BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 7, Number 4, December 1979

ON REARRANGEMENTS OF SERIES(1)

BY

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Abstract. This paper studies the effects upon the rearrangement of series by a permutation in terms of its run number.

1. The best known result on the rearrangements of series is Riemann's theorem which states that the terms of a conditionally convergent real series may be so rearranged that the resulting series will diverge to ∞, diverge to $-\infty$, converge to any preassigned real number, or oscillate between any two preassigned values, finite or infinite. See [3, p. 328]. Most subsequent works on rearrangements of series (e.g. [2]) are in the direction to allow the terms of the series taken from some vector space V, and to prove that under suitable conditions the sums of the convergent rearranged series will fill up some linear subspace of V.

In this paper we approach the problem from a different angle. Instead of considering a given series and its various rearrangements, we consider a given rearrangement to see how it acts on various series. The only result in this direction known to us is a theorem in [1] which we shall state as Theorem 1.

First let us introduce our terminology. By a *permutation* σ on the set N of positive integers we mean a one-one map of N onto itself. The series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is called the *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$ by σ. σ is called *convergence-preserving* if $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent whenever $\sum_{n=1}^{\infty} a_n$ is. σ is called *sum-preserving* if whenever both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{\sigma(n)}$ are convergent, they have the same sum.

Let $k \leq l$ be two positive integers. We denote the set $\{n \in N : k \leq n \leq l\}$ by $[k, l]$ and call it a *segment* in N. Let A

Received by the editors February 22, 1979 and in revised form June 15, 1979.
(1) Research sponsored by the National Science Council of the Republic of China.

be a finite set of positive integers. By a *run* in A we mean a maximal segment contained in A, maximality being with respect to set-theoretic inclusion. The runs in A form a partition of A. The total number of runs in A, i.e. the number of classes in this partition, is called the *run number* of A. Thus the set $\{3, 1, 9, 4, 6, 8, 7\}$ has 3 runs: $[1, 1]$, $[3, 4]$, $[6, 9]$. Its run number is 3.

We shall also denote the set $[k+1, l] = \{n \in N : k < n \leq l\}$ by $(k, l]$. The number of elements of a set A will be written as $\sharp A$.

If σ is a permutation of N, we shall denote by $\rho_n = \rho_n(\sigma)$ the run number of the set $\sigma([1, n])$. The sequence $\{\rho_n\}$ will be called the *sequence of run numbers* of σ.

With these notations we can restate Agnew's theorem in [1] in a slightly different form:

THEOREM 1. *A permutation σ on N is convergence-preserving if and only if the sequence $\{\rho_n\}$ of the run numbers of σ is bounded.*

Agnew's original proof depends on a theorem of Toeplitz on regular matrix transformations, and is not constructive. Also it does not tell what we can expect of σ in case it has an unbounded sequence of run numbers. Our Theorem 2 will be a strengthened form of the "if" part. It is rather elementary. We include it only for the sake of completeness. Our Theorem 3 will be a strengthened form of the "only if" part, and its proof will be constructive.

THEOREM 2. *A permutation σ on N with a bounded sequence of run numbers is both convergence-preserving and sum-preserving.*

THEOREM 3. *If the sequence $\{\rho_n\}$ of run numbers of a permutation σ on N is unbounded, then for any two non-negative numbers a and b there is a series $\sum_{n=1}^{\infty} a_n$ which converges to 0 while the partial sums of the rearranged series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ oscillates between a and $-b$.*

Actually this result is the best we can expect of σ; because there is a permutation σ which does not behave any better than this, as the following theorem shows:

THEOREM 4. *There exists a permutation σ of N with its sequence of run numbers unbounded such that for any series $\sum_{n=1}^{\infty} a_n$ which converges to* 0, *the lower limit of the partial sums of $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is always non-positive and the upper limit of those partial sums is always non-negative.*

What went wrong in this example is that the lower limit of $\{\rho_n(\sigma)\}$ is 1. In fact we have

THEOREM 5. *Let σ be a permutation on N and $\{\sigma_n\}$ be its sequence of run numbers. Then the following are equivalent*:

A. $\lim_{n\to\infty} \rho_n = \infty$.

B. *There exists a series $\sum_{n=1}^{\infty} a_n$ converging to* 0 *such that the lower limit of the partial sums of the rearranged series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is* 1.

The following theorem shows that the condition B in Theorem 5 is hard to improve:

THEOREM 6. *An example may be constructed to show that a permutation σ with*

$$\lim_{n\to\infty} \rho_n(\sigma) = \infty$$

may be sum-preserving.

Based on the construction of Theorem 6, we may construct another example to show

THEOREM 7. *A permutation τ on N may be sum-preserving even if both*

$$\lim_{n\to\infty} \rho_n(\tau) = \infty$$

and

$$\lim_{n\to\infty} \rho_n(\tau^{-1}) = \infty .$$

An unsolved problem left in this research is to find a necessary and sufficient condition for a permutation on N to be sum-preserving.

We thank Mrs. Wai-fong Chuan who informed us of the existence of the Reference [1] so that we know that Theorem 1 is not our creation. We have subsequently revised this paper.

2. **Proof of Theorem** 2. Assume that $\sum a_n = s$ and $\rho_n \leq M$ for all n. We want to show that $\sum a_{\sigma(n)} = s$. Given $\varepsilon > 0$. Let n_0 be a sufficiently large integer such that

$$\left| s - \sum_{j=1}^{n} a_j \right| < \frac{\varepsilon}{M}$$

whenever $n \geq n_0$ and

$$\left| \sum_{j=l}^{m} a_j \right| < \frac{\varepsilon}{M}$$

whenever $m \geq l \geq n_0$. Choose an integer n_1 such that $\sigma([1, n_1]) \supset [1, n_0]$. For $n > n_1$, let $\sigma([1, n]) = \bigcup_{i=1}^{\rho_n} R_i$, where R_i are the runs of $\sigma([1, n])$, arranged from left to right. We have

$$\sum_{k=1}^{n} a_{\sigma(k)} = \sum_{k \in R_1} a_k + \sum_{j=2}^{\rho_n} \sum_{k \in R_j} a_k .$$

Hence

$$\left| s - \sum_{k=1}^{n} a_{\sigma(k)} \right| \leq \left| s - \sum_{k \in R_1} a_k \right| + \sum_{j=2}^{\rho_n} \left| \sum_{k \in R_j} a_k \right| < M \cdot \frac{\varepsilon}{M} = \varepsilon .$$

Therefore

$$\sum a_{\sigma(k)} = s . \qquad \text{Q. E. D.}$$

3. **Proof of Theorem** 3. For each n, let $R(n, 1), R(n, 2), \cdots, R(n, \rho_n)$ be the runs in $\sigma([1, n])$, arranged from left to right. Since $\{\rho_n\}$ is unbounded, we can choose an increasing sequence of positive integers $n_1, n_2, \cdots$, such that

(3.1) $1 \in \sigma([1, n_1])$,

(3.2) $2 < \rho_{n_1} < \rho_{n_2} < \cdots$,

(3.3) $R(n_{j+1}, 1) \supset \sigma([1, n_j])$ for all $j \in N$.

For each j, let

$$P_j = \{\mathrm{Max}\, R(n_j, i) : i = 2, 3, \cdots, \rho_{n_j}\} ,$$

$$Q_j = \{\mathrm{Min}\, R(n_j, i) - 1 : i = 2, 3, \cdots, \rho_{n_j}\} .$$

From (3.3) we see that $Q_1, P_1, Q_2, P_2, \cdots$ are pairwise disjoint. For each positive integer k define

$$a_k = \begin{cases} a/(\rho_{n_j}-1) & \text{if } j \text{ is odd and } k \in P_j, \\ -a/(\rho_{n_j}-1) & \text{if } j \text{ is odd and } k \in Q_j, \\ -b/(\rho_{n_j}-1) & \text{if } j \text{ is even and } k \in P_j, \\ b/(\rho_{n_j}-1) & \text{if } j \text{ is even and } k \in Q_j, \\ 0 & \text{if otherwise.} \end{cases}$$

Then

$$\left|\sum_{k=1}^{\infty} a_k\right| \leq (a+b)/(\rho_{n_j}-1)$$

if $n \notin R(n_j, 1)$. Hence $\sum a_n = 0$. On the other hand it can be verified that

$$\sum_{k=1}^{n_j} a_{\sigma(k)} = \begin{cases} a & \text{if } j \text{ is odd}, \\ -b & \text{if } j \text{ is even}, \end{cases}$$

while

$$-b \leq \sum_{k=1}^{n} a_{\sigma(k)} \leq a$$

for all n. Therefore the series $\sum a_{\sigma(k)}$ diverges (unless $a = b = 0$) with its partial sums oscillating between $-b$ and a.

4. **Proof of Theorem** 4. Let $m_n = (n-1)n$. Then m_n is an increasing sequence of even numbers with $m_{n+1} - m_n = 2n$. We define $\sigma : N \to N$ as follows: for $n = 0, 1, 2, \cdots$ we define

$$\sigma(m_n + k) = \begin{cases} m_n + 2k & \text{if } 1 \leq k \leq n, \\ m_n + 2(k-n) - 1 & \text{if } n < k \leq 2n. \end{cases}$$

It may be readily verified that for this σ, $\rho_{m_n+n} = n+1$. Hence $\{\rho_n\}$ is unbounded and σ is not convergence-preserving. On the other hand $\rho_{m_n} = 1$. Hence if $\sum_{k=1}^{\infty} a_k = 0$, we have

$$\lim \sum_{k=1}^{m_n} a_{\sigma(k)} = \lim \sum_{k=1}^{m_n} a_k = 0.$$

This shows that 0 lies between the lower and the upper limits of the partial sums of $\sum a_{\sigma(n)}$. Q.E.D.

Another point worth noting is that the set of all convergence-preserving permutations is not a group. Thus, in the example of the above proof, the inverse of σ is given by

$$\sigma^{-1}(m_n + l) = \begin{cases} m_n + k & \text{if } l = 2k, \\ m_n + n + k & \text{if } l = 2k-1, \end{cases}$$

where $l \in [1, 2n]$. A moment's reflection reveals that the run numbers of $\sigma^{-1}([1, n])$ is either 1 or 2. Thus σ^{-1} is convergence-preserving while its inverse σ is not.

5. **Proof of Theorem** 5. B$\Rightarrow$A. Let p be an arbitrary positive integer. Since $\sum a_n = 0$, there is an m such that

$$\left|\sum_{k=1}^{n} a_k\right| < \frac{1}{p}$$

whenever $n \geq m$, and

$$\left|\sum_{k=r}^{s} a_k\right| < \frac{1}{p}$$

whenever $s \geq r \geq m$. Since the lower limit of the partial sums of $\sum a_{\sigma(n)}$ is 1, there is an m' such that

$$\sigma([1, m']) \supset [1, m] \quad \text{and} \quad \sum_{k=1}^{n} a_{\sigma(k)} \geq \frac{1}{2}$$

whenever $n \geq m'$. Now assume that $n \geq m'$. If $R_1, R_2, \cdots, R_{\rho_n}$ are the runs in $\sigma([1, n])$, then by our choice of m,

$$\left|\sum_{k \in R_j} a_k\right| < \frac{1}{p}$$

for each $j = 1, 2, \cdots, \rho_n$. Hence

$$\frac{1}{p}\rho_n \geq \sum_{j=1}^{\rho_n} \left|\sum_{k \in R_j} a_k\right| \geq \sum_{k=1}^{n} a_{\rho(k)} \geq \frac{1}{2},$$

so $\rho_n \geq p/2$ whenever $n \geq m'$. This proves that $\lim \rho_n = \infty$.

A$\Rightarrow$B. Since each additional term $\sigma(n+1)$ can alter the run number ρ_n at most by 1, so if we define $\varepsilon(1) = 1$ and

$$\varepsilon(n) = \rho_n - \rho_{n-1}, \qquad n > 1,$$

the value of $\varepsilon(n)$ is either 1, or 0, or -1.

(5.1) In the nonzero terms of the sequence $\{\varepsilon(\sigma^{-1}(n))\}_n$, 1 and -1 appear alternately with 1 appearing first.

Indeed, we note that

$$\varepsilon(\sigma^{-1}(1)) = \begin{cases} 1 & \text{if } \sigma^{-1}(1) < \sigma^{-1}(2), \\ 0 & \text{if } \sigma^{-1}(1) > \sigma^{-1}(2), \end{cases}$$

and for $n > 1$,

$$\varepsilon(\sigma^{-1}(n)) = \begin{cases} 1 & \text{if } \sigma^{-1}(n) < \sigma^{-1}(n-1) \text{ and } \sigma^{-1}(n) < \sigma^{-1}(n+1)\,, \\ -1 & \text{if } \sigma^{-1}(n) > \sigma^{-1}(n-1) \text{ and } \sigma^{-1}(n) > \sigma^{-1}(n+1)\,, \\ 0 & \text{if } \sigma^{-1}(n) \text{ lies between } \sigma^{-1}(n-1) \text{ and } \sigma^{-1}(n+1). \end{cases}$$

(5.1) follows from these observations.

Now let $N_+ = \varepsilon^{-1}(1)$, $N_0 = \varepsilon^{-1}(0)$ and $N_- = \varepsilon^{-1}(-1)$. Since $\lim \rho_n = \infty$, both N_+ and N_- are infinite sets. For each $n \in \sigma(N_+)$, let

$$F(n) = \mathrm{Min}\,\{k \in \sigma(N_-) : k > n\}\,.$$

By (5.1), F is a bijection from $\sigma(N_+)$ onto $\sigma(N_-)$. If we put $S = \sigma^{-1} \circ F \circ \sigma$, we have

(5.2) $S : N_+ \to N_-$ is a bijection.

Furthermore, from the observations made in the proof of (5.1), we can derive

(5.3) $S(n) > n$ for all $n \in N_+$.

In the sequel we shall construct a series $\sum a_n$ such that $\sum a_n = 0$ and $\liminf_n \sum_{k=1}^{n} a_{\sigma(k)} = 1$. In the definition of a_n we always require that

(C1) $a_{\sigma(n)} = -a_{\sigma(S^{-1}(n))}$ for all $n \in N_-$,

(C2) $a_{\sigma(k)} = 0$ for all $k \in N_0$.

So we need only to define $a_{\sigma(m)}$ for $m \in N_+$. It will be clear from the definition that $a_{\sigma(m)} \geq 0$ for all $m \in N_+$ and $\lim a_k = 0$.

For any finite subset I of N, let $\nu(I) = \sum_{n \in I} \varepsilon(n)$. If l, $k \in N$, let

$$\lambda(l, k) = \max\,\{n \geq l : \nu([l, n]) = k\}\,.$$

(5.4) $\lambda(l, k)$ is a well-defined positive integer.

In fact we note that (1) $\nu([l, n])$ changes its value at most by 1 when n is replaced by $n+1$, (2) $\nu([l, l]) = \varepsilon(l) = 1$ or 0 or -1, and (3) $\lim_n \nu([l, n]) = \infty$. (5.4) follows from these facts.

As immediate consequences of the definition of $\lambda(l, k)$, we also have

(5.5) If $m > \lambda' = \lambda(l, k)$, then $\nu((\lambda', m]) > 0$.

(5.6) $\lambda(l, k+1) > \lambda(l, k)$.

Next we introduce two construction procedures, called Procedures I and II respectively. At present we only describe what we can achieve with these procedures, leaving the detailed constructions toward the end of the proof.

PROCEDURE I. *Assume that we are given* $\lambda' = \lambda(l, \alpha')$ *and that* $a_{\sigma(j)}$ *have been defined for all* $j \in [1, \lambda']$ *satisfying*

$$\sum_{j=1}^{\lambda'} a_{\sigma(j)} = 1$$

as well as the conditions (C1) *and* (C2). *The present procedure constructs a* $\lambda'' = \lambda(\lambda', \alpha'')$ *and defines* $a_{\sigma(j)}$ *for all* $j \in (\lambda', \lambda'']$ *such that*

(0) $S(N_+ \cap [1, \lambda']) \subset [1, \lambda'']$;

(1) $\sum_{j=1}^{\lambda''} a_{\sigma(j)} = 2$;

(2) (C1) *and* (C2) *are satisfied*;

(3) $\sum_{j=1}^{d} a_{\sigma(j)} \geq 1$ *whenever* $d \in [\lambda', \lambda'']$;

(4) $\mathrm{Max}_{j \in (\lambda', \lambda'']} |a_{\sigma(j)}| \leq \mathrm{Max}_{j \in [1, \lambda'] \cap N_+, S(j) > \lambda'} |a_{\sigma(j)}|$.

PROCEDURE II. *Assume that we are given* $\lambda'' = \lambda(l, \alpha'')$ *and that* $a_{\sigma(j)}$ *have been defined for all* $j \in [1, \lambda'']$ *satisfying*

$$\sum_{j=1}^{\lambda''} a_{\sigma(j)} = 2$$

as well as the conditions (C1) *and* (C2). *This procedure constructs a* $\lambda' = \lambda(\lambda'', \alpha')$ *and defines* $a_{\sigma(j)}$ *for all* $j \in (\lambda'', \lambda']$ *such that*

(0′) $S(N_+ \cap [1, \lambda'']) \subset [1, \lambda']$;

(1′) $\sum_{j=1}^{\lambda'} a_{\sigma(j)} = 1$;

(2′) (C1) *and* (C2) *are satisfied*;

(3′) $\sum_{j=1}^{d} a_{\sigma(j)} \geq 1$ *whenever* $d \in [\lambda'', \lambda']$;

(4′) $\mathrm{Max}_{j \in (\lambda'', \lambda'] \cap N_+} |a_{\sigma(j)}| \leq \frac{1}{2} \mathrm{Max}_{j \in [1, \lambda''] \cap N_+, S(j) > \lambda''} |a_{\sigma(j)}|$.

With these procedures available, we construct the series $\sum a_{\sigma(n)}$ in the following manner: We start with $\lambda_1 = \lambda(1, 1)$ and define

$$a_{\sigma(j)} = \varepsilon(j) \qquad \text{for } j \in [1, \lambda_1].$$

Clearly

$$\sum_{j=1}^{\lambda_1} a_{\sigma(j)} = 1$$

and (C1) and (C2) are satisfied. By applying Procedure I and Procedure II alternately, we obtain an infinite sequence of integers

$$0 < \lambda_1 < \lambda_2 < \cdots$$

and a sequence of numbers $a_{\sigma(j)}$ such that

(1) $\lim_{j\to\infty} a_{\sigma(j)} = 0$;

(2) $a_{\sigma(j)} = -a_{\sigma(S(j))}$ if $j \in N_+$;

(3) $a_{\sigma(j)} = 0$ if $j \in N_0$;

(4) $\sum_{j=1}^{d} a_{\sigma(j)} \geq 1$ if $d \geq \lambda_1$;

(5) $\sum_{j=1}^{\lambda_k} a_{\sigma(j)} = 1$ for all odd k.

From (1), (2) and (3) we know that $\sum a_n = 0$. From (4) and (5) we know that

$$\liminf_{n\to\infty} \sum_{j=1}^{n} a_{\sigma(j)} = 1 .$$

This will complete the proof of Theorem 5 if we carry out the constructions in Procedures I and II, which we do presently:

Procedure I. By (5.6) we can choose an α'' so large that if we put $\lambda'' = \lambda(\lambda', \alpha'')$, then

$$[1, \lambda''] \supset S(N_+ \cap [1, \lambda'])$$

and

$$\frac{1}{\alpha''} \leq \mathrm{Max}_{j\in[1,\lambda']\cap N_+, S(j)>\lambda'} |a_{\sigma(j)}| .$$

Set

$$P = N_+ \cap (\lambda', \lambda''] ,$$
$$X = N_- \cap (\lambda', \lambda''] ,$$
$$Y = \{n \in X : S^{-1}(n) > \lambda'\} ,$$
$$Z = X\backslash Y = \{n \in X : S^{-1}(n) \leq \lambda'\} ,$$
$$B = S^{-1}(Y) \subset P ,$$
$$C = \text{the set of the first } \#Z \text{ elements in } P\backslash B ,$$
$$A = B \cup C .$$

By (5.5), $(\#P) - (\#X) = \nu((\lambda', \lambda'']) > 0$. Also, by (5.2), $\#B = \#Y$. Thus $\#(P\backslash B) > \#(X\backslash Y) = \#Z$, and C is well defined. By definition, $\#A = \#X$; let T be the order-preserving map of X onto A.

(5.7) $T(X) < x$ for all $x \in X$.

In fact (5.7) follows immediately if we can show that for every $x \in X$,

(5.8) $\sharp(A \cap (\lambda', x]) \geq \sharp(X \cap (\lambda', x])$.

Now if $\sharp((P \backslash B) \leq (\lambda', x]) \leq \sharp Z$, then

$$(P \backslash B) \cap (\lambda', x] \subset C \cap (\lambda', x] .$$

Thus

$$\begin{aligned} \sharp(A \cap (\lambda', x]) &= \sharp(C \cap (\lambda', x]) + \sharp(B \cap (\lambda', x]) \\ &\geq \sharp(P \cap (\lambda', x]) \geq \sharp(X \cap (\lambda', x]) , \end{aligned}$$

the last inequality follows from

$$\sharp(P \cap (\lambda', x]) - \sharp(X \cap (\lambda', x]) = \nu((\lambda', x]) > 0 .$$

If $\sharp(P \backslash B) \cap (\lambda', x]) > \sharp Z$, then

$$C \subset (\lambda', x] .$$

Thus

$$\begin{aligned} \sharp(A \cap (\lambda', x]) &= \sharp Z + \sharp(B \cap (\lambda', x]) \\ &\geq \sharp Z + \sharp(Y \cap (\lambda', x]) \geq \sharp(X \cap (\lambda', x]) . \end{aligned}$$

This proves (5.8).

We note that (5.8) is actually the possibility condition of the marriage problem as was mentioned in [4, §10].

For each $z \in Z$ consider the finite sequence of positive integers $v_1, v_2, \cdots, v_{2l+1}$ defined inductively by

$$\begin{array}{ll} v_1 = z , & v_2 = T(v_1) , \\ v_3 = S(v_2) , & v_4 = T(v_3) , \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ v_{2l+1} = S(v_{2l}) , & \end{array}$$

with $v_1, v_3, \cdots, v_{2l-1} \in X$, $v_2, v_4, \cdots, v_{2l} \in A$. We carry on the process until $v_{2l+1} > \lambda''$. Let

$$V(z) = \{v_1, v_2, \cdots, v_{2l}\} .$$

Remember that $a_{\sigma(z)} = -a_{\sigma(S^{-1}(z))} \leq 0$ has already been defined for $z \in Z$, and by (C1) and (C2),

$$\sum_{z \in Z} a_{\sigma(z)} = -\sum_{j=1}^{\lambda_1} a_{\sigma(j)} = -1 .$$

Now for $j \in (\lambda', \lambda'']$ we define $a_{\sigma(j)}$ as follows:

(C3) $a_{\sigma(v)} = \varepsilon(v)\,|a_{\sigma(z)}|$, $v \in V(z)$, $z \in Z$.

(C4) $a_{\sigma(j)} = 1/\alpha''$, $j \in P\backslash A$.

(C5) $a_{\sigma(j)} = 0$ otherwise.

If $d \in (\lambda', \lambda'']$, then

$$\sum_{i=1}^{d} a_{\sigma(j)} = \sum_{j=1}^{\lambda'} a_{\sigma(j)} + \sum_{\substack{j \in P\backslash A \\ j \leq d}} a_{\sigma(j)} + \sum_{z \in Z} \sum_{v \in V(z)} a_{\sigma(v)}\,.$$

Fix z and d temporarily. Let

$$Q_+ = Q_+(d, z) = \{v \in V(z) \cap A : v \leq d\}\,,$$

$$Q_- = Q_-(d, z) = \{v \in V(z) \cap X : v \leq d\}\,.$$

By (5.7), if $k \in Q_-$, then $T(k) \in Q_+$, so $\# Q_+ \geq \# Q_-$, and

$$\sum_{\substack{v \in V(z) \\ v \leq d}} a_{\sigma(v)} = \sum_{v \in Q_+} a_{\sigma(v)} + \sum_{v \in Q_-} a_{\sigma(v)}$$

$$= |a_{\sigma(z)}|\,(\# Q_+ - \# Q_-) \geq 0\,.$$

Therefore

$$\sum_{j=1}^{d} a_{\sigma(j)} \geq \sum_{j=1}^{\lambda'} a_{\sigma(j)} + \sum_{\substack{j \in P\backslash A \\ j \leq d}} a_{\sigma(j)}$$

$$\geq \sum_{j=1}^{\lambda'} a_{\sigma(j)} = 1\,.$$

This is the condition (3). Since $\# Q_+(z, \lambda'') = \# Q_-(z, \lambda'')$, and $\#(P\backslash A) = \alpha''$, we also have

$$\sum_{j=1}^{\lambda''} a_{\sigma(j)} = \sum_{j=1}^{\lambda'} a_{\sigma(j)} + \sum_{j \in P\backslash A} a_{\sigma(j)} = 2\,,$$

which is (1). The conditions (2) and (4) are immediate.

Procedure II. Choose α' so large that, if we put $\lambda' = \lambda(\lambda'', \alpha')$, then

$$[1, \lambda'] \supset S(N_+ \cap [1, \lambda''])\,.$$

Next we define $P, X, Y, Z, B, C, A, T, V(z), Q_+ = Q_+(d, z)$, and $Q_- = Q_-(d, z)$ *mutatis mutandis* as in Procedure I. For $j \in (\lambda'', \lambda']$ we define $a_{\sigma(j)}$ by

(C6) $a_{\sigma(v)} = \frac{1}{2}\,\varepsilon(v)\,|a_{\sigma(z)}|$, if $v \in V(z)\backslash\{z\}$, $z \in Z$.

(C7) $a_{\sigma(j)} = 0$, if otherwise.

Now if $d \in (\lambda'', \lambda']$, then

$$\sum_{j=1}^{d} a_{\sigma(j)} = \sum_{j=1}^{\lambda''} a_{\sigma(j)} + \sum_{z\in Z} \sum_{\substack{v\in V(z)\\ v\leq d}} a_{\sigma(v)} .$$

Since

$$\begin{aligned}\sum_{\substack{v\in V(z)\\ v\leq d}} a_{\sigma(v)} &= \sum_{v\in Q_+} a_{\sigma(v)} + \sum_{v\in Q_-} a_{\sigma(v)} \\ &= \begin{cases} \frac{1}{2}\,|a_{\sigma(z)}|\,(\# Q_+ - \# Q_-) & \text{if} \quad z \notin Q_- , \\ \frac{1}{2}\,|a_{\sigma(z)}|\,(\# Q_+ - \# Q_- - 1) & \text{if} \quad z \in Q_- . \end{cases}\end{aligned}$$

So

$$\sum_{j=1}^{d} a_{\sigma(j)} \geq \sum_{j=1}^{\lambda''} a_{\sigma(j)} + \frac{1}{2} \sum_{z\in Z} a_{\sigma(z)} .$$

As

$$\sum_{j\in Z} a_{\sigma(z)} = -\sum_{j=1}^{\lambda''} a_{\sigma(j)} = -2 ,$$

we have $\sum_{j=1}^{d} a_{\sigma(j)} \geq 1$ for $d \in (\lambda'', \lambda']$. This proves (3′). The conditions (1′), (2′) and (4′) are all clear. Q. E. D.

6. **Proof of Theorem** 6. We define $s_0 = 0$. From $k=1$ on we define successively the following:

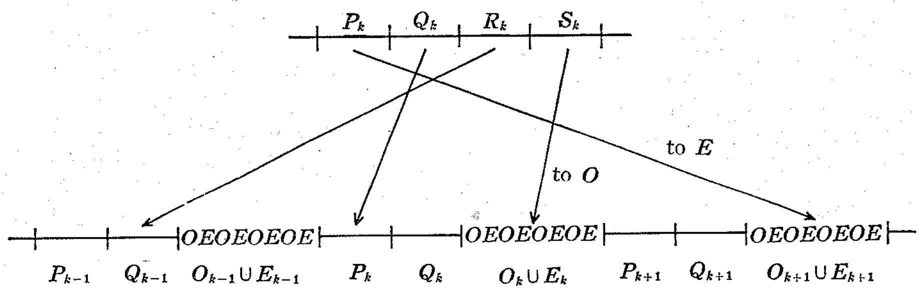
$$\begin{aligned} p_k &= s_{k-1} + k , & P_k &= (s_{k-1},\ p_k] ; \\ q_k &= p_k + k , & Q_k &= (p_k,\ q_k] ; \\ r_k &= q_k + k - 1 , & R_k &= (q_k,\ r_k] ; \\ s_k &= r_k + k - 1 , & S_k &= (r_k,\ s_k] ; \end{aligned}$$

R_1 and S_1 being interpreted as empty sets. We also define the set of even integers in $R_k \cup S_k$ as E_k and the set of odd integers in $R_k \cup S_k$ as O_k.

Next we define the map $\sigma : N \to N$ by defining it on the segments P_k, Q_k, R_k and successively. We let

$$\begin{aligned} \sigma(P_k) &= E_{k+1} , & \sigma(Q_k) &= P_k , \\ \sigma(R_k) &= Q_{k-1} , & \sigma(S_k) &= O_k , \end{aligned}$$

and on each of these segments σ should be order-preserving. We illustrate this definition in the following figure:



The definition of σ can also be written elementwise as follows:

$$\sigma(n) = \begin{cases} q_{k+1} + 2(n - s_{k-1}) & \text{if} \quad n \in P_k\,, \\ n - k & \text{if} \quad n \in Q_k\,, \\ n - 5k - 5 & \text{if} \quad n \in R_k\,, \\ q_k + 2(n - r_k) - 1 & \text{if} \quad n \in S_k\,. \end{cases}$$

When $n \in (p_k, p_{k+1}]$, each point in $\sigma(P_k)$ is a run in $\sigma([1, n])$. Therefore $\rho_n \geq k$. This shows that

$$\lim \rho_n = \infty\,.$$

To prove that σ is sum-preserving, assume that both $\sum a_n$ and $\sum a_{\sigma(n)}$ are convergent. We have

$$\sum_{n=1}^{s_k} a_{\sigma(n)} = \sum_{n=1}^{s_k} a_n + \sum_{n\in P_k} a_{\sigma(n)} - \sum_{n\in Q_k} a_n\,.$$

By Cauchy's criterion, the last two terms tend to 0 as $k \to \infty$. Therefore $\sum a_{\sigma(n)} = \sum a_n$.

Actually the last result can also be deduced from the fact that the sequence of run numbers of σ^{-1} is bounded. We prefer the above reasoning because we shall use it again in the proof of Theorem 7.

7. **Proof of Theorem** 7. We use the notations in the proof of Theorem 6. Let

$$N_1 = \bigcup_{k=1}^{\infty} (2s_{k-1},\ 2s_{k-1} + 4k - 2]$$

and

$$N_2 = N \backslash N_1\,.$$

Let ϕ_1 and ϕ_2 be the order-preserving one-one maps of N onto N_1 and onto N_2 respectively. Define the permutation τ on N as follows:

$$\tau(n) = \begin{cases} \phi_1 \circ \sigma \circ \phi_1^{-1}(n) & \text{if} \quad n \in N_1\,, \\ \phi_2 \circ \sigma^{-1} \circ \phi_2^{-1}(n) & \text{if} \quad n \in N_2\,. \end{cases}$$

Then $\tau | N_1$ and $\tau | N_2$ behave like σ and σ^{-1} respectively. Hence both τ and τ^{-1} have their sequences of run numbers tending to ∞.

τ is sum-preserving because

$$\sum_{n=1}^{2s_k} a_{\tau(n)} = \sum_{n=1}^{2s_k} a_n - \sum_{n \in \phi_1(Q_k)} a_n + \sum_{n \in \phi_1(P_k)} a_{\tau(n)} + \sum_{n \in \phi_2(Q_k)} a_{\tau(n)} - \sum_{n \in \phi_2(P_k)} a_n\,,$$

and the last four terms tend to 0 as $k \to \infty$.

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