangle. In particular, at the points $(t, a+b-s)$ and $(a+b-t, s)$, the Green's function has the same value. A continuous function $K(t, s)$ defined in $[a, b] \times [a, b]$ satisfying $K(t, s) = K(a+b-s, a+b-t)$ is said to be centrally symmetric. These functions occur naturally in the theory of rearrangements and convex bodies. The interested reader is referred to [4, 9].

As applications of our previous developement, consider the boundary problem

$$\text{(4.12)} \qquad \begin{cases} Lu = qu\,, & q \in C[a, b]\,, \\ Ek[u](a) + Fk[u](b) = 0\,. \end{cases}$$

THEOREM 4.7. *Suppose* (4.1) *is selfadjoint, C-selfadjoint and incompatible. If* $u(t)$ *is a nontrivial solution of* (4.12), *then*

$u(a+b-t)$ is a nontrivial solution of the following boundary problem

$$\begin{cases} Lv = q_1 v\,, \\ Ek[v](a) + Fk[v](b) = 0\,, \end{cases}$$

where $q_1(t) = q(a+b-t)$ for $t \in [a, b]$.

The proof amounts to veriying that $u(a+b-t)$ satisfies the integral equation

$$u(a+b-t) = \int_a^b G(t, s)\, q_1(s)\, u(a+b-s)\, ds\,,$$

where $G(t, s)$ is the Green's function for the associated homogeneous problem (4.1), and follows immediately from the change of variable and Corollary 4.6.

COROLLARY 4.8. *Suppose* (4.1) *is selfadjoint, C-selfadjoint and incompatible. Suppose further that q, in* (4.12) *is even. If $u(t)$ is a nontrivial solution of* (4.12), *so is $cu(t) + du(a+b-t)$ for any real numbers c and d.*

5. **Concluding remarks.** With minor modifications, our theory developed so far can be extended to include ordinary differential operators with complex coefficients. Furthermore, in view of the discussions in M. A. Naimark's book [10, Chapter 3], we can also develope a parallel theory for differential operators in spaces of vector functions, that is, operators of the form

$$LY = P_n(t)\, Y^{(n)} + P_{n-1}(t)\, Y^{(n-1)} + \cdots + P_0(t)\, Y = 0$$

where Y is a vector of m functions, and $P_0, P_1, \cdots, P_n$ are matrix functions. The possibility of a parallel theory for partial differential operators is yet to be investigated. However, for certain elliptic partial differential operators defined on symmetrical domains, the possibility seems quite real.

Special attention has been paid to the class of operators satisfying $L = L^\circ$, i.e. C-selfadjoint operators. They are sums of M_{2j} and M_{2j-1} as defined in Theorem 2.5. Had we defined C-selfadjoint operators by $L = -L^\circ$, however, we would have arrived at operators which are sums of operators similar to M_{2j} and M_{2j-1}, except that $p(t)$ is an *odd* function instead of even. All the theorems stated for C-selfadjoint operators can similarly be stated for this new class of operators, with appropriate sign changes of course.

Finally, we point out that eigenvalue problems have not been discussed at all. In particular, it seems worthwhile to study C-selfadjoint eigenvalue problems of the form

$$Lu = \lambda u ,$$
$$Ek[u](a) + Fk[u](b) = 0 ,$$

such that its corresponding homogeneous C-selfadjoint problem is incompatible. However, these problems are equivalent to eigenvalue problems of integral operators of the form

$$Tu(t) = \int_a^b G(t, s)\, u(s)\, ds ,$$

where $G(t, s)$ is centrally symmetric. But spectral properties of such operators defined in indefinite product spaces have been fairly well developed (see Bognár [1]). In particular, the well-known orthogonal expansion theorems associated with selfadjoint eigenvalue problems have been extended to general results associated with C-selfadjoint eigenvalue problems. These and others we shall therefore leave to the reader to explore.

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