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ON THE MAXIMAL EXCESS IN BOUNDARY CROSSINGS OF RANDOM WALKS RELATED TO FLUCTUATION THEORY AND LAWS OF LARGE NUMBERS

BY

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Abstract. Let $X, X_1, X_2, \cdots$ be i.i.d. with $EX=0$ and let $S_n = X_1 + \cdots + X_n$. Let $\{b(n)\}$ be a sequence of nonnegative numbers. In this paper, two different methods for analyzing moments of the maximal excess $M(\{b(n)\}) = \sup_{n\geq 0}(S_n - b(n))$ are reviewed and generalized. The first method relates the moments of $M(\{b(n)\})$ to certain convergence rate problems associated with the strong law and analogous limit theorems for the driftless random walk $\{S_n\}$. As an application of this method, a general theorem relating the finiteness of moments of $M(\{b(n)\})$ to moment restrictions on X^+ is obtained, generalizing earlier results of Kiefer and Wolfowitz, Chow and Lai, Cohn, and Scott. It is also shown that under minimal moment conditions on X^+, $EM^p(\{\varepsilon(n\log n)^{1/2}\}) < \infty$ or $= \infty$ according as $\varepsilon >$ or $< (pEX^2)^{1/2}$. The second method works only for linear boundaries $b(n) = \varepsilon n$ $(\varepsilon > 0)$ and is based on the Wiener-Hopf equation for the negative-drift random walk $\{S_n - \varepsilon n\}$. This method yields some sharp asymptotic estimates for the integral moments of $M(\{\varepsilon n\})$, and some applications of these asymptotic estimates to fluctuation theory are given.

1. Introduction and summary. Let $X, X_1, X_2, \cdots$ be i. i. d. random variables with $EX=0$ and let $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$. This assumption and notation will be adopted throughout the rest of the paper unless stated otherwise. Let $\{b(n)\}_{n=0,1,\dots}$ be a sequence of nonnegative numbers such that $b(0)=0$. Define

$$M(\{b(n)\}) = \sup_{n\geq 0}(S_n - b(n)), \tag{1.1}$$

$$\tilde{M}(\{b(n)\}) = \sup_{n\geq 0}(|S_n| - b(n)). \tag{1.2}$$

The Marcinkiewicz-Zygmund strong law of large numbers can

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be stated in terms of $M(\{b(n)\})$ as follows: For $\alpha > \frac{1}{2}$, if $E|X|^{1/\alpha} < \infty$, then

$$P[M(\{\varepsilon n^{\alpha}\}) < \infty] = 1 \quad \text{for all} \quad \varepsilon > 0. \tag{1.3}$$

Let $\log_1 x = \log x$ and $\log_k x = \log(\log_{k-1} x)$ for $k \geq 2$. The classical law of the iterated logarithm can be stated in terms of $M(\{b(n)\})$ as follows: If $EX^2 = \sigma^2 > 0$, then

$$\begin{aligned} P[M(\{c(2n\log_2 n)^{1/2}\}) < \infty] &= 1 \quad \text{if} \quad c > \sigma, \\ &= 0 \quad \text{if} \quad c < \sigma. \end{aligned} \tag{1.4}$$

To obtain a stronger conclusion than almost sure finiteness of the random variable $M(\{b(n)\})$ as in (1.2) or (1.3), it is natural to ask what conditions would guarantee the finiteness of its moments. For the special case $b(n) = \varepsilon n$, Kiefer and Wolfowitz [7] showed that for $p > 0$

$$E(X^+)^{p+1} < \infty \Longleftrightarrow EM^p(\{\varepsilon n\}) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.5}$$

Later Cohn [5] considered $b(n) = \varepsilon n^{\alpha}$ with $\alpha > \frac{1}{2}$ and showed that for $p > 0$

$$E|X|^{p+(1/\alpha)} < \infty \Longleftrightarrow E\widetilde{M}^p(\{\varepsilon n^{\alpha}\}) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.6}$$

This result was also proved independently around the same time by Chow and Lai [2] using other methods. Recently Scott [13], by sharpening the methods of Cohn, succeeded in extending (1.6) to $b(n) = (nl(n))^{\alpha}$, where $\alpha > \frac{1}{2}$ and $l(t) = (\log_k(t + e^k))^{\gamma}$ with $k = 1$ or 2 and $\gamma > 0$. He showed that for $p > 0$, if $E|X|^{1/\alpha} < \infty$, then

$$E\{|X|^{p+(1/\alpha)}/(l(|X|))\} < \infty \Longleftrightarrow E\widetilde{M}^p(\{n^{\alpha} l(n)\}) < \infty. \tag{1.7}$$

As pointed out by Scott [13], most of these investigations were motivated by applications to optimal stopping problems (cf. Section 4.8 of [4]) and queuing theory.

In §4 below, we shall give a much more definitive generalization of (1.6). While the approach of Cohn and Scott does not seem to be able to handle boundaries other than $n^{\alpha}l(n)$, the method which we developed in [2] to analyze $M(\{\varepsilon n^{\alpha}\})$ can be modified to handle much more general boundaries. In §4, we shall use this approach to show that an intrinsic form of results of the type (1.5), (1.6) and (1.7) is the following.

THEOREM 1. *Let* $X, X_1, X_2, \cdots$ *be i.i.d. random variables such that* $EX = 0$ *and* $E|X|^q < \infty$ *for some* $1 \leq q \leq 2$, *and let* $S_n = X_1 + \cdots + X_n$. *Let* $b(t)$ *be a positive function on* $[1, \infty)$ *satisfying the following conditions*:

$$b(t) \text{ is nondecreasing and } \lim_{t\to\infty} t^{-\delta} b(t) = \infty \text{ for some } \delta > 1/q, \tag{1.8a}$$

$$\limsup_{t\to\infty} b(\lambda t)/b(t) < \infty \text{ and } \liminf_{t\to\infty} b(\lambda t)/b(t) > 1 \text{ for all large } \lambda. \tag{1.8b}$$

Let $p > 0$. *For* $x \geq 0$ *define*

$$\beta(x) = \inf\{t : b(t) \geq x\}, \qquad B_p(x) = \int_0^x y^{p-1}\beta(y)\,dy. \tag{1.9}$$

Set $b(0) = S_0 = 0$. *Then*

$$EB_p(X^+) < \infty \Longleftrightarrow EM^p(\{b(n)\}) < \infty. \tag{1.10}$$

We note that if $b(t) \sim \{t(\log_k t)^r\}^\alpha$, where $\alpha > \frac{1}{2}$, r is any real number and $k = 1, 2, \cdots$, then $\beta(x) \sim Cx^{1/\alpha}(\log_k x)^{-r}$ and $B_p(x) \sim C' x^{p+(1/\alpha)}(\log_k x)^{-r}$, where C and C' are positive constants. Hence Theorem 1 includes (1.5), (1.6) and (1.7) as special cases and it can even handle the one-sided result involving $M(\{b(n)\})$ which implies the corresponding two-sided version for $\tilde{M}(\{b(n)\})$.

The condition (1.8a) excludes boundaries of the type $n^{1/2}\psi(n)$, where $\psi(n)$ is slowly varying, e. g., $\psi(n) = (\log_2 n)^{1/2}$ or $\psi(n) = \log n$. In connection with the law of the iterated logarithm (1.4) it is particularly interesting to consider the moments of $M(\{c(2n\log_2 n)^{1/2}\})$. Parts (i) and (ii) of the following theorem imply, however, that all positive moments of $M(\{c(2n\log_2 n)^{1/2}\})$ are infinite. In fact, for $p > 0$, even $EM^p(\{\varepsilon(n\log n)^{1/2}\}) = \infty$ if $\varepsilon < (p\,EX^2)^{1/2}$. Part (iii) of the following theorem says that the equivalence (1.10), however, still holds for the "marginal" class of boundaries $b(n) = \varepsilon(n\log n)^{1/2}$ with $\varepsilon > (p\,EX^2)^{1/2}$.

THEOREM 2. *Let* $X, X_1, X_2, \cdots$ *be i.i.d. and let* $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$. *Let* $p > 0$.

(i) *If X is symmetric, then*

(1.11) $$EM^p(\{\varepsilon(n\log n)^{1/2}\})=\infty \qquad \text{for all } 0<\varepsilon<(p\,EX^2)^{1/2}.$$

(*Note that EX^2 may be infinite in* (1.11).)

(ii) *In general (without assuming X to be symmetric) the relation* (1.11) *still holds if*

(1.12) $$EX=0 \quad \text{and} \quad E\{|X|^{p+2}/(1+\log^+|X|)\}<\infty.$$

(iii) *If* (1.12) *holds, then for every $\varepsilon>(p\,EX^2)^{1/2}$,*

(1.13) $$E\widetilde{M}^p(\{\varepsilon(n\log n)^{1/2}\})<\infty.$$

Conversely, if (1.13) *holds for some $\varepsilon>0$, then* (1.12) *holds.*

(iv) *Suppose $EX=0$ and $EX^2<\infty$. If*

(1.14) $$\int_{[X>e]} X^{p+2}(\log X)^{-1}\,dP<\infty,$$

then for every $\varepsilon>(p\,EX^2)^{1/2}$,

(1.15) $$EM^p(\{\varepsilon(n\log n)^{1/2}\})<\infty.$$

Conversely, if (1.15) *holds for some $\varepsilon>0$, then* (1.14) *holds.*

In connection with the Kiefer-Wolfowitz theorem (1.5), Kingman [8] obtained the following useful inequality:

(1.16) $$EM(\{\varepsilon n\})\le \frac{1}{2}\varepsilon^{-1}EX^2.$$

This inequality turns out to be asymptotically sharp when $\varepsilon\downarrow 0$, as Robbins, Siegmund and Wendel [12] later showed that if $0<EX^2=\sigma^2<\infty$, then as $\varepsilon\downarrow 0$,

(1.17) $$EM(\{\varepsilon n\})\sim\frac{1}{2}\sigma^2\varepsilon^{-1}.$$

The approach which we developed in [2] and which we shall review briefly in §4 enabled us to generalize Kingman's inequality to boundaries of the type $b(n)=\varepsilon n^\alpha$ and also to higher moments. It was shown in [2] that given $\alpha>\frac{1}{2}$ and $p>0$, there exists a universal constant $A_{p,\alpha}$ depending only on p and α such that

(1.18) $$EM^p(\{\varepsilon n^\alpha\})\le A_{p,\alpha}\,\varepsilon^p\{E(X^+/\varepsilon)^{p+(1/\alpha)} + (E(X/\varepsilon)^2)^{p\alpha/(2\alpha-1)}\}.$$

The above inequality (1.18) in turn enabled us to generalize in [2] the Robbins-Siegmund-Wendel theorem (1.17) as follows: For $\alpha > \frac{1}{2}$ and $p > 0$, if $\infty > EX^2 = \sigma^2 > 0$ and $E(X^+)^{p+(1/\alpha)} < \infty$, then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{p/(2\alpha-1)} EM^p(\{\varepsilon n^\alpha\}) = \sigma^{2p\alpha/(2\alpha-1)} E(\sup_{t\geq 0} (W(t) - t^\alpha))^p , \tag{1.19}$$

where $W(t)$, $t \geq 0$, is the standard Wiener process.

The asymptotic result (1.17) often turns out to be an adequate numerical approximation for $EM(\{\varepsilon n\})$ when ε is small (cf. [8, page 324]). As will be shown in §2, (1.17) can in fact be sharpened as

$$EM(\{\varepsilon n\}) - \frac{1}{2} \sigma^2 \varepsilon^{-1} = O(1) \tag{1.20}$$

under the minimal assumption that $0 < EX^2 = \sigma^2 < \infty$. More generally, we shall show in §2 that if p is a positive integer and $E|X|^{p+1} < \infty$, then as $\varepsilon \downarrow 0$,

$$EM^p(\{\varepsilon n\}) - 2^{-p}(p!)(\sigma^2/\varepsilon)^p = O(\varepsilon^{-(p-1)}) , \tag{1.21}$$

which therefore yields a refinement of (1.19) with $\alpha = 1$. In this special case $\alpha = 1$, $M(\{\varepsilon n\})$ is the maximum of the random walk $\{S_n - \varepsilon n\}$ with negative drift, and the asymptotic relations (1.20) and (1.21) have a number of interesting implications in fluctuation theory which we shall discuss in §3.

The approach which we used to prove (1.18) and (1.19) in [2] is based on a maximal inequality and the invariance principle for driftless random walks. It does not, however, seem to be powerful enough to give more refined results of the type (1.20) and (1.21). Instead of thinking in terms of driftless random walks, Kingman's approach to prove (1.16) in [8] deals directly with the negative-drift random walk and is based on the following simple and useful lemma which is commonly referred to as the Wiener-Hopf equation (cf. [6, page 385]).

LEMMA 1. *Let* $Y, Y_1, Y_2, \cdots$ *be i.i.d. and* $U_n = Y_1 + \cdots + Y_n$ $(U_0 = 0)$. *Suppose* $\lim_{n\to\infty} U_n = -\infty$. *Let* $M = \max_{n\geq 0} U_n$. *Then*

$$M \overset{\mathscr{L}}{=} (M + Y)^+ , \tag{1.22}$$

where "$\overset{\mathscr{L}}{=}$" *denotes equality in distribution.*

In §2 we shall make use of Lemma 1 and a refinement of Kingman's argument to obtain a recursion formula for EM^k in terms of $EM^{k-1}, \cdots, EM$ and the moments of Y. This recursion formula not only gives an easy alternative proof of the asymptotic relation (1.19) when p is an integer and $\alpha = 1$, but it also proves the sharper results (1.20) and (1.21).

For notational convenience, we shall sometimes use Vinogradov's symbol $\ll$ instead of Landau's O in the following sections.

2. Application of the Wiener-Hopf equation to integral moments of the maximum of a random walk with negative drift.

THEOREM 3. *Let* $Y, Y_1, \cdots$ *be i.i.d. with* $EY = -\varepsilon < 0$. *Let* $U_n = \sum_1^n Y_i$ $(U_0 = 0)$ *and let* $M = \max_{n\geq 0} U_n$. *Let* k *be a positive integer. If* $E|Y|^{k+1} < \infty$, *then*

$$EM = (2\varepsilon)^{-1} \{EY^2 - E((M+Y)^-)^2\} , \tag{2.1}$$

and in general,

$$\begin{aligned} EM^k = \{(k+1)\varepsilon\}^{-1} \Big\{ & \binom{k+1}{2}(EM^{k-1})(EY^2) \\ & + \binom{k+1}{3}(EM^{k-2})(EY^3) + \cdots \\ & + EY^{k+1} + (-1)^k E((M+Y)^-)^{k+1} \Big\} . \end{aligned} \tag{2.2}$$

Proof. First assume in addition that $E(Y^+)^{k+2} < \infty$. Then $EM^{k+1} < \infty$ by the Kiefer-Wolfowitz theorem (1.5). Therefore using the Wiener-Hopf equation (1.22),

$$EM^{k+1} = E((M+Y)^+)^{k+1} . \tag{2.3}$$

Since $x^{k+1} = (x^+)^{k+1} + (-x^-)^{k+1}$, it then follows that

$$
\begin{aligned}
&E((M+Y)^+)^{k+1}\\
&\qquad = E(M+Y)^{k+1} + (-1)^k E((M+Y)^-)^{k+1}\\
&\qquad = EM^{k+1} - (k+1)\,\varepsilon(EM^k) + \binom{k+1}{2}(EY^2)(EM^{k-1})\\
&\qquad\qquad + \binom{k+1}{3}(EY^3)(EM^{k-2}) + \cdots\\
&\qquad\qquad + EY^{k+1} + (-1)^k E((M+Y)^-)^{k+1}.
\end{aligned}
\tag{2.4}
$$

From (2.3) and (2.4), (2.2) is established if $E(Y^+)^{k+2} < \infty$ is also assumed.

We now drop the condition $E(Y^+)^{k+2} < \infty$ and use a truncation argument. We note that all the terms in (2.2) are finite under the assumption $E|Y|^{k+1} < \infty$. (In particular, $E((M+Y)^-)^{k+1} \le E(Y^-)^{k+1} < \infty$.) For $c > 0$, define $Y_j(c) = Y_j I_{[Y_j \le c]}$, $U_n(c) = Y_1(c) + \cdots + Y_n(c)$ and $M_c = \max_{n\ge 0} U_n(c)$. By choosing c large enough, $EY(c) = -\varepsilon(c) < 0$. Since $(Y_1(c))^+$ is bounded, (2.2) holds for M_c, i.e.,

$$
\begin{aligned}
EM_c^k = \{(k+1)\,\varepsilon(c)\}^{-1}\Big\{&\binom{k+1}{2}(EM_c^{k-1})(EY_1^2(c)) + \cdots\\
&+ EY_1^{k+1}(c) + (-1)^k E((M_c + Y_1(c))^-)^{k+1}\Big\}.
\end{aligned}
\tag{2.5}
$$

Letting $c \to \infty$ in (2.5) then gives the desired conclusion. ▮

It is obvious that (2.3) and (2.4) also hold for $k = 0$, and so it follows that

$$E(M+Y)^- = -EY. \tag{2.6}$$

Hence for $k = 1, 2, \cdots$,

$$E((M+Y)^-)^k \ge |EY|^k. \tag{2.7}$$

Kingman's inequality (1.16) follows from (2.1) and (2.7) (with $k = 2$). The bound (2.7), however, is too crude to yield asymptotic results of the type (1.20) and (1.21). A more careful analysis in [3] yields sharp bounds for $E((M+Y)^-)^k$ which are of the right order of magnitude for proving (1.20) and (1.21). In particular, these bounds lead to the following lemma.

LEMMA 2. *Let* $X, X_1, X_2, \cdots$ *be i.i.d. with* $EX = 0$ *and let* $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$. *For* $\varepsilon > 0$, *define*

(2.8) $$M_\varepsilon = M(\{\varepsilon n\}) = \max_{n\geq 0}(S_n - \varepsilon n), \quad Y_\varepsilon = X - \varepsilon.$$

Let $p > 0$. *If* $E(X^-)^{p+1} < \infty$ *and* $P[X \neq 0] > 0$, *then*

(2.9) $$E((M_\varepsilon + Y_\varepsilon)^-)^p \ll \varepsilon \quad as \quad \varepsilon \downarrow 0.$$

The proof of Lemma 2 is given in [3], where it is also shown that $E((M_\varepsilon + Y_\varepsilon)^-)^p \gg \varepsilon$ (under the assumption $0 < E(X^-)^2 < \infty$), which is considerably sharper than the result given by (2.7). Making use of Theorem 3 and Lemma 2, we obtain (1.20) and more generally (1.21) as an immediate corollary.

COROLLARY 1. *With the same notations as in Lemma 2, let k be a positive integer. If* $E|X|^{k+1} < \infty$, *then as* $\varepsilon \downarrow 0$,

(2.10) $$EM_\varepsilon^k = \{2^{-k}(k!) + O(\varepsilon)\}(EX^2/\varepsilon)^k.$$

Proof. Putting (2.9) in Theorem 3 and noting that $EY_\varepsilon^j = EX^j + O(\varepsilon)$ for $j = 1, \cdots, k+1$, (2.10) follows by induction in the case $P[X \neq 0] > 0$. The case $X = 0$ a.s., however, is trivial. ∎

3. Relation between the maximum and the ascending ladder variable of a negative-drift random walk and some applications of Theorem 3.

Let $Y, Y_1, Y_2, \cdots$ be i.i.d. with $EY = -\varepsilon < 0$ and let $U_n = Y_1 + \cdots + Y_n$ $(U_0 = 0)$. Define

(3.1) $$M = \max_{n\geq 0} U_n, \quad \tau_+ = \inf\{n \geq 1 : U_n > 0\} (\inf \emptyset = \infty).$$

As is well known in fluctuation theory, the maximum M is closely related to the ascending ladder variable U_{τ_+}. The following theorem gives a recursive relation for the kth moment of $U_{\tau_+} I_{[\tau_+ < \infty]}$ in terms of moments of lower order and EM^k. The recursive relation for EM^k given in Theorem 3 therefore in turn provides a recursive estimate for $EU_{\tau_+}^k I_{[\tau_+<\infty]}$.

THEOREM 4. *Let $Y, Y_1, Y_2, \cdots$ be i.i.d. with $EY = -\varepsilon < 0$ and let $U_n = Y_1 + \cdots + Y_n$ $(U_0 = 0)$. Define M and τ_+ as in* (3.1). *Then for* $r > 0$,

(3.2) $$EU_{\tau_+}^r I_{[\tau_+<\infty]} < \infty \Longleftrightarrow EM^r < \infty \Longleftrightarrow E(Y^+)^{r+1} < \infty.$$

Let k be a positive integer and assume that $E(Y^+)^{k+1} < \infty$. *Then*

$$EU_{\tau_+} I_{[\tau_+<\infty]} = P[M=0](EM)\,, \tag{3.3}$$

$$\begin{aligned} EU^2_{\tau_+} I_{[\tau_+<\infty]} &= P[M=0]\{EM^2 - 2(EU_{\tau_+} I_{[\tau_+<\infty]}/P[M=0])^2\} \\ &= P[M=0]\{EM^2 - 2(EM)^2\}\,, \end{aligned} \tag{3.4}$$

and in general,

$$\begin{aligned} &EU^k_{\tau_+} I_{[\tau_+<\infty]}/P[M=0] \\ &\quad = EM^k - \sum_C \frac{k!}{k_1!\cdots k_\nu!} \cdot \frac{(EU^{k_1}_{\tau_+} I_{[\tau_+<\infty]})\cdots(EU^{k_\nu}_{\tau_+} I_{[\tau_+<\infty]})}{P^\nu[M=0]}\,, \end{aligned} \tag{3.5}$$

where $\sum_C$ *denotes summation over the set* C *of all ordered partitions* $(k_1,\cdots, k_\nu)$ *of* k *(i.e.,* $k=k_1+\cdots+k_\nu$*) with* $\nu\geq 2$ *and* $k_i\geq 1$ $(i=1,\cdots,\nu)$.

REMARK. For the special case $r=1$, the equivalence (3.2) was established by Taylor [15, page 742]. As an immediate consequence of (3.2), we obtain the following generalization of a theorem of Taylor [15] concerning stopping times of negative-drift random walks: If $EY<0$, then for $r>0$,

$$E(Y^+)^{r+1} < \infty \Longrightarrow E(U^+_\tau)^r I_{[\tau<\infty]} < \infty \tag{3.6}$$

for every random variable τ having a possibly defective distribution on the positive integers.

Proof of Theorem 4. Since $EY<0$, $P[\tau_+=\infty]=P[M=0]>0$. Define the succesive ladder indices $T_1=\tau_+$ and for $i\geq 2$

$$\begin{aligned} T_i &= \inf\{n>T_{i-1} : U_n - U_{T_{i-1}} > 0\} \quad \text{on } [T_{i-1}<\infty]\,, \\ &= \infty \quad \text{on } [T_{i-1}=\infty]\,. \end{aligned} \tag{3.7}$$

Obviously

$$M = U_{T_1} I_{[T_1<\infty]} + (U_{T_2} - U_{T_1}) I_{[T_2<\infty]} + \cdots\,. \tag{3.8}$$

From (3.8) and (1.5), (3.2) follows easily.

Let $T_0=0$ and $\tau_i=T_i-T_{i-1}$ for $i\geq 1$. It follows from (3.8) that

$$\begin{aligned} EM &= EU_{\tau_1} I_{[\tau_1<\infty]} + E(U_{\tau_1+\tau_2} - U_{\tau_1}) I_{[\tau_1<\infty,\,\tau_2<\infty]} + \cdots \\ &= \{EU_{\tau_1} I_{[\tau_1<\infty]}\}\{1 + P[\tau_1<\infty] + P[\tau_1<\infty,\,\tau_2<\infty] + \cdots\} \\ &= \{EU_{\tau_+} I_{[\tau_+<\infty]}\}/P[\tau_+=\infty]\,, \end{aligned} \tag{3.9}$$

since $P[\tau_1<\infty,\cdots,\tau_i<\infty]=P^i[\tau_+<\infty]$. Hence (3.3) holds.

We now use a similar argument to prove (3.5). Raising both sides of (3.8) to the kth power and expanding, we obtain that

$$\begin{aligned}EM^k &= \{EU^k_{\tau_+} I_{[\tau_+<\infty]}\}\{1+P[\tau_+<\infty]+P^2[\tau_+<\infty]+\cdots\}\\ &\quad+\sum_{C}\frac{k!}{k_1!\cdots k_\nu!}\{EU^{k_1}_{\tau_+} I_{[\tau_+<\infty]}\}\cdots\{EU^{k_\nu}_{\tau_+} I_{[\tau_+<\infty]}\}\\ &\quad\times\sum_{1\le j_1<\cdots<j_\nu}(P[\tau_+<\infty])^{j_\nu-\nu}.\end{aligned} \tag{3.10}$$

Letting $p=P[\tau_+<\infty]$, we note that for every fixed $\nu\geq 2$,

$$\begin{aligned}\sum_{1\le j_1<\cdots<j_\nu} p^{j_\nu-\nu} &= (1-p)^{-1}\sum_{1\le j_1<\cdots<j_{\nu-1}} p^{j_{\nu-1}-(\nu-1)}\\ &=\cdots=(1-p)^{-\nu}.\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), (3.5) holds. ∎

A useful estimate of the quantity $P[M=0]$ which appears in the formulas (3.3)–(3.5) of Theorem 4 is provided by the following basic result in fluctuation theory (cf. [6, page 379]):

$$P[M=0]=P[\tau_+=\infty]=1/E\tau_-, \tag{3.12}$$

where $\tau_-=\inf\{n\geq 1: U_n\leq 0\}$. By Wald's lemma,

$$\varepsilon E\tau_- = -EU_{\tau_-}\geq EY^-, \tag{3.13}$$

and therefore in view of (3.12), we obtain the following upper bound:

$$P[M=0]\leq \varepsilon/EY^-. \tag{3.14}$$

Making use of (3.3), (3.14) and Kingman's inequality (1.16), we have the following simple proof of a well-known result of Spitzer [14] concerning the ladder variable of a driftless random walk.

COROLLARY 2. *Let* $X, X_1, X_2,\cdots$ *be i.i.d. with* $EX=0$ *and let* $S_n=X_1+\cdots+X_n$. *Define* $T=\inf\{n\geq 1: S_n>0\}$. *Then*

$$0<EX^2<\infty \Longrightarrow ES_T<\infty. \tag{3.15}$$

Proof. For $\varepsilon>0$, let $Y_i=Y_i(\varepsilon)=X_i-\varepsilon$, $U_n=U_n(\varepsilon)=Y_1+\cdots+Y_n$, and define M and τ_+ as in (3.1). Assume that $0<EX^2<\infty$. Clearly as $\varepsilon\downarrow 0$, (3.16) $$U_{\tau_+} I_{[\tau_+<\infty]} \longrightarrow S_T \quad \text{a. s.}$$

Therefore in view of Fatou's lemma, it suffices to show that

(3.17) $$EU_{\tau_+} I_{[\tau_+<\infty]} = O(1) \quad \text{as} \quad \varepsilon \downarrow 0 .$$

Since $P[M=0] \ll \varepsilon$ by (3.14) and $EM \ll \varepsilon^{-1}$ by Kingman's inequality (1.16), (3.17) follows immediately from (3.3). ∎

The result (3.15) was first proved by Spitzer [14] using much deeper fluctuation thoeretic tools and a strictly analytic argument involving certain generating functions associated with the driftless random walk $\{S_n\}$. The above proof, which is based on considering the negative-drift random walk $\{S_n - \varepsilon n\}$ instead, is much more elementary and probabilistic in nature. Spitzer's analytic arguments have been generalized by Lai [10] who showed that more generally for $k = 1, 2, \cdots$,

(3.18) $$0 < E\,|X|^{k+1} < \infty \Longrightarrow ES_T^k < \infty .$$

By making use of (1.20) and (1.21) instead of Kingman's inequality (1.16), we are able to modify the argument used in Corollary 2 to give a simple proof of (3.18) for the case $k=2$.

COROLLARY 3. *With the same notations as in Corollary 2,*

(3.19) $$0 < E\,|X|^3 < \infty \Longrightarrow ES_T^2 < \infty .$$

Proof. As in proof of Corollary 2, it suffices to show that

(3.20) $$EU_{\tau_+}^2 I_{[\tau_+<\infty]} = O(1) \quad \text{as} \quad \varepsilon \downarrow 0$$

under the assumption $0 < E|X|^3 < \infty$. By (3.4), the left-hand side of (3.20) is equal to $P[M=0]\{EM^2 - 2(EM)^2\}$. Let $\sigma^2 = EX^2$. By (1.20), $EM = \frac{1}{2}\sigma^2/\varepsilon + O(1)$, and by (1.21), $EM^2 \sim \frac{1}{2}(\sigma^2/\varepsilon)^2$. Since $P[M=0] \ll \varepsilon$, (3.20) follows immediately. ∎

To be able to extend the above argument to prove (3.18) for higher values of k, we would need more detailed asymptotic expansions for $EM^i(\{\varepsilon n\})$ $(i = 1, \cdots, k)$ than (2.11) and the argument would become much more involved. In [3] we present an alternative approach also based on the negative-drift random walk $\{S_n - \varepsilon n\}$ to prove (3.18) for all values of k (k need not even be an integer).

While Corollary 1 establishes the asymptotic expansion (1.20) for EM_ε to the $O(1)$ term when $EX^2 < \infty$, a similar application of Theorem 3 turns out to yield also an asymptotic expansion for $\operatorname{Var} M_\varepsilon$ to the $O(1)$ term when $E|X|^3 < \infty$. Both these asymptotic approximations are of interest in queuing theory, since the mean and the variance of M_ε are important entities in the analysis of queues. Moreover, by making use of the well-known series representations of EM_ε and $\operatorname{Var} M_\varepsilon$ in terms of $E(S_n - \varepsilon n)^+$ and $E\{(S_n - \varepsilon n)^+\}^2$ (cf. [8]), we have the following interesting corollary on the asymptotic behavior of the series $\sum_1^\infty n^{-1} E(S_n - \varepsilon n)^+$ and $\sum_1^\infty n^{-1} E\{(S_n - \varepsilon n)^+\}^2$.

COROLLARY 4. *With the same notations as in Corollary 1, if $EX = 0$ and $EX^2 = \sigma^2 < \infty$, then as $\varepsilon \downarrow 0$,*

$$\sum_1^\infty n^{-1} E(S_n - \varepsilon n)^+ (= EM_\varepsilon) = \frac{1}{2}\sigma^2 \varepsilon^{-1} + O(1). \tag{3.21}$$

If furthermore $E|X|^3 < \infty$, then as $\varepsilon \downarrow 0$,

$$\begin{aligned}\sum_1^\infty n^{-1} E\{(S_n - \varepsilon n)^+\}^2 &(= \operatorname{Var} M_\varepsilon) \\ &= \frac{1}{4}\sigma^4 \varepsilon^{-2} + \frac{1}{3}(EX^3)\varepsilon^{-1} + O(1).\end{aligned} \tag{3.22}$$

Proof. We need only show (3.22). Let $Y_\varepsilon = X - \varepsilon$. By (2.1) and (2.2) (with $k = 2$),

$$\begin{aligned}\operatorname{Var} M_\varepsilon = &(4\varepsilon^2)^{-1}(EY_\varepsilon^2)^2 + (3\varepsilon)^{-1}(EY_\varepsilon^3) \\ &+ (3\varepsilon)^{-1} E((M_\varepsilon + Y_\varepsilon)^-)^3 - (4\varepsilon^2)^{-1}\{E((M_\varepsilon + Y_\varepsilon)^-)^2\}^2.\end{aligned}$$

Hence applying (2.9), we obtain (3.22). ∎

4. **Relation of the maximal excess to convergence rates for driftless random walks and the proof of Theorems 1 and 2.** In the preceding "Wiener-Hopf equation" approach to analyze $M(\{\varepsilon n\})$, the linearity of the boundary (which can then be transformed into the negative drift of a random walk) is crucial to the argument, and the method obviously fails to analyze $M(\{b(n)\})$ for nonlinear boundaries $b(n)$. To handle general boundaries $b(n)$, we shall make use of a maximal inequality for driftless random walks and the

following simple lemma which generalizes a similar result in [2] for the case $b(n)=\varepsilon n^{\alpha}$.

LEMMA 3. *Let* $S_0=0$, $S_1, S_2, \cdots$ *be any sequence of random variables (not necessarily sample sums). Let* $b(t)$ *be a positive non-decreasing function on* $[1, \infty)$ *such that* $\lim_{t\to\infty} b(t)=\infty$. *For* $x\geq 0$, *define* $\beta(x)=\inf\{t: b(t)\geq x\}$. *Set* $b(0)=0$ *and define* $M(\{b(n)\})$ *as in* (1.1). *Let* $c>1$. *Then for* $t>0$,

$$\begin{aligned} &P[S_n > cb(n) \text{ for some } n>\beta(t/(c-1))] \\ &\qquad \leq P[M(\{b(n)\})>t] \leq \sum_{j=1}^{\infty} P\Big[\max_{n\leq \beta(c^j t)} S_n > c^{j-1} t\Big]. \end{aligned} \tag{4.1}$$

Consequently, for $p>0$,

$$EM^p(\{b(n)\}) \leq \{pc^p/(c^p-1)\} \int_0^{\infty} u^{p-1} P\Big[\max_{n\leq \beta(cu)} S_n > u\Big] du\,. \tag{4.2}$$

Moreover, letting $L=L(\{cb(n)\})=\sup\{n\geq 1: S_n > cb(n)\}(\sup\emptyset=0)$,

$$EM^p(\{b(n)\}) \geq p(c-1)^p \int_0^{\infty} u^{p-1} P[L>\beta(u)]\, du\,. \tag{4.3}$$

Proof. Since $b(n)<c^j t \Rightarrow n\leq \beta(c^j t)$, it follows that

$$\begin{aligned} P[M(\{b(n)\})>t] &\leq \sum_{j=1}^{\infty} P\Big[\max_{(c^{j-1}-1)t\leq b(n)\leq (c^j-1)t} (S_n - b(n))>t\Big] \\ &\leq \sum_{j=1}^{\infty} P\Big[\max_{n\leq \beta(c^j t)} S_n > c^{j-1} t\Big]. \end{aligned}$$

To complete the proof of (4.1), we note that

$$\begin{aligned} &P[S_n > cb(n) \text{ for some } n>\beta(t/(c-1))] \\ &\qquad \leq P[S_n - b(n) > (c-1)\, b(n) \text{ for some } n \\ &\qquad\qquad\qquad \text{with } b(n)\geq t/(c-1)] \\ &\qquad \leq P[M(\{b(n)\})>t]. \end{aligned}$$

From (4.1), (4.2) and (4.3) follow easily. ∎

Throughout the rest of this section, we shall specialize Lemma 3 to the case of a driftless random walk $S_n = X_1+\cdots+X_n$, $X, X_1, X_2, \cdots$ being i.i.d. with $EX=0$. Our generalization (1.18) of Kingman's inequality was obtained in [2] by using Lemma 3 with $b(n)=\varepsilon n^{\alpha}$ and the following maximal inequality: For $x>0$, $1\leq q\leq 2$ and $n, k=1, 2, \cdots$,

$$(4.4)\qquad P\Big[\max_{j\le n} S_j > x\Big] \le n P[X > x/(2k)] + \{(4k/x)^q n E|X|^q\}^k$$

(cf. [2, page 55] and [11, page 68]). We shall now use Lemma 3 and (4.4) to prove Theorem 1.

Proof of Theorem 1. In view of (4.2) and (4.3), to prove the equivalence (1.10), it suffices to show the following implications:

$$(4.5)\qquad EB_p(X^+) < \infty \Longrightarrow \int_0^\infty u^{p-1} P\Big[\max_{j\le \beta(cu)} S_j > u\Big] du < \infty \quad \text{for all } c > 0,$$

$$(4.6)\qquad \int_0^\infty u^{p-1} P[L(\{cb(n)\}) > \beta(u)]\, du < \infty \quad \text{for some } c > 1 \Rightarrow EB_p(X^+) < \infty .$$

We first note that in view of (1.8b),

$$(4.7)\qquad EB_p(X^+) < \infty \Longleftrightarrow \int_0^\infty u^{p-1} \beta(cu) P[X > u]\, du < \infty \quad \text{for all (or equivalently for some) } c > 0 .$$

From (1.8a), it follows that $\beta(cu) \ll u^{1/\delta}$. Since $q > 1/\delta$, by setting $n = [\beta(cu)]$ and choosing k large enough in (4.4), it is easy to see that the implication (4.5) holds.

To prove the implication (4.6), we note that $n^{-\delta} S_n \to 0$ a.s. since $E|X|^q < \infty$ and $1/\delta < q \le 2$, and so in view of (1.8a), $S_n/b(n) \to 0$ a.s. Therefore by an argument similar to that used in Lemma 3 of [2], letting

$$L^*(\{2cb(n)\}) = \sup\{n \ge 1 : X_n > 2cb(n)\} (\sup \varnothing = 0),$$

we obtain that for all large m,

$$(4.8)\qquad P[L(\{cb(n)\}) > m] \ge \frac{1}{2} P[L^*(\{2cb(n)\}) > m] .$$

By Lemma 4 below,

$$(4.9)\qquad P[L^*(\{2cb(n)\}) > m] \ge m P[X > 2cb(2m)]/\{1 + m P[X > 2cb(2m)]\} .$$

In view of (1.8a), we have for all large m

$$(4.10)\qquad m P[X > 2cb(2m)] \le m P[X > m^{1/q}] \longrightarrow 0 \quad \text{as } m \to \infty,$$

since $E|X|^q < \infty$. By (1.8b), $b(2[\beta(u)]) \leq \gamma u$ for some $\gamma > 0$ and all large u. Therefore from (4.8), (4.9) and (4.10), it follows that

$$\int_0^\infty u^{p-1} P[L(\{cb(n)\}) > \beta(u)]\, du < \infty$$
$$\Longrightarrow \int_0^\infty u^{p-1} \beta(u)\, P[X > 2c\gamma u]\, du < \infty .$$

Hence in view of (4.7), the desired implication (4.6) holds. ∎

LEMMA 4. *Let $X, X_1, X_2, \cdots$ be i.i.d. random variables. Then for $\varepsilon > 0$ and $m = 1, 2, \cdots$,*

$$P\Big[\max_{j\leq m} X_j > \varepsilon\Big] \geq m P[X > \varepsilon]/\{1 + m P[X > \varepsilon]\} . \tag{4.11}$$

Moreover, if $\{b(n)\}$ is a nondecreasing sequence of real numbers and $L^*(\{b(n)\}) = \sup\{n \geq 1 : X_n > b(n)\}$, then for $m = 1, 2, \cdots$,

$$\begin{aligned} &P[L^*(\{b(n)\}) > m] \\ &\qquad \geq m P[X > b(2m)]/\{1 + m P[X > b(2m)]\} . \end{aligned} \tag{4.12}$$

Proof. The inequality (4.11) follows easily from

$$\begin{aligned} P\Big[\max_{j\leq m} X_j > \varepsilon\Big] &= P[X_1 > \varepsilon] + P[X_2 > \varepsilon]\, P[X_1 \leq \varepsilon] + \cdots \\ &\qquad + P[X_m > \varepsilon]\, P^{m-1}[X \leq \varepsilon] \\ &\geq m P[X > \varepsilon]\, P\Big[\max_{j\leq m} X_j \leq \varepsilon\Big]. \end{aligned}$$

To prove (4.12), we apply (4.11) and note that

$$\begin{aligned} P[L^*(\{b(n)\}) > m] &\geq P[X_n > b(n) \text{ for some } m < n \leq 2m] \\ &\geq P\Big[\max_{j\leq m} X_j > b(2m)\Big]. \quad ∎ \end{aligned}$$

REMARK. The preceding proof of Theorem 1 also gives the following convergence rate result: With the same notations and assumptions as in Theorem 1,

$$\begin{aligned} EB_p(X^+) < \infty \Longleftrightarrow \int_0^\infty u^{p-1} P\Big[\max_{j\leq \beta(cu)} S_j > u\Big]\, du < \infty \\ \text{for all (or equivalently for some) } c > 0 . \end{aligned} \tag{4.13}$$

In the literature on the strong law and related limit theorems, most convergence rate results are, however, concerned with the convergence or divergence of the integral

$$(4.14)\qquad \int_1^\infty t^{p-1} P\Big[\max_{j\le t} S_j > b(t)\Big]\,dt$$

instead. Let us assume that $b(t)$ is strictly increasing and differentiable. Then applying a change of variable $cu = b(t)$ to the integral in (4.13), we obtain that

$$(4.15)\qquad \begin{aligned}&\int_0^\infty u^{p-1} P\Big[\max_{j\le\beta(cu)} S_j > u\Big]\,du\\ &\qquad = c^{-p}\int_0^\infty (b(t))^{p-1}\, b'(t)\, P\Big[\max_{j\le t} S_j > b(t)/c\Big]\,dt\,.\end{aligned}$$

Viewing the integral above in comparison with the integral in (4.14), it is interesting also to compare (4.13) with the corresponding result for the integral in (4.14). In [11], it was proved, again by a simple application of the maximal inequality (4.4), that with the same notations and assumptions as in Theorem 1,

$$(4.16)\qquad E(\beta(X^+))^{p+1} < \infty \Longleftrightarrow \int_1^\infty t^{p-1} P\Big[\max_{j\le t} S_j > b(t)\Big]\,dt < \infty\,.$$

Proof of Theorem 2. We shall make use of Lemma 3 together with the ideas and results developed in [1], [9] and [11]. Setting $b(t) = \varepsilon(t\log t)^{1/2}$ in Lemma 3, we obtain from (4.3) that

$$(4.17)\qquad EM^p(\{\varepsilon(n\log n)^{1/2}\}) \ge \varepsilon^p(c-1)^p E(L\log L)^{p/2} \quad \text{for } c>1,$$

where $L = \sup\{n : S_n > c\varepsilon(n\log n)^{1/2}\}(\sup\emptyset = 0)$.

To prove (i), let $0<\varepsilon<(pEX^2)^{1/2}$. Choose $c>1$ such that $c\varepsilon < (pEX^2)^{1/2}$. Since X is symmetric, it then follows from Theorem 4(i) of [11] that $EL^{p/2} = \infty$. Hence in view of (4.17), $EM^p(\{\varepsilon(n\log n)^{1/2}\}) = \infty$.

We next proceed to prove (iv). Suppose $EX = 0$ and $EX^2 = \sigma^2 < \infty$. Assume that the one-sided moment condition (1.14) holds. In view of (4.2) and (4.15), to prove that (1.15) holds for all $\varepsilon > p^{1/2}\sigma$, it suffices to show that

$$(4.18)\qquad \int_1^\infty t^{(p/2)-1}(\log t)^{p/2} P\Big[\max_{j\le t} S_j > \varepsilon(t\log t)^{1/2}\Big]\,dt < \infty \quad \text{for all } \varepsilon > p^{1/2}\sigma\,.$$

Since the maximal inequality (4.4) clearly fails to yield the desired convergence rate result (4.18) for this "marginal" class of boundaries

$\varepsilon(t \log t)^{1/2}$, we shall use another approach which involves the analysis of the limiting behavior of delayed sums (see Lemma 5 below). For $r,\ t \geq 1$, define

$$S_{r,t} = \sum_{r<j\leq[r]+t} X_j\,, \qquad \bar{S}_{r,t} = \max_{1\leq j\leq t} S_{r,j}\,. \tag{4.19}$$

($S_{r,t}$ is called a delayed sum of the sequence $\{X_j\}$.) Set $\delta=2/(2+p)$ and $\psi(t) = \{(1-\delta)\log t\}^{-(1-\delta)}$ in Lemma 5 below and define $n_k = [k^{1/(1-\delta)}/(\log k)]$. By Lemma 5,

$$P[\bar{S}_{n_k,\, n_k^\delta \psi(n_k)} > \theta\{n_k^\delta \psi(n_k) \log n_k\}^{1/2} \text{ i. o.}] = 0 \quad \text{for all} \quad \theta > \{2(1-\delta)\}^{1/2}\sigma\,. \tag{4.20}$$

Clearly $n_k^\delta \psi(n_k) \sim k^{\delta/(1-\delta)}/(\log k)$ and $\{\bar{S}_{n_k,\, n_k^\delta \psi(n_k)} : k \geq k_0\}$ is a set of independent random variables for large k_0. Therefore by the Borel-Cantelli lemma, (4.20) is equivalent to

$$\int_1^\infty P\Big[\max_{j\leq g(u)} S_j > \theta\{g(u)(\log u)/(1-\delta)\}^{1/2}\Big]\, du < \infty \quad \text{for all} \quad \theta > \{2(1-\delta)\}^{1/2}\sigma\,, \tag{4.21}$$

where $g(u) = u^{\delta/(1-\delta)}/(\log u)$. Applying a change of variable $t=g(u)$ to the integral in (4.21) and noting that $\log t \sim (\delta/(1-\delta)) \log u$ and $2(1-\delta)/\delta = p$, it is easy to see that (4.18) holds.

Conversely assume that (1.15) holds for some $\varepsilon > 0$. To prove the moment condition (1.14) on X^+, we can make use of Lemma 4 and essentially repeat the argument used in Theorem 1.

We shall now prove (iii). Clearly the preceding proof establishes that if (1.12) holds then (1.13) also holds for every $\varepsilon > (p EX^2)^{1/2}$. Conversely, assume that (1.13) holds for some $\varepsilon > 0$. To prove that (1.12) holds, we need only show that $EX = 0$ and $EX^2 < \infty$, since we can then apply (iv) to obtain the desired moment condition for X^+ and X^- separately. Take $c > 1$ and note that as in (4.17),

$$\infty > E\widetilde{M}^p(\{\varepsilon(n \log n)^{1/2}\}) \geq \varepsilon^p(c-1)^p E(\tilde{L} \log \tilde{L})^{p/2}\,,$$

where $\tilde{L} = \sup\{n : |S_n| > c\varepsilon(n \log n)^{1/2}\}$. Hence $E\tilde{L}^{p/2} < \infty$, and so by Theorem 3 of [11], $EX = 0$ and $EX^2 < \infty$ as desired.

To prove (ii), we make use of (iii) and a truncation argument involving Esseen's error estimate in the normal approximation. The details are similar to [11, page 66] and are omitted. ∎

In the following lemma, the special case $\psi(t) = 1$ has been established in [2], and this special case is closely related to convergence rate results of the type (4.16) for the "marginal" boundaries $b(t) = \varepsilon(t\log t)^{1/2}$ and also to the finiteness of moments of L, where L is as defined in (4.17) (cf. [2] and [11]). As we have seen in proof of Theorem 2, the special case $\psi(t) = c(\log t)^{-(1-\delta)}$, however, is involved in the corresponding problem for $M(\{b(n)\})$. Our proof of Lemma 5 is based on a modification of the ideas used in the proof of Theorem 1 of [9] and Theorem 3 of [2] for the special case $\psi(t) = 1$. The details of the proof are omitted here.

LEMMA 5. *Let $X, X_1, X_2, \cdots$ be i.i.d. random variables such that $EX = 0$ and $EX^2 = \sigma^2 < \infty$. For r, $t \geq 1$, define $S_{r,t}$ and $\bar{S}_{r,t}$ as in* (4.19). *Let ψ be a positive continuous function on* $[1, \infty)$ *satisfying the following two conditions*:

(4.22a) ψ *is slowly varying*,

(4.22b) $(\log t)^{-\rho} \ll \psi(t) \ll (\log t)^{\rho}$ *as* $t \to \infty$ *for some* $\rho > 0$.

Then for every $0 < \delta < 1$, *the following statements are equivalent*:

(4.23) $$\sum_{1}^{\infty} P[X > \varepsilon\{n^{\delta}\psi(n)\log n\}^{1/2}] < \infty$$ *for some* (*or equivalently for all*) $\varepsilon > 0$;

(4.24) $$\limsup_{n\to\infty} \{2(1-\delta)\, n^{\delta}\psi(n)\log n\}^{-1/2} \bar{S}_{n, n^{\delta}\psi(n)} \leq \sigma \quad a.s.;$$

(4.25) $$\limsup_{n\to\infty} \{2(1-\delta)\, n^{\delta}\psi(n)\log n\}^{-1/2} S_{n, n^{\delta}\psi(n)} \leq \sigma \quad a.s.$$

In particular, if $\psi(t) \sim c(\log t)^{-(1-\delta)}$ for some constant c, then (4.23) *is equivalent to the moment condition*

(4.26) $$\int_{[X>e]} X^{2/\delta}(\log X)^{-1}\, dP < \infty.$$

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