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ON DIVERGENCE OF VILENKIN-FOURIER SERIES

BY

JAU-D. CHEN[1][2]

Abstract. Given a Vilenkin system of functions with respect to the sequence of primes $\{p_1, p_2, \cdots, p_k, \cdots\}$ with $\sup_k\{p_k\} < \infty$, there exists no $f \in L[0, 1]$ whose Vilenkin-Fourier partial sums $\{S_n f(x)\}$ are bounded divergent everywhere. This result is an analogue to trigonometric-Fourier series.

1. In 1926 Kolmogorov (see [1, 7, 11]) showed that there exist $f \in L[0, 2\pi]$ for which the trigonometric-Fourier series (abbreviated as TFS) is unbounded divergent everywhere, and in 1936 Marcinkiewicz [1, 8, 11] constructed an example of a TFS that is bounded divergent almost everywhere. For Vilenkin-Fourier series (abbreviated as VFS), there are various properties parallel to those of TFS [10, 3]. In both cases, Stein [9, Theorem 6, 7], on the basis of a general theorem, has established the existence of integrable functions whose Fourier series diverge almost everywhere.

In the present article, we show that there exists no everywhere bounded divergent VFS with respect to bounded Vilenkin systems.

2. Let $\{p_1, p_2, \cdots, p_k, \cdots\}$ be a sequence of prime integers, and let $m_0 = 1$, $m_k = p_1 p_2 \cdots p_k$ for $k \geq 1$.

For each $x \in [0, 1)$, there is a representation

$$x = \sum_{k=1}^{\infty} \frac{d_k(x)}{m_k},$$

where $0 \leq d_k(x) < p_k$. The representation is unique if the terminating form is chosen for rationals of the form l/m_k. Let's define a system of functions $\{\phi_0, \phi_1, \cdots, \phi_j, \cdots\}$ as follows. Set

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$$\phi_{j-1}(x) = (\omega_j)^{d_j(x)},\ \omega_j = \exp\left(\frac{2\pi}{p_j}\right)i;\ \phi_{j-1}(x+1) = \phi_{j-1}(x).$$

For each positive integer n, there is a canonical representation for n:

$$n = r_0 m_0 + \cdots + r_j m_j + \cdots + r_k m_k,$$

where $0 \le r_j < p_{j+1}$ for $j = 0, \cdots, k$.

Set

$$\psi_0(x) \equiv 1,\ \psi_n(x) = \phi_0^{r_0}(x)\phi_1^{r_1}(x)\cdots\phi_j^{r_j}(x)\cdots\phi_k^{r_k}(x).$$

This system of functions $\{\psi_j\}$ is called a Vilenkin system, and ψ_j is the j-th Vilenkin function with respect to the sequence $\{p_1, p_2, \cdots, p_k, \cdots\}$. If the sequence of primes is fixed, we simply say that $\{\psi_j\}$ is a Vilenkin system in which ψ_j is the j-th Vilenkin function.

It is well known, any Vilenking system is a complete orthnormal system in $L^2[0, 1]$.

For $f \in L[0, 1]$, with period 1, the series

$$S[f, x] \sim \sum_{j=0}^{\infty} c_j \psi_j(x)$$

where

$$c_j = \int_0^1 f(x)\overline{\psi_j(x)}\,dx \qquad (j \ge 0)$$

is called the Vilenkin-Fourier series of f.

The Vilenkin system $\{\psi_j\}$ is called bounded or unbounded accoring as $p = \sup_k\{p_k\} < \infty$ or $= \infty$. In the sequel, we confine ourselves to a bounded Vilenkin system.

Let $S_n f(x)$ and $\sigma_n f(x)$ be respectively the nth partial sum and $(C, 1)$ sum for the VFS of f at x. In order to prove our main theorem, we need the following two lemmas.

LEMMA 1. *Suppose G is a measurable subset of* $[0, 1]$ *and let* $M \ge 0$ *be such that for each* n

$$|S_n f(x)| \le M \qquad \text{for a.e. } x \in G.$$

Then $|f(x)| \le M$ *for a.e.* x *in* G.

Proof. By Fejér's theorem for the VFS [10], for a.e. $x \in [0, 1]$, we have

$$\sigma_n f(x) \longrightarrow f(x) \qquad \text{as} \quad n \to \infty .$$

If $|S_n f(x)| \leq M$ for each n and for a.e. $x \in G$, then

$$|\sigma_n f(x)| \leq M \qquad \text{for each } n \text{ and for a.e. } x \in G .$$

So

$$|f(x)| \leq M \qquad \text{for a.e. } x \in G .$$

LEMMA 2. *Suppose* $f \in L[0, 1]$ *with almost everywhere divergent* $\{S_n f(x)\}$. *Let*

$$A_m = \{x \in [0, 1] : |S_n f(x)| \leq m \textit{ for each } n\} .$$

Then the set A_m *is nowhere dense.*

Proof. Suppose, on the contrary,

$$\bar{A}_m \supset G ,$$

where G is a nonempty interval. Then for each $x \in G$ with $x \neq l/m_k$ for all positive integers l, k and $m_k = p_1 \cdot p_2 \cdots p_k$ there exists a subinterval $I_n(x)$ of G such that $x \in I_n(x)$ and $S_n f(\cdot)$ is constant in $I_n(x)$, which implies

$$|S_n f(x)| \leq m \qquad \text{for a.e. } x \in G, \text{ and for each } n .$$

By Lemma 1, we have

$$|f(x)| \leq m \qquad \text{for a.e. } x \in G .$$

So

$$f \in L^2(G) .$$

By the principle of localization [10] and a theorem of Gosselin on the almost everywhere convergence of VFS for $L^2[0, 1]$ functions with respect to bounded Vilenkin systems [5], this result implies that

$$\{S_n f(x)\} \text{ is convergent for a.e. } x \in G ,$$

in contardiction to the a.e. divergence of $\{S_n f(x)\}$.

3. Now we state and prove our main theorem.

THEOREM 1. *There exists no* $f \in L[0, 1]$ *such that* $\{S_n f(x)\}$ *is bounded divergent everywhere.*

Proof. Suppose there is an $f \in L[0, 1]$ such that $\{S_n f(x)\}$ is bounded divergent everywhere. For each $m \geq 1$, let

$$A_m = \{x \in [0, 1] : |S_n f(x)| \leq m \text{ for each } n\}.$$

Then, by Lemma 2, the set A_m is nowhere dense for each m. So

$$A = \bigcup_{m=1}^{\infty} A_m$$

is a set of first category.

Let $B = [0, 1] \sim A$. Then $B \neq \emptyset$. But if $x \in B$, we have

$$\overline{\lim_{n \to \infty}} |S_n f(x)| = \infty,$$

a contradiction.

In view of a theorem of Carleson and Hunt [2, 6], we note that Lemma 2 is obvious for TFS. Since in 1967, Edwards [4, p. 157] wrote that it was apparently still unknown whether in the Marcinkiewicz example of an a.e. boundedly divergent TFS the "almost everywhere" can be replaced by "everywhere", it seems that the following theorem is worth stating explicitly.

THEOREM 2. *There exists no everywhere bounded divergent TFS.*

Finally, we remark that the method of proof in Lemma 2, based on the a.e. convergence of VFS for $L^2[0, 1]$ functions with respect to bounded Vilenkin systems, may not be applicable to the unbounded cases. It would be interesting to know whether our theorem holds for unbounded Vilenkin series.

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DEPARTMENT AND INSTITUTE OF MATHEMATICS, NATIONAL TAIWAN NORMAL UNIVERSITY, TAIPEI, TAIWAN

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, NANKANG, TAIPEI, TAIWAN