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ON THE UNIFORM INTEGRABILITY OF $|b^{-1/p} W_{M_b}|^p$, $0<p<2$

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Abstract. Let $Y_1, Y_2, \cdots$ be a sequence of ramdom variables with $E|Y_1|^p<\infty$ for some $0<p<2$. Let $W_n=Y_1+\cdots+Y_n$. Pyke and Root [6] and later Chow [3] proved the L_p-convergence for $n^{-1/p}W_n$ under some conditions. In this paper we consider a family of stopping times $\{M_b\}_{b\geq 1}$ and prove the unifom integrability of $|b^{-1/p}W_{M_b}|^p$.

Let $Y_1, Y_2, \cdots$ be a sequence of i.i.d. random variables with $E|Y_1|^p<\infty$ for some $0<p<2$, $W_n=Y_1+\cdots+Y_n$. Pyke and Root [6] prove that $E|W_n/n^{1/p}|^p\to 0$ as $n\to\infty$ provided a further condition $EY_1=0$ is assumed when $1\leq p<2$. Later Chow [3] relaxes the condition of i.i.d. and obtains a significant generalization. In this paper we will generalize the result of Pyke and Root in a different direction. Consider a family of stopping times $\{M_b\}_{b\geq 1}$ with respect to $F_n=\sigma(Y_1,\cdots,Y_n)$. We prove that under certain hypothesis $\{|W_{M_b}/b^{1/p}|^p\}_{b\geq 1}$ is uniformly integrable. This result together with the corresponding law of large numbers (see e.g. [5]) obviously imply the L_p-convergence.

To prove the main theorem we need the following lemma which is a consequence of Burkholder's inequality [1], (see also [4]).

LEMMA. *Let $Y_1, Y_2, \cdots$ be a sequence of independent random variables with mean* 0. *Assume that* $\sup_n E|Y_n|^p\leq \mu_p<\infty$. *Let N be a stopping time with respect to $F_n=\sigma(Y_1,\cdots,Y_n)$. Let J be a positive integer. Then we have*

$$\textit{For}\quad 1<p<2,\qquad E\left|\sum_{J+1}^{N} Y_i\right|^p\leq C_p\cdot\mu_p E(N-J)^+,$$

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where C_p *is a constant depending only on* p.

THEOREM. *Let* $Y_1, Y_2, \cdots$ *be a sequence of independent random variables. Assume that* $\{|Y_n|^p\}$ *is uniformly integrable for some* $0<p<2$. *Let* $\{M_b, b \geq b_0 \geq 1\}$ *be a family of stopping times with respect to* $F_n = \sigma(Y_1, Y_2, \cdots, Y_n)$. *Assume that* $\{M_b/b\}$ *is uniformly integrable. Let* $W_{M_b} = \sum_1^{M_b} Y_i$. *Then*

(1) *if* $0<p\leq 1$, *then* $\{|W_{M_b}|^p/b\}$ *is uniformly integrable.*

(2) *if* $1<p<2$ *and* $EY_i=0$ *for all* i, *then* $\{|W_{M_b}|^p/b\}$ *is uniformly integrable.*

Proof. Let K be a positive number. Let $J_b = [Kb]$, $M_b' = \min\{M_b, J_b\}$.

(1) $0<p\leq 1$. Let

$$Y_i' = Y_i I_{[|Y_i|\leq K]},$$

$$Y_i'' = Y_i - Y_i',$$

$$W'_{M_b'} = \sum_1^{M_b'} Y_i',$$

$$W''_{M_b'} = \sum_1^{M_b'} Y_i''.$$

Then

$$W_{M_b} = W'_{M_b'} + W''_{M_b'} + \sum_{J_b+1}^{M_b} Y_i.$$

We are going to prove that $|W'_{M_b'}|^p/b$ is uniformly integrable for any given K and that both $E(|W''_{M_b'}|^p/b)$ and $E(|\sum_{J_b+1}^{M_b} Y_i|^p/b)$ tend to 0 uniformly in b as K goes to infinity. These will imply the uniform integrability of $\{|W_{M_b}|^p/b\}$.

The uniform integrability of $|W'_{M_b'}|^p/b$ comes from the fact that it is uniformly bounded.

A little calculation with Wald's equation in mind gives

$$E\frac{1}{b}|W''_{M_b'}|^p \leq E\left(\frac{M_b}{b}\right)\cdot(\sup E|Y_i''|^p) \tag{1}$$

and

$$E\frac{1}{b}\left|\sum_{J_b+1}^{M'_b} Y_i\right|^p \leq E\left(\frac{M_b}{b}-K\right)^+\cdot(\sup E|Y_i|^p). \tag{2}$$

The hypotheses of the theorem imply that $\{E(M_b/b)\}$ is uniformly bounded and that both $E(M_b/b-K)^+$ and $\sup E|Y_i''|^p$ tend to 0 as K goes to infinity. Hence (1) and (2) tell us respectively that $E(|W''_{M'_b}|^p/b)$ and $E(|\sum_{J_b+1}^{M_b} Y_i|^p/b)$ tend to 0 uniformly in b as K goes to infinity.

Therefore, the proof is completed for this part.

(2) $1<p<2$. Let

$$Y_i' = Y_i I_{[|Y_i|\le K]} - E(Y_i I_{[|Y_i|\le K]}),$$

$$Y_i'' = Y_i - Y_i',$$

$$W'_{M'_b} = \sum_1^{M'_b} Y_i',$$

$$W''_{M'_b} = \sum_1^{M'_b} Y_i''.$$

Then

$$W_{M_b} = W'_{M'_b} + W''_{M'_b} + \sum_{J_b+1}^{M_b} Y_i.$$

The procedure of this proof is the same as that for the case $0<p\le 1$.

In fact, we have

(3) $E|(1/b^{1/p})W'_{M'_b}|^{\tilde p} \le (b/b^{\tilde p/p}) C_{\tilde p}\cdot(\sup E|Y_i'|^{\tilde p})$. $E(M_b/b)$, where $p<\tilde p<2$.

(4) $E(1/b)|W''_{M'_b}|^p \le C_p\cdot(\sup E|Y_i''|^p)E(M_b/b)$,

(5) $E(1/b)|\sum_{J_b+1}^{M_b} Y_i|^p \le C_p\cdot(\sup E|Y_i|^p)E(M_b/b-K)^+$.

The meaning of the notations in the above inequalities is explained in the Lemma.

With a similiar reasoning as in the case $0<p\le 1$, we know that (3), (4) and (5) imply the uniform integrability of $|W_{M_b}|^p/b$. The proof is therefore completed.

REMARK. Yu [7] proved that if $p\ge 2$, $EY_i=0$, then the uniform integrability of $\{(M_b/b)^{p/2}\}$ implies the uniform integrability of $\{|W_{M_b}/\sqrt{b}|^p\}$. The proof of which makes use of the fancy inequality of Burkholder—Davis—Gundy, see [2].

A related problem is also studied in [8].

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