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BOUNDARY VALUE PROBLEMS FOR nth ORDER ORDINARY DIFFERENTIAL EQUATIONS

BY

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Abstract. For nth order nonlinear ordinary differential equations when the boundary conditions are prescribed at two points, several existence and uniqueness results are obtained. In some cases it is shown that the results are best possible. An iterative method is given to the maximal solution. A comparison result is also given.

1. **Introduction.** In this paper we shall consider the nth order nonlinear differential equation

$$x^{(n)} + f(t, x, x', \cdots, x^{(r)}) = 0, \tag{1}$$

together with

$$\begin{aligned} &x^{(i)}(a) = 0, \quad i = 0, 1 \cdots, n-2, \\ &x^{(p)}(b) = 0 \end{aligned} \tag{2}$$

or

$$\begin{aligned} &x^{(p)}(a) = 0, \\ &x^{(i)}(b) = 0, \quad i = 0, 1, \cdots, n-2, \ (a < b), \end{aligned} \tag{3}$$

where $0 \le r \le p \le n-1$. It will always be assumed that the function $f(t, x, x', \cdots, x^{(r)})$ is continuous in $(t, x, x', \cdots, x^{(r)})$, at least in the interior of its domain.

In §2, we have used Schauder's fixed point theorem to discuss the existence of solutions for the above boundary value problems. In §3, several versions of the contraction mapping principle are used when the function f satisfies a Lipschitz condition over $[a, b] \times R^{r+1}$ and over a compact region (defined appropriately in the results) to prove the existence and uniqueness of the solutions, In §4, a weight function technique previously used by Collatz [9]

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is employed to find better results than those obtained in §3. For some particular cases best possible results are obtained, In §5, several weight functions are used to prove the existence and uniqueness of the solutions. For the boundary value problem (1), (2) best possible length of the interval is obtained. In §6, an iterative procedure is given which provides the existence and uniqueness, as well as an error bound which is of special type; namely, it satisfies the same boundary conditions as the solution. In the last section 7, a different successive approximation method is used to find the maximal solution of the boundary value problem (1), (2) for $0 \leq r \leq n-2$. A comparison theorem is also given.

The results obtained in this paper generalize the known results for the second and third order differential equations. Also it seems that some results obtained here cannot be improved further.

2. Existence and uniqueness.

LEMMA 1. *The Green's function of the boundary value problem*

$$(4)\qquad \begin{aligned} -x^{(n)}(t) &= 0, \\ x^{(i)}(a) &= 0, \qquad i=0, 1, \cdots, n-2, \\ x^{(p)}(b) &= 0, \qquad (0 \leq p \leq n-1) \end{aligned}$$

and all its derivatives with respect to t upto order p are non-negative.

Proof. It can easily be verified that the Green's function of the boundary value problem (4) is

$$(5)\qquad G(t, s) = \frac{1}{(n-1)!}\begin{cases} G_2(t, s) = G_1(t, s) - (t-s)^{n-1}, \\ \qquad\qquad a \leq s \leq t \leq b, \\ G_1(t, s) = (t-a)^{n-1}\left(\dfrac{b-s}{b-a}\right)^{n-p-1}, \\ \qquad\qquad a \leq t \leq s \leq b. \end{cases}$$

Hence the j-th derivative $(j \leq p)$ of $G(t, s)$ with respect to t is

$$(6)\qquad \frac{\partial^j G(t, s)}{\partial t^j} = \frac{1}{(n-j-1)!}\begin{cases} (t-a)^{n-j-1}\left(\dfrac{b-s}{b-a}\right)^{n-p-1} - (t-s)^{n-j-1}, \\ \qquad\qquad a \leq s \leq t \leq b, \\ (t-a)^{n-j-1}\left(\dfrac{b-s}{b-a}\right)^{n-p-1}, \\ \qquad\qquad a \leq t \leq s \leq b. \end{cases}$$

So it is sufficient to prove $\partial^j G_2(t, s)/\partial t^j \geq 0$ when $a \leq s \leq t \leq b$. Since it is true if $t = s$, we consider only $a \leq s < t \leq b$. Because

$$\frac{t-a}{t-s} \geq 1, \quad \frac{b-a}{b-s} \geq 1$$

and

$$\frac{t-a}{t-s} \geq \frac{b-a}{b-s},$$

we have

$$\left(\frac{t-a}{t-s}\right)^{n-j-1} \geq \left(\frac{b-a}{b-s}\right)^{n-p-1}$$

and hence

$$(t-a)^{n-j-1}\left(\frac{b-s}{b-a}\right)^{n-p-1} - (t-s)^{n-j-1} \geq 0.$$

LEMMA 2. *The Green's function of the boundary value problem*

$$\begin{aligned} &-x^{(n)}(t) = 0, \\ (7) \qquad &x^{(p)}(a) = 0, \qquad (0 \leq p \leq n-1), \\ &x^{(i)}(b) = 0, \qquad i = 0, 1, \cdots, n-2 \end{aligned}$$

and its jth derivative $(0 \leq j \leq p)$ *with respect to* t *are non-negative if* $(n+j)$ *is even and nonpositive if* $(n+j)$ *is odd.*

Proof. The Green's function of the boundary value problem (7) is

$$(8) \qquad H(t, s) = \frac{(-1)^n}{(n-1)!}\begin{cases} H_2(t, s) = (b-t)^{n-1}\left(\dfrac{s-a}{b-a}\right)^{n-p-1}, & a \leq s \leq t \leq b, \\ H_1(t, s) = H_2(t, s) - (s-t)^{n-1}, & a \leq t \leq s \leq b. \end{cases}$$

Hence

$$(9) \qquad \frac{\partial^j H(t, s)}{\partial t^j} = \frac{(-1)^{n+j}}{(n-j-1)!}\begin{cases} (b-t)^{n-j-1}\left(\dfrac{s-a}{b-a}\right)^{n-p-1}, & a \leq s \leq t \leq b, \\ (b-t)^{n-j-1}\left(\dfrac{s-a}{b-a}\right)^{n-p-1} - (s-t)^{n-j-1}, & a \leq t \leq s \leq b. \end{cases}$$

Since

$$(b-t)^{n-j-1}\left(\frac{s-a}{b-a}\right)^{n-p-1}-(s-t)^{n-j-1}\geq 0\,, \tag{10}$$

if $a\leq t\leq s\leq b$ as in Lemma 1, the result follows.

LEMMA 3.

$$\begin{aligned}\int_a^b\left|\frac{\partial^j G(t,s)}{\partial t^j}\right|ds &= \int_a^b\frac{\partial^j G(t,s)}{\partial t^j}\,ds\\ &=\frac{1}{(n-j-1)!}\left[\frac{b-a}{n-p}(t-a)^{n-j-1}-\frac{1}{n-j}(t-a)^{n-j}\right]\\ &=\phi_j(t)\,.\end{aligned} \tag{11}$$

$$\begin{aligned}&\int_a^b\left|\frac{\partial^j H(t,s)}{\partial t^j}\right|ds\\ &=\frac{1}{(n-j-1)!}\left[\frac{b-a}{n-p}(b-t)^{n-j-1}-\frac{1}{n-j}(b-t)^{n-j}\right]\\ &=\psi_j(t)\,.\end{aligned} \tag{12}$$

$$\max_{a\leq t\leq b}\phi_j(t)=\max_{a\leq t\leq b}\psi_j(t)=\alpha_j\,, \tag{13}$$

where

$$\begin{aligned}\alpha_j &=\frac{(n-j-1)^{n-j-1}}{(n-j)!}\times\left(\frac{b-a}{n-p}\right)^{n-j} && \text{if}\quad n-1>p=j\,,\\ &=(b-a) && \text{if}\quad n-1=p=j\,,\\ &=\frac{(p-j)(b-a)^{n-j}}{(n-p)(n-j)!} && \text{if}\quad n-1\geq p\geq j+1\,.\end{aligned} \tag{14}$$

Proof. See the appendix.

THEOREM 4. *Let $M_j>0$ $(j=0,1,\cdots,r)$ be given real numbers and let Q be the maximum of $|f(t,x,x',\cdots,x^{(r)})|$ on the compact set*

$$\{(t,x,x',\cdots,x^{(r)}): a\leq t\leq b,\ |x^{(j)}|\leq M_j,\ j=0,1,\cdots,r\}.$$

Then if

$$\begin{aligned}(b-a)&\leq(n-p)\left(\frac{(n-j)!\,M_j}{a(n-j-1)^{n-j-1}}\right)^{1\,(n-j)}\\ & && \text{if}\quad n-1>p=j\,,\\ &\leq\frac{M_j}{Q} && \text{if}\quad n-1=p=j\,,\\ &\leq\left(\frac{(n-j)!\,(n-p)\,M_j}{Q(p-j)}\right)^{1/(n-j)} && \text{if}\quad n-1\geq p\geq j+1\,,\\ & && j=0,1,\cdots,r\end{aligned} \tag{15}$$

each of the boundary value problems (1), (2); (1), (3) *has a solution of class* $C^{(n)}$ *on* $[a, b]$. *Furthermore, given* $\varepsilon > 0$ *there is a solution* $x(t)$ *such that*

$$(16) \qquad |x^{(j)}(t)| < \varepsilon, \qquad j = 0, 1 \cdots, r,$$

on $[a, b]$ *provided* $b - a$ *is sufficiently small.*

Proof. Here we shall give the proof for the boundary value problem (1), (2). If the integral equation

$$(17) \qquad x(t) = \int_a^b G(t, s) f(s, x(s), \cdots, x^{(r)}(s))\, ds$$

has a solution of class $C^{(r)}$ on $[a, b]$, then it will also be a solution for (1), (2).

The set

$$B[a, b] = \{x(t) \in C^{(r)}[a, b] : \|x^{(j)}\| \leq M_j,\ j = 0, 1, \cdots, r\},$$

where

$$\|x^{(j)}\| = \max_{a \leq t \leq b} |x^{(j)}(t)|,$$

is a closed convex subset of the Banach space $C^{(r)}[a, b]$. The mapping $T : C^{(r)}[a, b] \rightarrow C^{(n)}[a, b]$ defined by

$$(18) \qquad (Tx)(t) = \int_a^b G(t, s) f(s, x(s), x'(s), \cdots, x^{(r)}(s))\, ds$$

is completely continuous. For $x \in B[a, b]$, we have

$$\begin{aligned} |(Tx)^{(j)}(t)| &\leq Q \int_a^b \frac{\partial^j G(t, s)}{\partial t^j}\, ds \\ &= Q\phi_j(t) \leq \alpha_j \qquad j = 0, 1, \cdots, r, \end{aligned}$$

where we have used Lemmas 1 and 3. Thus condition (15) implies that T maps $B[a, b]$ into itself. It then follows from the Schauder fixed point theorem that T has a fixed point in $B[a, b]$. The fixed point is a solution of the stated boundary value problem.

If $x(t)$ is a solution of (1), (2) with $x \in B[a, b]$, then $|x^{(j)}(t)| \leq Q\alpha_j$ $(j = 0, 1, \cdots, r)$ on $[a, b]$ and the last assertion of the therem follows.

COROLLARY 5. *Assume the function* $f(t, x, x', \cdots, x^{(r)})$ *satisfies the condition that*

$$|f(t, x, x', \cdots, x^{(r)})| \le C_0 + \sum_{j=0}^{r} C_{j+1} |x^{(j)}|^{\alpha(j)},$$

where $0 < \alpha(j) < 1$ *for* $j = 0, 1, \cdots, r$. *Then each of the boundary value problems* (1), (2); (1), (3) *has a solution.*

3. Uniqueness results.

THEOREM 6. *Let the function* $f(t, x, x', \cdots, x^{(r)})$ *satisfy a Lipschitz condition of the form*

$$\begin{aligned} &|f(t, x, x', \cdots, x^{(r)}) - f(t, y, y', \cdots, y^{(r)})| \\ &\le \sum_{j=0}^{r} L_j |x^{(j)} - y^{(j)}| \end{aligned} \tag{19}$$

on $[a, b] \times R^{r+1}$. *Then if*

$$\alpha = \sum_{j=0}^{r} L_j \alpha_j < 1, \tag{20}$$

each of the boundary value problems (1), (2); (1), (3) *has one and only one solution.*

Proof. Here also we shall give the proof for the boundary value problem (1), (2). Let the space S consist of continuously r times differentiable functions on $[a, b]$ with norm

$$\|u\| = \max_{a \le t \le b} \sum_{j=0}^{r} L_j |u^{(j)}(t)|.$$

We shall prove that the operator T defined as in (18) on S is a contracting operator on S. To this end we have

$$\begin{aligned} &|(Tx)^{(j)}(t) - (Ty)^{(j)}(t)| \\ &\le \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} \sum_{i=0}^{r} L_i |x^{(i)}(s) - y^{(i)}(s)| ds \\ &\le \|x - y\| \max_{a \le t \le b} \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} ds \\ &\le \|x - y\| \alpha_j, \qquad j = 0, 1, \cdots, r. \end{aligned} \tag{21}$$

Multiplying (21) by L_j and summing over all j, we obtain

$$\begin{aligned} \|Tx - Ty\| &\le \|x - y\| \sum_{j=0}^{r} L_j \alpha_j \\ &= \alpha \|x - y\|. \end{aligned}$$

Since $\alpha < 1$, the operator T is contracting and hence the problem (1), (2) has one and only one solution.

To prove the next result, we need the following:

LEMMA 7. (Falb and Jong [10]). *Let T map a ball $B = \{w: \|w - y_0\| \leq \mu\}$ of a Banach space S into S. If there is an $\alpha \in (0, 1)$ such that for all $u, v \in B$*

$$\|Tu - Tv\| \leq \|u - v\| \tag{22}$$

and if

$$\|Ty_0 - y_0\| \leq \mu(1 - \alpha), \tag{23}$$

then T has a unique fixed point y in B. If T maps the ball B into itself, condition (23) *can be omitted.*

In the next theorem we shall show that the function f need not satisfy a Lipschitz condition on $[a, b] \times R^{r+1}$ but it is sufficient if f satisfies Lipschitz condition on D

$$D = \Big\{(t, x(t)) : a \leq t \leq b,\ x(t) \in C^{(r)}[a, b], \quad |x^{(j)}(t)| \leq N \frac{\alpha_j}{\alpha_0},\ j = 0, 1, \cdots, r\Big\}. \tag{24}$$

THEOREM 8. *Let the function $f(t, x, x', \cdots, x^{(r)})$ satisfy a Lipschitz condition* (19) *on D, where N satisfies either*

$$m\alpha_0 \leq N(1 - \alpha) \tag{25}$$

if $m = \max |f(t, 0, 0, \cdots, 0)|$ for $a \leq t \leq b$, or merely

$$M\alpha_0 \leq N \tag{26}$$

if $M = \max_D |f(t, x, x', \cdots, x^{(r)})|$.

Then each of the boundary value problems (1), (2); (1), (3) *has one and only one solution $x(t) \in D$.*

Proof. Let the space S consist of r times continuously differentiable functions on $[a, b]$ with the norm

$$\|u\| = \max_{0 \leq j \leq r} \left\{ \frac{\alpha_0}{\alpha_j} \max_{a \leq t \leq b} |u^{(j)}(t)| \right\}.$$

Let $x(t) \equiv 0$ and B be the ball $\{w \in S : \|w\| \leq N\}$. Then if $u(t)$, $v(t) \in B$, we have from the operator T defined as in (18) on B

$$|(Tu)^{(j)}(t)-(Tv)^{(j)}(t)|$$
$$\leq \sum_{i=0}^{r}\left\{L_i \max_{a\leq t\leq b}|u^{(i)}(t)-v^{(i)}(t)|\ x\cdot\int_a^b \frac{\partial^j G(t,s)}{\partial t^j}\,ds\right\}$$
$$\leq \alpha_j \sum_{i=0}^{r} L_i \frac{\alpha_i}{\alpha_0}\|u-v\|$$

and hence

$$\frac{\alpha_0}{\alpha_j}|(Tu)^{(j)}(t)-(Tv)^{(j)}(t)| \leq \sum_{i=0}^{r} L_i\,\alpha_i\|u-v\|$$
$$\leq \alpha\|u-v\| \qquad j=0,1,\cdots,r,$$

from which it follows that

$$\|Tu-Tv\|\leq\alpha\|u-v\|.$$

To apply Lemma 7, we need to show that (23) holds. Let (25) hold; then we have

$$|(Tx_0)^{(j)}(t)|\leq\int_a^b \frac{\partial^j G(t,s)}{\partial t^j}|f(s,0,0,\cdots,0)|\,ds$$
$$\leq m\alpha_j \qquad j=0,1,\cdots,r.$$

Hence we have

$$\|(Tx_0)-x_0\|\leq N(1-\alpha).$$

Next let (26) hold; then for any $u\in B$ we have

$$|u^{(j)}(t)|\leq\frac{\alpha_j}{\alpha_0}N$$

and by the hypothesis $M=\max_D|f(t,u,u',\cdots,u^{(r)})|$. It follows that

$$|(Tu)^{(j)}(t)|\leq\int_a^b\frac{\partial^j G(t,s)}{\partial t^j}|f(s,u(s),u'(s),\cdots,u^{(r)}(s))|\,ds$$
$$\leq M\alpha_j \qquad j=0,1,\cdots,r.$$

Hence

$$\|Tu\|\leq N.$$

This completes the proof of Theorem 8.

For $n=2$ see Bailey [6] and for $n=3$ see Agarwal and Krishnamoorthy [4].

4. Some improved uniqueness results.

THEOREM 9. *Let the function* $f(t, x, x', \cdots, x^{(r)})$ *satisfy a Lipschitz condition* (19) *on* $[a, b] \times R^{r+1}$. *Then if*

$$\sup_{a\le t\le b} \frac{1}{W(t)} \sum_{j=0}^{r} L_j \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} W(s)\, ds = \alpha < 1, \tag{27}$$

the boundary value problem (1), (2) has one and only one solution. Also, if

$$\sup_{a\le t\le b} \frac{1}{W(t)} \sum_{j=0}^{r} L_j \int_a^b \left| \frac{\partial^j H(t, s)}{\partial t^j} \right| W(s)\, ds = \alpha < 1, \tag{28}$$

the boundary value problem (1), (3) has one and only one solution, where $W(t)$ is a positive, or possibly non-negative, continuous function on $[a, b]$.

Proof. Let S be the space of r times continuously differentiable functions. The norm on the space S is defined as follows:

$$\|u\| = \sup_{a\le t\le b} \frac{1}{W(t)} \sum_{j=0}^{r} L_j \,|u^{(j)}(t)|.$$

We have from the operator T defined as in (18) on S

$$\begin{aligned} &|(Tx)^{(j)}(t) - (Ty)^{(j)}(t)| \\ &\quad \le \int_a^b \left\{ \frac{\partial^j G(t, s)}{\partial t^j} \,|f(s, x(s), x'(s), \cdots, x^{(r)}(s)) \right. \\ &\qquad\qquad \left. - f(s, y(s), y'(s), \cdots, y^{(r)}(s))| \right\} ds \\ &\quad \le \int_a^b \left\{ \frac{\partial^j G(t, s)}{\partial t^j} W(s) \sum_{i=0}^{r} L_i \frac{|x^{(i)}(s) - y^{(i)}(s)|}{W(s)} \right\} ds \\ &\quad \le \|x - y\| \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} W(s)\, ds \end{aligned}$$

and hence

$$\begin{aligned} \|Tx - Ty\| &\le \left\{ \sup_{a\le t\le b} \frac{1}{W(t)} \sum_{j=0}^{r} \int_a^b L_j \frac{\partial^j G(t, s)}{\partial t^j} W(s)\, ds \right\} \|x - y\| \\ &= \alpha \|x - y\|. \end{aligned}$$

Since $\alpha < 1$, we can apply the contraction mapping principle, which will ensure the existence of a unique solution of the boundary value problem (1), (2), and the proof for the boundary value problem (1), (3) is similar.

The following particular cases of Theorem 9 are of independent interest.

4.1 If we take $W(t) \equiv 1$, then Theorem 9 is the same as Theorem 6.

4.2 Let $n=2$, $p=r=0$, and

$$W(t) = \frac{(b-t)(t-a)}{2};$$

then condition (27) takes the form

$$\frac{5}{48} L_0(b-a)^2 < 1,$$

which is better than the result

$$\frac{(\sqrt{3}-1)}{4\sqrt{3}} L_0(b-a)^2 < 1$$

obtained by Coles and Sherman [8]. Also, if we take

$$W(t) = \sin \pi \left(\frac{t-a}{b-a}\right),$$

then condition (27) takes the form

$$\frac{1}{\pi^2} L_0(b-a)^2 < 1,$$

which is the same as obtained by Bailey et al. [6].

4.3 Let $n=3$, $p=r=0$, and

$$W(t) = \frac{1}{6}(t-a)^2(b-t);$$

then condition (27) takes the form

$$\frac{1}{60} L_0(b-a)^3 < 1,$$

which is better than

$$\frac{3}{160} L_0(b-a)^3 < 1$$

obtained by Agarwal [1, 2]. Also, if we take $W(t)$ as the nontrivial solution of the homogeneous differential equation

$$W'''(t) + L_0W(t) = 0$$

satisfying $W(a) = W'(a) = W(b) = 0$ with $W(t) > 0$, $t \in (a, b)$, then condition (27) takes the form $(b-a) < (b-a)_1$, where $(b-a)_1$ is the first positive root of the equation in l:

$$2\sin\left(\frac{\sqrt{3}}{2}L_0^{1/3}l - \frac{\pi}{6}\right) + \exp\left(-\frac{3}{2}L_0^{1/3}l\right) = 0,$$

which is the same as obtained by Agarwal [3].

4.4 Let $r = 0$ and let $W(t)$ be the nontrivial solution of the homogeneous differential equation

$$W^{(n)}(t) + L_0 W(t) = 0$$

satisfying $W^{(i)}(a) = 0$, $i = 0, 1, \cdots, n-2$, $W^{(p)}(b) = 0$ with $W^{(p)}(t) > 0$, $t \in (a, b)$ and let $a + l_p(L_0)$ denote the first point after a where the pth derivative of $W(t)$ vanishes. Then condition (27) takes the form $(b-a) < l_p(L_0)$ and this result is best possible. For, the boundary value problem

$$\begin{aligned} &u^{(n)}(t) + L_0 u(t) = 0, \\ &u^{(i)}(a) = 0, && i = 0, 1, \cdots, n-2, \\ &u^{(p)}(a + l_p(L_0)) = 0, && 0 \le p \le n-1, \end{aligned}$$

has the trivial as well as nontrivial solution; also if $u^{(p)}(a + l_p(L_0)) \neq 0$, then there is no solution.

5. Several weight functions.

THEOREM 10. *Let the function* $f(t, x, x', \cdots, x^{(r)})$ *satisfy a Lipschitz condition* (19) *on* $[a, b] \times R^{r+1}$. *Then if*

$$\frac{1}{W_j(t)} \int_a^b \left\{\frac{\partial^j G(t, s)}{\partial t^j} \sum_{i=0}^{r} L_i W_i(s)\right\} ds \le \alpha < 1 \tag{29}$$

$$j = 0, 1, \cdots, r,$$

the boundary value problem (1), (2) *has one and only one solution. Also, if*

$$\frac{1}{W_j(t)} \int_a^b \left\{\left|\frac{\partial^j H(t, s)}{\partial t^j}\right| \sum_{i=0}^{r} L_i W_i(s)\right\} ds \le \alpha < 1, \tag{30}$$

the boundary value problem (1), (3) *has one and only one solution,* *where* $W_i(t)$ $(i=0, 1, \cdots, r)$ *are positive, or possibly non-negative, continuous functions on* $[a, b]$.

Proof. Let S be the space of r times continuously differentiable functions. The norm on the space S is defined as follows:

$$\|u\| = \max_{0\le j\le r}\left\{\max_{a\le t\le b}\frac{|u^{(j)}(t)|}{W_j(t)}\right\}.$$

We have from the operator T defined as in (18) on S

$$\begin{aligned}|(Tx)^{(j)}(t)-(Ty)^{(j)}(t)| &\le \int_a^b\left\{\frac{\partial^j G(t,s)}{\partial t^j}\sum_{i=0}^r L_i W_i(s)\left|\frac{|x^{(i)}(s)-y^{(i)}(s)|}{W_i(s)}\right.\right\}ds\\ &\le \|x-y\|\int_a^b\left\{\frac{\partial^j G(t,s)}{\partial t^j}\right\}\sum_{i=0}^r L_i\, W_i(s)\Big\}\,ds\end{aligned}$$

and hence

$$\|Tx-Ty\| \le \left[\frac{1}{W_j(t)}\int_a^b\left\{\frac{\partial^j G(t,s)}{\partial t^j}\sum_{i=0}^r L_i\, W_i(s)\right\}ds\right]\|x-y\|,$$
$$j=0, 1, \cdots, r.$$

Since $\alpha<1$, we can apply the contraction mapping principle. The proof for the problem (1), (3) is similar.

5.1 Let $n=3$, $r=p=2$, and

$$W_0(t)=\frac{1}{2}(t-a)^2(b-a)-\frac{1}{6}(t-a)^3,$$
$$W_1(t)=-\frac{1}{2}(t-a)^2+(t-a)(b-a),$$
$$W_2(t)=(b-t).$$

Then condition (29) takes the form

$$\frac{1}{3}L_0(b-a)^3+\frac{1}{2}L_1(b-a)^2+\frac{1}{2}L_2(b-a)<1,$$

which is the same as obtained by Agarwal [2].

5.2 We attempt to find suitable $W_j(t)$ $(j=0, 1, \cdots, r)$ by requiring equality in (29). For this we choose $W_j(t)=W^{(j)}(t)$, and choose $W(t)$ to satisfy

$$W^{(n)}(t)+\frac{1}{\alpha}\sum_{j=0}^r L_j W^{(j)}(t)=0 \tag{31}$$

with $W^{(p)}(t) > 0$, $t \in (a, b]$, so that $W^{(i)}(t) > 0$, $t \in (a, b]$ for all $i = 0, 1, \cdots, p$; and if $p = n-1$, $W^{(p)}(t) > 0$, $t \in [a, b]$.

Now let $u(t)$ be the solution of

$$u^{(n)}(t) + \sum_{j=0}^{r} L_j u^{(j)}(t) = 0 \tag{32}$$

satisfying $u^{(i)}(a) = 0$ $(i = 0, 1, \cdots, n-2)$, $u^{(p)}(t) > 0$ on $(a, b]$; and if $p = n-1$ then $u^{(n-1)}(t) > 0$ on $[a, b]$.

Since the solutions of (31) depend continuously on α, we choose α sufficiently close to but less than 1, so that (31) has a solution $W(t)$ which satisfies $W^{(i)}(a) = 0$ $(i = 0, 1, \cdots, n-2)$ and whose pth derivative is arbitrarily close to the pth derivative of $u(t)$. Since $u^{(p)}(t) > 0$ on $(a, b]$, and $u^{(p)}(t) > 0$ on $[a, b]$ if $p = n-1$, $W^{(p)}(t)$ can be taken to be strictly positive on $(a, b]$; and if $p = n-1$ we can take it to be so on $[a, b]$. With such α and $W_j(t) = W^{(j)}(t)$ $(j = 0, 1, \cdots, r)$, equality holds in (29), and Theorem (10) ensures the existence of a unique solution of the problem (1), (2).

From the above observation, if we let $a + l_p(L_0, L_1, \cdots, L_r)$ denote the first point after a where—pth derivative of $u(t)$ vanishes, then if $(b-a) < l_p$ $(L_0, L_1, \cdots, L_r)$ there exists a unique solution of (1), (2) and this result is best possible. For if equality holds, then there exists a trivial as well as a nontrivial solution of (32). Also, if $u^{(p)}(a + l_p(L_0, L_1, \cdots, L_r)) \neq 0$, then there is no solution.

For $n = 2$, the same results are obtained by Bailey et al. [6] and for $n = 3$ by Agarwal [3].

6. **Convergence of successive approximation.** Hereafter in this section we shall denote $l_p(L_0, L_1, \cdots, L_r)$ by $l_p(L)$ in short. First we shall prove that $l_p(\mu L)$ is a decreasing function of μ and hence there is $\mu > 1$ such that $l_p(\mu L) = b - a$. For this, we prove

THEOREM 11. *Let $u(t)$, $u_1(t)$ be two functions satisfying*

$$D_n u(t) \equiv u^{(n)}(t) + \sum_{j=0}^{r} L_j u^{(j)}(t) = 0,$$

$$u_1^{(n)}(t) + \sum_{j=0}^{r} L_j u_1^{(j)}(t) \leq 0,$$

$$u^{(i)}(a) = u_i^{(1)}(a), \qquad i = 0, 1, \cdots, n-1.$$

Then

$$u^{(p)}(t) \geq u_1^{(p)}(t),\ t \in [a,\ a + l_p(L)].$$

Proof. Inequality $D_n u_1(t) \leq 0$ with some $\phi(t) \geq 0$ can be written as $D_n u_1(t) + \phi(t) = 0$, and hence if $u(t)$ is the solution of the equation $D_n u(t) = 0$ with $u^{(j)}(a) = u_1^{(j)}(a)$ $(j = 0, 1, \cdots, n-1)$, then we have

$$u_1(t) = u(t) - \int_a^t w(t-s)\ \phi(s)\, ds,$$

where $w(t)$ is the solution of $D_n w(t) = 0$ with $w^{(j)}(a) = 0$, $j = 0, 1, \cdots, n-2$ and $w^{(n-1)}(a) = 1$. Thus we have

$$u_1^{(p)}(t) - u^{(p)}(t) = -\int_a^t \frac{\partial^p w(t-s)}{\partial t^p}\ \phi(s)\, ds.$$

But since $(\partial^p w(t-s))/\partial t^p \geq 0$ as long as $a \leq s \leq t \leq a + l_p(L)$, the result follows immediately.

Now if we take $v(t)$ as the solution of the equation

$$v^{(n)}(t) + \mu \sum_{j=0}^{r} L_j v^{(j)}(t) = 0 \qquad (\mu > 1)$$

and $u(t)$ as the solution of (32) with $u^{(i)}(a) = v^{(i)}(a) = 0$, $i = 0, 1, \cdots, n-2$ and $u^{(n-1)}(a) = v^{(n-1)}(a) = 1$, then from Theorem 11, $u^{(p)}(t) \geq v^{(p)}(t)$, $t \in [a,\ a + l_p(L)]$. Hence $v^{(p)}(t)$ must vanish before $a + l_p(L)$ since $u^{(p)}(t)$ vanishes at $a + l_p(L)$. This simple fact and the uniqueness of the solutions of initial value problems imply that for some $\mu > 1$, $l_p(\mu\ L) = b - a$. Thus the problem

$$\begin{aligned} &\psi^{(n)}(t) + \mu \sum_{j=0}^{r} L_j \psi^{(j)}(t) = 0 \qquad (\mu > 1), \\ &\psi^{(i)}(a) = 0, \qquad i = 0, 1, \cdots, n-2, \\ &\psi^{(p)}(b) = 0, \end{aligned} \tag{33}$$

has a nontrivial solution with $\psi^{(p)}(t) > 0$, $t \in (a, b)$, and if $p = n-1$ then on $[a, b)$. To fix the choice of $\psi(t)$ we also require $\psi^{(n-1)}(a) = 1$. Then it is always possible to choose C to be the smallest possible constant such that

$$\int_a^b \frac{\partial^j G(t,s)}{\partial t^j}\, ds \leq C\left(\frac{1}{\mu}\right) \psi^{(j)}(t). \tag{34}$$

THEOREM 12. *Let* $f(t, x, x', \cdots, x^{(r)})$ *satisfy Lipschitz condition* (19) *on* D

$$D = \{(t, x(t)) : a \le t \le b,\ x(t) \in C^{(r)}[a, b],$$
$$|x^{(j)}(t)| \le m \cdot C \cdot \frac{1}{\mu}\left(1 - \frac{1}{\mu}\right)^{-1} \psi^{(j)}(t),$$
$$j = 0, 1, \cdots, r\}.$$

Then an infinite sequence $\{x_n(t)\}$ $(n = 0, 1, 2, \cdots)$ *can be obtained in* D *by the successive approximations*

$$x_0(t) \equiv 0,$$
$$x_{n+1}(t) = \int_a^b G(t, s) f(s, x_n(s), x_n'(s), \cdots, x_n^{(r)}(s))\, ds \qquad n = 0, 1, 2, \cdots \tag{35}$$

and as $n \to \infty$ *it converges to a unique solution of the boundary value problem* (1), (2) *in* D.

Proof. First we shall prove by induction that an infinite sequence $\{x_n(t)\}$ $(n = 0, 1, 2, \cdots)$ can be obtained in D by the successive approximations (35). Since $x_0(t) \in D$, let us assume that $x_1(t), x_2(t), \cdots, x_{n-1}(t)$ have been obtained in D. Then $x_n(t)$ is obtained by (35) since $x_{n-1}(t) \in D$. In order to complete the induction it is sufficient to prove $x_n(t) \in D$. Now by (35) and (34) we have

$$\begin{aligned} |x_1^{(j)}(t) - x_0^{(j)}(t)| &\le \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} |f(s, 0, 0, \cdots, 0)|\, ds \\ &\le m \int_a^b \frac{\partial^j G(t, s)}{\partial t^j}\, ds \\ &\le m \cdot C \cdot \frac{1}{\mu} \cdot \psi^{(j)}(t) \qquad j = 0, 1, \cdots r. \end{aligned} \tag{36}$$

Hence $x_1(t) \in D$.

Recall that the problem (33) is equivalent to

$$\psi^{(j)}(t) = \mu \int_a^b \left\{ \frac{\partial^j G(t, s)}{\partial t^j} \sum_{i=0}^{r} L_i \psi^{(i)}(s) \right\} ds \qquad j = 0, 1, \cdots, r.$$

Hence by (35) and (36), we have

$$|x_2^{(j)}(t)-x_1^{(j)}(t)| \leq \int_a^b \left\{\frac{\partial^j G(t,s)}{\partial t^j} \sum_{i=0}^r L_i |x_1^{(i)}(s)-x_0^{(i)}(s)|\right\} ds$$

$$\leq m\cdot C\cdot\left(\frac{1}{\mu}\right)\int_a^b \left\{\frac{\partial^j G(t,s)}{\partial t^j} \sum_{i=0}^r L_i \psi^{(i)}(s)\right\} ds$$

$$\leq m\cdot C\cdot\left(\frac{1}{\mu}\right)^2 \psi^{(j)}(t) \qquad j=0,1,\cdots,r.$$

Continuing in this way, we get

$$|x_n^{(j)}(t)-x_{n-1}^{(j)}(t)| \leq m\cdot C\cdot\left(\frac{1}{\mu}\right)^n \psi^{(j)}(t) \qquad j=0,1,\cdots,r. \tag{37}$$

Then from these inequalities we successively have

$$|x_n^{(j)}(t)-x_0^{(j)}(t)| \leq \sum_{i=0}^{n-1} |x_{n-i}^{(j)}(t)-x_{n-i-1}^{(j)}(t)|$$

$$\leq \sum_{i=0}^{n-1} m\cdot C\cdot\left(\frac{1}{\mu}\right)^{n-i}\psi^{(j)}(t)$$

$$\leq m\cdot C\cdot\left(\frac{1}{\mu}\right)\left(1-\frac{1}{\mu}\right)^{-1}\psi^{(j)}(t),$$

$$(\text{since } \mu>1) \qquad j=0,1,\cdots,r$$

and hence $x_n(t)\in D$. This completes the induction. Also, since $\mu>1$, estimates (37) ensure that the sequence $\{x_n(t)\}$ converges to a limit, say $x(t)$. Hence we have proved that the boundary value problem (1), (2) has at least one solution in D. Since it is easy to prove the uniqueness part it is left to the reader.

6.1 From Theorem 12, the error estimates are obtained as

$$|x^{(j)}(t)-x_n^{(j)}(t)| \leq m\cdot C\cdot\left(\frac{1}{\mu}\right)^{n+1}\left(1-\frac{1}{\mu}\right)^{-1}\psi^{(j)}(t) \qquad j=0,1,\cdots,r. \tag{38}$$

For $n=2$, the same results are obtained by Bailey et al. [6] and for $n=3$ Agarwal [5]. The error bound obtained in (38) satisfies the same boundary conditions as the original problem.

7. **Maximal solution and comparison result.** Here we shall consider $0\leq r\leq n-2$. Let B denote the Banach space of r times continuously differentiable functions on $[a, b]$ with the norm

$$\|x\| = \max_{0 \le j \le r} \{ \max_{a \le t \le b} |x^{(j)}(t)| \}$$

and let $K \subset B$ denote the closed convex cone of functions having nonnegative derivatives upto order r. Let S denote those elements of K which satisfy the boundary conditions (2). Further we define

$$S_\rho = \{x \in S: \|x\| \le \rho\}.$$

THEOREM 13. *Let the following assumptions hold.*

A. *$g(t, u_0, u_1, \cdots, u_r)$ is continuous nonnegative and satisfies $g(t, \bar{u}_0, \bar{u}_1, \cdots, \bar{u}_r) \ge g(t, u_0, u_1, \cdots, u_r)$ for all $(t, u_0, u_1, \cdots, u_r)$, $(t, \bar{u}_0, \bar{u}_1, \cdots, \bar{u}_r) \in [a, b] \times R_+^{r+1}$ such $\bar{u}_0 \ge u_0$, $\bar{u}_1 \ge u_1, \cdots, \bar{u}_r \ge u_r$.*

B. *There exists a $\rho > 0$ such that for $t \in [a, b]$*

$$\int_a^b \frac{\partial^j G(t, s)}{\partial t^j} g(s, \rho, \rho, \cdots,) \, ds \le \rho, \qquad j = 1, 2, \cdots, r.$$

For $t \in [a, b]$ define the iterates

$$x_0(t) = \int_a^b G(t, s) \, g(s, \rho, \rho, \cdots, \rho) \, ds,$$

$$x_n(t) = \int_a^b G(t, s) \; g(s, x_{n-1}(s), x'_{n-1}(s), \cdots, x^{(r)}_{n-1}(s)) \, ds, \qquad n = 1, 2, \cdots. \tag{39}$$

Then the sequence $\{x_n(t)\}$ converges to the maximal solution of the equation

$$x^{(n)} + g(t, x, x', \cdots, x^{(r)}) = 0 \tag{40}$$

satisfying boundary conditions (2) in S_ρ.

Proof. From Theorem 4, solutions of the boundary value problem (40), (2) are the solutions of the operator equation

$$Tx(t) = \int_a^b G(t, s) \; g(s, x(s), x'(s), \cdots, x^{(r)}(s)) \, ds, \tag{41}$$

where T is defined on S_ρ.

Now clearly $x_0(t) \in S_\rho$, since $x_0^{(j)}(t) \le \rho$, $j = 0, 1, \cdots, r$. Using A and B, we find

$$x_1^{(j)}(t) = \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} f(s, x_0(s), x_0'(s), \cdots, x_0^{(r)}(s)) \, ds$$

$$\le \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} f(s, \rho, \rho, \cdots, \rho) \, ds = x_0^{(j)}(t), \qquad j = 0, 1, \cdots, r.$$

Using an inductive argument it is easily seen that

$$\rho \geq x_0^{(j)}(t) \geq x_1^{(j)}(t) \geq \cdots \geq x_n^{(j)}(t).$$

Next from the uniform continuity of $\partial^j G(t, s)/\partial t^j$ on $[a, b] \times [a, b]$, $0 \leq j \leq r$, given $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that for $|t_1 - t_2| < \delta(\varepsilon)$,

$$\|G(t_1, s) - G(t_2, s)\| \leq \frac{\varepsilon}{(b-a) \max\limits_{a \leq t \leq b} g(t, \rho, \rho, \cdots, \rho)}.$$

But for any n, $n = 0, 1, \cdots$.

$$\begin{aligned} &|x_n^{(j)}(t_1) - x_n^{(j)}(t_2)| \\ &\leq \int_a^b \left\{ \left| \frac{\partial^j G(t_1, s)}{\partial t_1^j} - \frac{\partial^j G(t_2, s)}{\partial t_2^j} \right| g(s, x_{n-1}(s), x'_{n-1}(s), \cdots, \right. \\ &\qquad\qquad x_{n-1}^{(r)}(s))\, ds \\ &\leq \int_a^b \|G(t_1, s) - G(t_2, s)\| g(s, \rho, \rho, \cdots, \rho)\, ds \end{aligned}$$

whenever $|t_1 - t_2| < \delta(\varepsilon)$ uniformly in n for all $j = 0, 1, \cdots, r$. This shows that $\{x_n(t)\}$ is equicontinuous. Hence by Arzela's theorem $\{x_n(t)\}$ is precompact and being monotonic converges uniformly to some $x(t) \in S_\rho$. Form the continuity of g, $x(t)$ is a solution of the operator equation (41) or equivalently of (40), (2). Now let $y(t)$ be any other solution of (40), (2) in S_ρ. Then

$$\begin{aligned} &x_0^{(j)}(t) - y^{(j)}(t) \\ &= \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} \{g(s, \rho, \rho, \cdots, \rho) - g(s, y(s), y'(s) \cdots, y^{(r)}(s))\}\, ds \\ &\geq 0. \end{aligned}$$

In fact, induction shows that for each $n = 0, 1, 2, \cdots$,

$$x_n^{(j)}(t) - y^{(j)}(t) \geq 0 \qquad j = 0, 1, \cdots, r.$$

Therefore, in the limit, $x^{(j)}(t) - y^{(j)}(t) \geq 0$. This completes the proof of Theorem 13.

THEOREM 14. *Let the assumptions of Theorem* 13 *hold. Suppose* $Z(t)$ *is a solution of boundary value problem* (1), (2) *in* B *with* $\|Z\| < \rho$, *and let*

$$|f(t, u_0, u_1, \cdots, u_r)| \leq g(t, |u_0|, |u_1|, \cdots, |u_r|),$$

for all $(t, u_0, u_1, \cdots, u_r)$ *in* $[a, b] \times R^{r+1}$. *Then* $|Z^{(j)}(t)| \leq x^{(j)}(t)$ *for* $a \leq t \leq b$, *where* $x(t)$ *is the maximal solution of*(40), (2).

Proof. $n = 1, 2, \cdots$, set

$$y_n(t) = \int_a^b G(t, s) g(s, y_{n-1}(s), y'_{n-1}(s), \cdots, y_{n-1}^{(r)}(s))\, ds$$

with $y_0(t) \equiv |Z(t)|, \cdots, y_0^{(r)}(t) \equiv |Z^{(r)}(t)|$. Then

$$y_1^{(j)}(t) = \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} g(s, |Z(s)|, |Z'(s)|, \cdots, |Z^{(r)}(s)|)\, ds$$
$$\geq \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} |f(s, Z(s), Z'(s), \cdots, Z^{(r)}(s))|\, ds \geq |Z^{(j)}(t)|$$
$$j = 0, 1, \cdots, r.$$

Inductively, it can readily be seen that

$$|Z^{(j)}(t)| \leq y_1^{(j)}(t) \leq y_2^{(j)}(t) \leq \cdots \leq y_n^{(j)}(t) < \rho.$$

As in Theorem 13, $\{y_n(t)\}$ is equicontinuous and, in view of the montonicity, converges uniformly to a solution $y(t) \in S_\rho$ of (40), (2). But the maximality of $x(t)$ completes the proof of Theorem 13.

For $n = 2$, similar results of Theorems 13 and 14 have been obtained earlier by Chandra and Fleishman [7].

APPENDIX

Proof of the Lemma 3.

From Lemma 1 and the form of the Green's function $G(t, s)$, we have

$$\int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| ds = \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} ds$$
$$= \frac{1}{(n-j-1)!} \left[\int_a^t \left\{ (t-a)^{n-j-1} \left(\frac{b-s}{b-a} \right)^{n-p-1} - (t-s)^{n-j-1} \right\} ds + \int_t^b (t-a)^{n-j-1} \left(\frac{b-s}{b-a} \right)^{n-p-1} \right] ds$$
$$= \frac{1}{(n-j-1)!} \left\{ \left[- \frac{(t-a)^{n-j-1}}{(b-a)^{n-p-1}} \frac{(b-s)^{n-p}}{n-p} \right]_a^b + \left[\frac{(t-s)^{n-j}}{n-j} \right]_a^t \right\}$$
$$= \frac{1}{(n-j-1)!} \left\{ (b-a) \frac{(t-a)^{n-j-1}}{n-p} - \frac{1}{n-j} (t-a)^{n-j} \right\}$$

which proves (11). Similarly from Lemma 2 and the form of the Green's function $H(t, s)$ the relation (12) follows. Now (13) and hence (14) follow from the following observations:

(a) $\phi_j(t)$ attains maximum at

(i) $t = t_1$, if $j = p \leq n-1$

where

$$t_1 = a + \frac{(n-j-1)(b-a)}{(n-p)},$$

(ii) $t = b$, if $j+1 \leq p \leq n-1$.

(b) $\psi_j(t)$ attains maximum at

(iii) $t = t_2$, if $j = p \leq n-1$

where

$$t_2 = b - \frac{(n-j-1)(b-a)}{(n-p)},$$

(iv) $t = a$, if $j+1 \leq p \leq n-1$.

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