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ON TOTAL CURVATURE OF IMMERSED MANIFOLDS, IV: SPECTRUM AND TOTAL MEAN CURVATURE

BY

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Abstract. In this paper we give best possible answers to a problem proposed in [10] concerning total mean curvatures of isometric imbeddings from some compact Riemannian manifolds into a euclidean space of arbitrary dimension.

1. Introduction. In the classical theory of surfaces in a euclidean m-space E^m, the two most important curvatures are the so called Gauss curvature K and the mean curvature α. The Gauss curvature is an isometric invariant and the integral of the Gauss curvature gives the well-known Gauss-Bonnet formula. For the mean curvature of a compact surface M in E^m the total mean curvature

(1.1) $$\int_M \alpha^2 \, dV$$

is a conformal invariant [9], where dV is the volume element of M. In particular, if M is the boundary of a convex domain D in E^3, the total mean curvature (1.1) is also a spectral invariant [15], i.e., it depends only on the eigenvalues of the Laplace-Beltrami operator Δ on D.

In the first part of this series [4], it is proved that, for any compact manifold M of n dimensions immersed in E^m, the total mean curvature satisfies (see, also [16]).

(1.2) $$\int_M \alpha^n dV \geqq c_n,$$

where $\alpha = |H|$, H the mean curvature vector, and c_n the volume of unit n-sphere $S^n(1)$. The equality sign of (1.2) holds if and only if M is imbedded as an ordinary n-sphere in an $(n+1)$-dimensional linear subspace of E^m.

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It is an interesting problem to improve the inequality (1.2) for some special manifolds. For example, it is not known that whether the total mean curvature of every topological 2-torus in E^m satisfies

$$\int_M \alpha^2 dV \geqq 2\pi^2. \tag{1.3}$$

However, when M is a flat torus, (1.3) holds if (a) $n=4$ [6], or (b) M is pseudo-umbilical [11], or (c) the normal connection is flat [8]. In [17], Willmore proved that (1.3) also holds if M is a tube surface of the same radius in E^3.

In [10], the author proposed the following problem.

PROBLEM. *Let M be a given n-dimensional compact Riemannian manifold and $x: M \to E^m$ an isometric imbedding of M in E^m. What is the total mean curvature of x?*

In the third part of this series, we obtained some partial answers to this problem. In this part, we shall continue to study this problem. The method we used in this part bases on the spectral theory of Riemannian manifolds. As two typical examples we shall show that the total mean curvature of any isometric imbedding of the $2n$-torus T^{2n} in E^m is $\geqq (2\pi^2/n)^n$ and the total mean curvature of any isometric imbedding of the Klein bottle $K(a, b)$ in E^m is $> 2\pi^2$.

2. **Estimate of Total Mean Curvature by λ_p.** Let M be a compact Riemannian manifold of dimension n and let Δ be the Laplace-Beltrami operator acting on differentiable functions $C^\infty(M)$ on M. It is known that Δ is an elliptic operator. The operator Δ has an infinite sequence

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_p < \cdots \uparrow \infty \tag{2.1}$$

of eigenvalues. Let $V_i = \{f \in C^\infty(M) \colon \Delta f = \lambda_i f\}$ be the eigenspace with eigenvalue λ_i. Then the dimension of each V_i is finite and is called the *multiplicity* of λ_i. It is known that if we define $\langle f, g\rangle$ by $\int_M fg\, dV$, for $f, g \in C^\infty(M)$, then $\langle\, ,\rangle$ defines a pre-Hilbert structure on $C^\infty(M)$ and the decomposition $\sum V_i$ is orthogonal with respect to this structure. Moreover, $\sum_{i=0}^\infty V_i$ is dense in $C^\infty(M)$. Since M is compact, V_0 is 1-dimensional and consists of constant functions.

For each function $f \in C^\infty(M)$, let f_i be the projection of f on the subspace V_i, $i = 0, 1, 2, \cdots$. We say that a function $f \in C^\infty(M)$

is of *order* $\geqq p$ if $f_i = 0$ for $i = 0, 1, \cdots, p-1$, i.e., f has no components in $V_0, V_1, \cdots, V_{p-1}$. A function $h \in C^\infty(M)$ is said to be of order p if $h \in V_p$. It is clear that the zero function 0 is of order p for each p and there is no immersion of order 0 because $n > 0$.

In the following an isometric immersion

$$x = (x_1, \cdots, x_m) : M \longrightarrow E_m$$

is said to be of *order* $\geqq p$ (respectively, of *order* p) if each coordinate function of $x = (x_1, \cdots, x_m)$ is of order $\geqq p$ (respectively, of order p).

We prove the following

THEOREM 1. *If $x : M \to E^m$ is an imbedding of order $\geqq p$ $(p \geqq 1)$ from a compact n-dimensional manifold M into E^m, then the total mean curvature of M satisfies*

$$\int_M \alpha^n \, dV \geqq \left(\frac{\lambda_p}{n}\right)^{n/2} v(M), \quad \textit{for} \quad n > 1, \tag{2.2}$$

where $v(M)$ is the volume of M. The equality holds if and only if x is an imbedding of order p.

Proof. Suppose that $x : M \to E^m$ is an imbedding of order $\geqq p$; $p \geqq 1$. Then each coordinate function x_i is of order $\geqq p$; $p \geqq 1$. Since $(x_i)_t$ is the component of x_i in V_t, if we denote by $\langle \, , \rangle$ the inner product on the pre-Hilbert space $C^\infty(M)$ and

$$a_{i_t} = \langle x_i, (x_i)_t / \|(x_i)_t\| \rangle = \int_M x_i \, (x_i)_t \, dV \Big/ \left(\int_M |(x_i)_t^2| \, dV \right)^{1/2},$$

then a similar argument as given in [1, p. 186] yields

$$\begin{aligned} 0 &\leqq \|dx_i - \sum_{t \geqq p} a_{i_t} \, d(x_i)_t\|^2 \\ &= \|dx_i\|^2 - 2 \sum_{t \geqq p} a_{i_t} \langle dx_i, d(x_i)_t \rangle + \sum_{t \geqq p} a_{i_t}^2 \| d(x_i)_t\|^2 \\ &= \|dx_i\|^2 - 2 \sum_{t \geqq p} a_{i_t} \langle x_i, \Delta(x_i)_t \rangle + \sum_{t \geqq p} a_{i_t}^2 \langle (x_i)_t, \Delta(x_i)_t \rangle \\ &= \|dx_i\|^2 - \sum_{t \geqq p} \lambda_t a_{i_t}^2, \end{aligned} \tag{2.3}$$

where $\| \, \|$ denotes the norm induced from from $\langle \, , \rangle$. From (2.3) we find

$$\|dx_i\|^2 \geqq \sum_{t \geqq p} \lambda_t a_{i_t}^2 \geqq \lambda_p \left(\sum_{t \geqq p} a_{i_t}^2 \right) = \lambda_p \|x_i\|^2,$$

i.e.,

$$\int_M |dx_i|^2 \, dV \geqq \lambda_p \int_M x_i^2 \, dV, \tag{2.4}$$

where $|dx_i|$ is the length of the 1-form dx_i on M. It is clear that the equality of (2.4) holds if and only if x_i is a function of order p. On the other hand, since

$$|dx|^2 = \sum_{i=1}^{m} |dx_i|^2 = n = \dim M,$$

(2.4) implies

$$n\, v(M) \geqq \lambda_p \int_M |x|^2 \, dV, \tag{2.5}$$

where $|x|$ is the length of x with respect to the euclidean metric of E^m. Let σ be the second fundamental form of M in E^m. Then the mean curvature vector H is given by $H = 1/n$ trace σ and the mean curvature $\alpha = |H|$. From (2.5) and the well-known Schwarz inequality, we have

$$\begin{aligned} n\, v(M)\Big(\int_M \alpha^2 \, dV\Big) &\geqq \lambda_p \Big(\int_M |x|^2 \, dV\Big)\Big(\int_M \alpha^2 \, dV\Big) \\ &\geqq \lambda_p \Big(\int_M \alpha |x| \, dV\Big)^2 \geqq \lambda_p \Big(\int_M (H\cdot x) \, dV\Big)^2, \end{aligned} \tag{2.6}$$

where $H\cdot x$ is the scalar product of H and x in E^m.

From Proposition 2.2 of [7] we have

$$v(M) + \int_M (H\cdot x) \, dV = 0. \tag{2.7}$$

Therefore, (2.6) and (2.7) give

$$\int_M \alpha^2 \, dV \geqq \frac{\lambda_p}{n} v(M). \tag{2.8}$$

Now by using the Hölder inequality we find

$$\frac{\lambda_p}{n} v(M) \leqq \int_M \alpha^2 \, dV \leqq \Big(\int_M \alpha^{2r} \, dV\Big)^{1/r} \Big(\int_M dV\Big)^{1/s}, \tag{2.9}$$

where

$$\frac{1}{r} + \frac{1}{s} = 1, \qquad r, s > 1. \tag{2.10}$$

Thus we have

$$\left(\int_M \alpha^{2r}\, dV\right)^{1/r} \geqq \left(\frac{\lambda_p}{n}\right) v(M)^{1/r}. \tag{2.11}$$

If $n=2$, (2.8) gives (2.2). If n is greater than 2, we set

$$2n = r, \tag{2.12}$$

then from (2.11) we get

$$\left(\int_M \alpha^n\, dV\right)^{n/2} \geqq \left(\frac{\lambda_p}{n}\right) v(M)^{n/2},$$

from which we get (2.2).

If the equality sign of (2.2) holds, the inequalities in the proof above become equalities. Thus every coordinate function x_i is of order p. Consequently, the imbedding is of order p.

Conversely, if the imbedding x is of order p, then we have

$$\Delta x = \lambda_p x. \tag{2.13}$$

A theorem of Takahashi [14] then implies that M is immersed in a hypersphere $S^{m-1}(r)$ of radius r centered at the origin as a minimal submanifold.

On the other hand, since $\Delta x = 2nH$ and M is minimal in $S^{m-1}(r)$, $x = r^2 H$. Hence (2.13) implies

$$\alpha^2 = \frac{1}{r^2} \quad \text{and} \quad \lambda_p = \frac{2n}{r^2}.$$

From these we see that the equality of (2.12) holds. This completes the proof of the theorem.

Our proof of Theorem 1 is a modification and generalization of the well known proof of the isoperimetric inequality due to A. Hurwitz given in 1902. (See, also [13]).

If $x: M \to E^m$ is any imbedding of M into E^m, then by a suitable translation of E^m we may choose the center of gravity as the origin of E^m. Under this coordinate system, x is an imbedding of order $\geqq 1$. Since inequality (2.2) is independent of the choice of the euclidean coordinate symem of E^m, Theorem 1 implies the following.

THEOREM 2. *Let $x: M \to E^m$ be an imbedding of a compact n-dimensional manifold M into E^m. Then the total mean curvature of x satisfies*

$$\int_M \alpha^n \, dV \geqq \left(\frac{\lambda_1}{n}\right)^{n/2} v(M).$$

The equality holds if and only if there exists a vector c in E^m such that $x - c$ is an imbedding of order 1.

These two theroms are inspired by [2, 13]. If $n = 2$, Theorem 2 is due to Bleecker-Weiner [2] and Reilley [13].

THEOREM 3. *Let M be a compact Riemannian manifold of dimension n. If every isometric imbedding of M in an euclidean space E^m has total mean curvature $\geqq B$ and if the pth eigenvalue of Δ on M satisfies $\lambda_p^n < (n)^{n/2} B/v(M)$, then M admits no isometric imbedding of order $t \leqq p$ in E^m.*

This theorem follows immediately from Theorem 1. For the later purpose, we mention the following result of Takahashi [14].

LEMMA 4. *Let $x: M \to S^m(r)$ be an isometric minimal imbedding from an n-dimensional compact Riemannian manifold M into an m-sphere of radius r. Then $r = \sqrt{n/\lambda_p}$ for some eigenvalue λ_p of the Laplace-Beltrami operator Δ on M.*

This lemma shows that, for any compact Riemannian manifold M, the radii of spheres in which M can be isometrically minimally imbedded are determined by its spectrum; Spec$(M) = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$. Because not every λ_p can be realized by such an imbedding, it seems to be interesting to determine, for a given p, whether M admits an isometric imbedding of order p. From Lemma 4 we have the following partial answer to this problem.

PROPOSITION 5. *Let M be an n-dimensional compact Riemannian manifold. If λ_p/n is less than the maximal sectional curvature of M, then M admits no isometric imbedding of order t in any euclidean space for $t \leqq p$.*

Proof. It M admits an isometric imbedding of order t in E^m, then $\Delta x = \lambda_t x$, where x is the imbedding. Thus M is imbedded in an $(m-1)$-sphere of radius $r=\sqrt{n/\lambda_t}$ as a minimal submanifold. Therefore, by the equation of Gauss, the sectional curvature of M is less than or equal to λ_t/n.

COROLLARY 6. *Let M be the product manifold $S^1(a)\times S^{2n-1}(b)$ with $a>b$. Then M admits no isometric imbedding of order p for any $p<\sqrt{2n}\,a/b$.*

Proof. Since M is the product manifold $S^1(a)\times S^{2n-1}(b)$, the maximal sectional curvature of M is $1/b^2$ and the spectrum of M is given by

$$\left\{\frac{k^2}{a^2}+\frac{l(2n+l-2)}{b^2}:k,\ l=0,1,2,\cdots\right\}. \tag{2.14}$$

Since $a>b$, the first few eigenvalues λ_k for $k<\sqrt{2n}\,a/b$ are given by

$$\lambda_k=\frac{k^2}{a^2}<\frac{2n}{b^2}.$$

Thus we find

$$\frac{\lambda_k}{2n}<\frac{1}{b^2}.$$

Consequently, Corollary 6 follows from Proposition 5.

3. **Applications.** In this section we shall give some applications of results obtained in §2.

THEOREM 7. *Let M be the product surface $S^1(a)\times S^1(b)$ with $a\geq b$. Then*

(i) *M admits no isometric imbedding of order $p<a/b$ in any euclidean space and*

(ii) *every isometric imbedding of order $\geq a/b$ satisfies*

$$\int_M \alpha^2\,dV\geq 2\pi^2\,a/b. \tag{3.1}$$

The equality of (3.1) holds if and only if a/b is an integer and the isometric imbedding is of order a/b.

Proof. Part (i) follows immediately from Corollary 6 by setting $n=1$.

If $x : M \to E^m$ is an isometric imbedding of order $\geqq a/b$, then Theorem 1 implies

$$\int_M \alpha^2 \, dV \geqq \frac{\lambda_t}{2} A, \tag{3.2}$$

where t is the smallest integer greater than or equal to a/b and A is the area of M. It is clear from (2.14) that

$$\lambda_t \geqq \frac{1}{b^2} \quad \text{and} \quad A = 4\pi^2 ab.$$

Thus Part (ii) follows from Theorem 1 and (3.2).

For the $2n$-torus $T^{2n} = S^1(1) \times \cdots \times S^1(1)$ ($2n$ times), we have the following best possible result.

THEOREM 8. *Let M be the $2n$-torus T^{2n}. Then every isometric imbedding $x : M \to E^m$ satisfies*

$$\int_M \alpha^{2n} \, dV \geqq \left(\frac{2\pi^2}{n}\right)^n. \tag{3.3}$$

The equality holds if and only if M is imbedded in a hypersphere of radius $r = \sqrt{2n}$ by an imbedding of order 1.

This theorem follows from Theorem 2, Lemma 4 and the fact $\lambda_1 = 1$.

The standard imbedding of T^{2n} in E^{4n} is an isometric imbedding of order 1.

Let a, b be two positive numbers. Consider the group Γ generated by the following two mappings of E^2:

$$(x, y) \longrightarrow (x, y + b),$$

$$(x, y) \longrightarrow \left(x + \frac{a}{2}, -y\right).$$

Then the quotient surface E^2/Γ with the induced metric gives rise to a flat compact non-orientable surface $K(a, b)$. Such a surface is called a *Klein bottle*.

THEOREM 9. *Let M be any compact flat surface in E^m. If the total mean curvature satisfies*

$$\int_M \alpha^2 \, dV \leqq 2\pi^2, \tag{3.4}$$

then M is homeomorphic to a torus T^2.

Proof. Since M is a compact flat surface. M is either homeomorphic to a torus T^2 or it is isometric to a Klein bottle $K(a, b)$ for some a, b [1, p. 6]. If M is isometric to a Klein bottle $K(a, b)$, Proposition B. II. 1 of [1] gives

$$\lambda_1 = 4\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \qquad A = \frac{ab}{2}.$$

Thus, from Theorem 2, we obtain

$$\int_M \alpha^2\, dV \geqq 2\pi^2. \tag{3.5}$$

Combining this with (3.4) we find that the equality of (3.5) holds. Consequently, M is imbedded in E^m by an imbedding of order 1. Thus M is pseudo-umbilical in E^m. Since M is flat, Theorem 2 of [11] implies that M is imbedded in E^4 as a Clifford torus. This contradicts our assumption.

Let RP^{2n} be the $2n$-dimensional real projective space of constant sectional curvature one. Then the spectrum of RP^{2n} is given by

$$\lambda_k = 2k(2n + 2k - 1), \qquad k \geqq 0. \tag{3.6}$$

The multiplicity S_k of λ_k is

$$\frac{(2n + 2k - 2)\cdots(2n + 1)2n}{(2k)!}(2n + 4k - 1). \tag{3.7}$$

For each k, let $\{f_1, \cdots, f_{s_k}\}$ be an orthonormal basis for V_k. We define a map ψ_k by

$$\psi_k = r(f_1, \cdots, f_{s_k}), \qquad r = \sqrt{n/(2n + 1)}.$$

Then ψ_k defines an equivariant isometric imbedding of RP^{2n} into E^{s_k}. ψ_k is called the k-th *standard imbedding* of RP^{2n}. For each k, all kth standard imbeddings are equivalent under euclidean motions. The first standard imbedding ψ_1 is called the *Veronese imbedding.*

For the real projective plane, the Veronese imbedding is given by

$$x_1 = \frac{1}{3} uv, \quad x_2 = \frac{1}{3} uw, \quad x_3 = \frac{1}{3} vw,$$

$$x_4 = \frac{1}{6}(u^2 - v^2), \quad x_5 = \frac{1}{6\sqrt{3}}(u^2 + v^2 - 2w^2),$$

where $u^2 + v^2 + w^2 = 1$.

THEOREM 10. *Every isometric imbedding of RP^{2n} into E^m satisfies*

$$\int_{RP^{2n}} \alpha^{2n}\, dV \geq \left(\frac{2n+1}{n}\right)^n \frac{c_{2n}}{2}. \tag{3.8}$$

The equality holds if and only if RP^{2n} is imbedded in a $E^{n(2n+3)}$ by the Veronese imbedding. Moreover, in this case, RP^{2n} is imbedded in a hypersphere of radius $\sqrt{n/(2n+1)}$.

Proof. Since $v(RP^{2n}) = c_{2n}/2$, (3.6) and Theorem 2 imply (3.8). If the equality of (3.8) holds, x is an imbedding of order 1. Thus Lemma 4 says that RP^{2n} is imbedded in a hypersphere of radius $\sqrt{n/(2n+1)}$ as a minimal submanifold. On the other hand, the minimal imbedding of RP^{2n} in spheres is the composition of the standard imbedding ψ_k with some symmetric, positive semi-definite linear map A of R^m [12]. In our case $\psi_k = \psi_1$, the linear map A is the identity map according to Theorem 1.4 of [12] ($s = 2$ in the notation of [12]). Thus x is the Veronese imbedding. The converse of this is trivial.

Let CP^{2n} and HP^{4n} be the (real) $2n$-dimensional complex projective space and (real) $4n$-dimensional quaternion projecive space with the standard metrics, respectively. Then by using the same method as above we have the following two results.

THEOREM 11. *Every isometric imbedding of CP^{2n} into E^m satisfies*

$$\int_{CP^{2n}} \alpha^{2n}\, dV \geq \left[\frac{2(n+1)\pi}{n}\right]^n \Big/ n!. \tag{3.9}$$

THEOREM 12. *Every isometric imbedding of HP^{4n} into E^m satisfies*

$$\int_{HP^{4n}} \alpha^{4n}\, dV \geq 2\left[\frac{(2n+3)\pi}{n}\right]^{2n} \Big/ (2n+1)! \tag{3.10}$$

REMARK. It seems to the author that inequalities (3.8), (3.9) and (3.10) hold for all imbeddings (not necessary isometric) in E^m

for RP^n, CP^{2n} and HP^{4n} respectively. But the author cannot prove this conjecture at this moment.

References

1. M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété Riemanniene*, Lecture Notes in Math. No. 194, Springer-Verlag, Berlin, 1971.

2. D. Bleecker and J. Weiner, *Extrinsic bounds on λ_1 of Δ on a compact manifold*, Comment. Math. Helv. **51** (1976), 601-609.

3. B. Y. Chen, *Geometry of submanifolds*, M. Dekker, New York, 1973.

4. ______, *On the total curvature of immersed manifolds*, I, Amer. J. Math. **93** (1971), 148-162.

5. ______, *On the total curvature of immersed manifolds*, II, Amer. J. Math. **94** (1972), 799-809.

6. ______, *On the total curvature of immersed manifolds*, III, Amer. J. Math. **95** (1973), 636-642.

7. ______, *G-total curvature of immersed manifolds*, J. Differential Geometry **7** (1972), 373-393.

8. ______, *Pseudo-umbilical submanifolds in a Riemannian manifold of constant curvature*, II, J. Math. Soc. Japan **25** (1973), 105-114.

9. ______, *An invariant of conformal mappings*, Proc. Amer. Math. Soc. **40** (1973), 563-564.

10. ______, *Mean curvature vector of a submanifold*, Proc. Symp. Pure Math. **27** (1975), 119-123.

11. ______, *Total mean curvature of immersed surfaces in E^m*, Trans. Amer. Math. Soc. **218** (1976), 333-341.

12. M. doCarmo and N. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math. **93** (1971), 43-62.

13. R. C. Reilley, *On the first eigenvalues of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52** (1977), 525-533.

14. T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380-385.

15. R. T. Waechter, *On hearing the shape of drum: An extension to higher dimensions*, Proc. Cambridge Philos. Soc. **72** (1972), 439-447.

16. T. J. Willmore, *Mean curvature of immersed surfaces*, An. Sti. Univ. "Al. I. Cuza," Iasi, Sect. Ia Mat. **14** (1968), 99-103.

17. ______, *Mean curvature of Riemannian immersions*, J. London Math. Soc. **3** (1971), 307-310.

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