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CONFORMAL MAPPINGS AND FIRST EIGENVALUE OF LAPLACIAN ON SURFACES

BY

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Abstract. In this note we give a simple relation between conformal mapping and the first eigenvalue of Laplacian for surfaces in Euclidean spaces.

1. **Statement of Main Theorem.** Let M be a compact Riemannian surface and Δ the Laplace-Beltrami operator acting on differentiable functions $C^\infty(M)$ on M. It is known that Δ is an elliptic operator. The operator Δ has an infinite sequence

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_p<\cdots\uparrow\infty \tag{1.1}$$

of eigenvalues. Let $V_p=\{f\in C^\infty(M):\Delta f=\lambda_p f\}$ be the eigenspace with eigenvalue λ_p. Then the dimension of V_p is finite, it is called the multiplicity of λ_p. Let $x:M\to E^m$ be an immersion of a surface M in E^m. Then the euclidean metric of E^m induces a Riemannian metric on M. In this paper we shall consider only the induced metric on M. As in [4], we shall call an immersion $x:M\to E^m$ to be of order p if all coordinate functions of $x=(x_1,\cdots,x_m)$ are in V_p, where $x_1,\cdots,x_m$ are the euclidean coordinates of x. In the following we shall denote by $\lambda_1(x)$ and $A(x)$ respectively the first eigenvalue λ_1 and the area of M with respect to the immersion x when it is necessary.

In this paper, we shall prove the following *conformal inequality* for λ_1.

THEOREM 1. *Let $x:M\to E^m$ be an imbedding of order 1 from a compact surface M in E^m. If φ is a conformal mapping of E^m with $A(x)=A(\varphi\circ x)$, then we have*

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(1.2) $$\lambda_1(\varphi \circ x) \leqq \lambda_1(x).$$

The equality holds when and only when φ *is a rigid motion of* E^m.

If c is a nonzero constant, then we have $\lambda_1(cx) = \lambda_1(x)/c^2$ and $A(cx) = c^2 A(x)$ for the similarity transformation cx. Thus the assumption on the area in Theorem 1 is necessary. Moreover the assumption is generic in the sense that if $A(x) \neq A(\varphi \circ x)$, then by choosing a suitable similarity transformation ψ of E^m we have $A(x) = A(\psi \circ \varphi \circ x)$.

It seems to the author that inequality (1.2) is the only conformal inequality we know so far for spectra. Some applications of Theorem 1 will be given in the last section. A typical example reads as follows: *If* M *is a cyclide of Dupin given by an inversion of an anchor ring in* E^3 *with circles of radii* a *and* b *satisfy* $a/b = \sqrt{2}$, *then* $\lambda_1 < 4\pi^2/A$.

2. **Proof of Theorem 1.** Let $x : M \to E^m$ be an imbedding of a compact surface M in E^m. Without loss of generality we may choose the center of gravity as the origin of E^m. Let $(x_1, \cdots, x_m)$ be the euclidean coordinates of E^m. Then we have $\int_M x_i \, dV = 0$. The minimal principle [1] then implies

(2.1) $$\int_M |dx_i|^2 dV \geqq \lambda_1 \int_M (x_i)^2 dV, \qquad i = 1, \cdots, m.$$

The equality holds if and only if each x_i is in V_1. On the other hand, since $|dx|^2 = \sum_{i=1}^m |dx_i|^2 = 2$, (2.1) gives

(2.2) $$2A(x) \geqq \lambda_1(x) \int_M |x|^2 dV.$$

Let $H = \frac{1}{2}$ trace σ be the mean curvature vector of x, σ the second fundamental form of x. Then we have [6]

(2.3) $$A(x) + \int_M (x \cdot H) \, dV = 0.$$

Thus (2.2), (2.3) and the Schwartz inequality imply

$$2A \int_M |H|^2 dV \geqq \lambda_1 \left(\int_M |x| \, |H| \, dV \right)^2$$
$$\geqq \lambda_1 \left(\int_M (x \cdot H)^2 dV \right)^2 \geqq \lambda_1 A^2.$$

Consequently, we obtain the following result of Reilly [7].

$$\int_M |H|^2 dV \geqq \frac{\lambda_1(x)}{2} A(x). \tag{2.4}$$

The equality holds if and only if $x-a$ is of order 1 for some vector a in E^m.

In the following, we denote by TMC(x) the total mean curvature of x, i. e.,

$$\mathrm{TMC}(x) = \int_M |H|^2 dV.$$

Suppose that $x: M \to E^m$ is an imbedding of order 1. Then we have

$$\mathrm{TMC}(x) = \frac{\lambda_1(x)}{2} A(x). \tag{2.5}$$

If φ is a conformal mapping of E^m into E^m then we have [3]

$$\mathrm{TMC}(\varphi \circ x) = \mathrm{TMC}(x). \tag{2.6}$$

Combining this with (2.4) and (2.5), we find

$$\lambda_1(\varphi \circ x) A(\varphi \circ x) \leqq \lambda_1(x) A(x). \tag{2.7}$$

The inequality holds if and only if $\varphi \circ x - b$ is of order 1 for some vector b in E^m. In particular, if $A(x) = A(\varphi \circ x)$, (2.7) gives

$$\lambda_1(\varphi \circ x) \leqq \lambda_1(x). \tag{2.8}$$

If the equality of (2.8) holds, then $\varphi \circ x - b$ is also of order 1 for some vector b in E^m. By using a translation on E^m, we may also assume that the center of gravity of $\varphi \circ x$ is the origin too. In this case, $b=0$, and $\varphi \circ x$ is of order 1. Consequently, we have

$$\Delta x = \lambda_1 x, \qquad \bar{\Delta}(\varphi \circ x) = \lambda_1(\varphi \circ x), \tag{2.9}$$

where $\lambda_1 = \lambda_1(x) = \lambda_1(\varphi \circ x)$ and Δ and $\bar{\Delta}$ are the Laplace-Beltrami operators on M with respect to x and $\varphi \circ x$, respectively. From (2.9) and a theorem of Takahashi [8], we see that M is imbedded by x and $\varphi \circ x$ into the same hypersphere $S^{m-1}(r)$ of radius $r = \sqrt{2/\lambda_1}$ as minimal surfaces.

Now, by a result of Haantjes [5], we know that conformal mapping on E^m are generated by translations, rotations, homothetic transformations and inversions centered at a fixed point. Since the centers of gravity of x and $\varphi \circ x$ are assumed to be at the same point 0, the conformal mapping φ is free of translation. Moreover, since M is imbedded both by x and $\varphi \circ x$ into the same hypersphere $S^{m-1}(r)$, φ is free of homothetic transformations (except the identity transformation). On the other hand, inversions centered at 0 are given in the following form:

$$\bar{x} = \frac{c^2}{(x \cdot x)} x$$

for nonzero constants c. Thus inversions centered at 0 always carry a hypersphere of radius r into a hypersphere of radius c^4/r^2. In our case, since both surfaces given by x and $\varphi \circ x$ lie in the same hypersphere $S^{m-1}(r)$. Thus φ is free of inversions too. Consequently, φ is given only by a rotation. Conversely, because the area and the spectrum of a surface are invariant under rigid motions (generated by translations and rotations), if φ is a rigid motion, the equality of (1.2) holds.

REMARK. Theorem 1 shows that the estimates on total mean curvature for surfaces in E^m given in [2, 7] are weak in general.

3. **Applications.** In this section we shall give the following applications of Theorem 1.

Let $S^1(1)$ be the unit circle in a plane E^2. Then the product surface $T^2 = S^1(1) \times S^1(1)$ is a flat surface in E^4 with area $A = 4\pi^2$ and $\lambda_1 = 1$ [1]. A surface in $E^m \supset E^4$ is called a *conformal Clifford torus* if it is the image of the Clifford torus under a conformal mapping of E^m. The anchor ring in E^3 given by

$$((\sqrt{2} + \cos u) a \cos v, (\sqrt{2} + \cos u) a \sin v, a \sin v)$$

$$0 \leqq u < 2\pi, \qquad 0 \leqq v < 2\pi,$$

is among the class of conformal Clifford tori. It is easy to see that the Clifford torus in E^4 is of order 1. There exists no conformal Clifford torus of order 1 in E^3. Theorem 1 implies the following.

THEOREM 2. *Let M be a conformal Clifford torus in E^m ($m \geqq 3$) with area $4\pi^2$. Then we have*

$$\lambda_1 \leqq 1. \tag{1.4}$$

The equality holds if and only if M is a Clifford torus.

Let (x, y, z) be the Euclidean coordinates of E^3 and $(u^1, u^2, u^3, u^4, u^5)$ be the Euclidean coordinates of E^5. We consider the mapping defined by

$$u^1 = \frac{1}{3}yz, \qquad u^2 = \frac{1}{3}zx, \qquad u^3 = \frac{1}{3}xy,$$

$$u^4 = \frac{1}{6}(x^2 - y^2), \qquad u^5 = \frac{1}{6\sqrt{3}}(x^2 + y^2 - 2z^2).$$

This defines an isometric immersion of $S^2(1)$ into $S^4(1/\sqrt{3})$ as a minimal surface. Two points (x, y, z) and $(-x, -y, -z)$ of $S^2(1)$ are mapped into the same point of $S^4(1/\sqrt{3})$ and this mapping defines an imbedding of the real projective plane into $S^4(1/\sqrt{3}) \subset E^5$. This real projective plane imbedded in E^5 is called the *Veronese surface*. It is known that Veronese surface satisfies $A = 2\pi$ and $\lambda_1 = 6$ [1]. A surface in E^m $(m \geqq 5)$ is called a *conformal Veronese surface* if it is the image of the Veronese surface under a conformal mapping of E^m. There is no conformal Veronese surface of order 1 in E^4. From Theorem 1 we have

THEOREM 3. *Let M be a conformal Veronese surface with area 2π in E^m. Then we have*

$$\lambda_1 \leqq 6. \tag{1.5}$$

The equality holds if and only if M is a Veronese surface.

REFERENCES

1. M. Berger, P. Gauduchon and E. Mazet, *Le Spectre d'une variété Riemannienne*, Lecture Notes in Math, no. 194, Springer-Verlag, Berlin, 1971.

2. D. Bleecker and J. Weiner, *Extrinsic bounds on λ_1 of Δ on a compact manifold*, Comm. Math. Helv. **51** (1976), 601-609.

3. B.Y. Chen, *An invariant of conformal mappings*, Proc. Amer. Math. Soc. **40** (1973), 563-564.

4. _______, *On total curvature of immersed manifolds*, IV, *Spectrum and total mean curvature*, Bull. Inst. Math. Acad. Sinica **7** (1979), 301-311.

5. J. Haantjes, *Conformal representations of an n-dimensional Euclidean space with a non-definite fundamental form on itself*, Proc. Kon. Ned. Akad. Amsterdam **40** (1937), 700-705.

6. C. C. Hsiung, *Isoperimetric inequalities for 2-dimensional Riemannian manifolds with boundary*, Ann. of Math. **73** (1961), 213–220.

7. R.C. Reilly, *On the first eigenvalues of the Laplacian for compact submanifolds of Euclidean space*, Comm. Math. Helv. **52** (1977), 525–533.

8. T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

9. J. White, *A global invariant of conformal mapping in space*, Proc. Amer. Math. Soc. **38** (1973), 162–164.

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