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A NOTE ON THE MAXIMUM OF PARTIAL SUMS OF IID RANDOM VARIABLES

BY

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Abstract. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables. Assume $E|X_1|^p < \infty$ for some $p > 2$, $EX_1 = 0$, $EX_1^2 = 1$. Let $S_n = X_1 + X_2 + \cdots + X_n$.

The law of iterated logarithm for $\max |S_j|$ was studied by Chung, Jain and Pruitt, etc. For $\max S_j$, the lim sup part can be derived easily from that of $\max |S_j|$. But for the lim inf part, one has to do it separately. We prove a theorem about this in this paper.

1. **Introduction.** Let $\{X_n\}$ be a sequence of real-valued, independent and identically distributed random variables defined on a probability space (Ω, F, P). Let $S_n = \sum_{i=1}^n X_i$, $EX_1 = 0$ and $EX_1^2 = 1$. It is well known that the law of iterated logarithm for $\max_{1\le k\le n}|S_k|$ is

$$\limsup_{n\to\infty} \left\{\max_{1\le k\le n} |S_k|/(2n \log\log n)^{1/2}\right\} = 1 \qquad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \left\{\max_{1\le k\le n} |S_k|/(n/\log\log n)^{1/2}\right\} = \pi/8^{1/2} \qquad \text{a.s.}$$

(see [1, 2, 4]). We also know that, for the lim sup part, $\max_{1\le k\le n} S_k$ has the same form as $\max_{1\le k\le n}|S_k|$, but this is not true for the lim inf part. In this paper we will study the lim inf part of $\max_{1\le k\le n} S_k$, and we obtain the following

THEOREM. *Let X_n be* i.i.d. *random variables with $E|X_1|^p < \infty$ for some $p > 2$, $EX_1 = 0$ and $EX_1^2 = 1$. Let $S_n = X_1 + \cdots + X_n$. Then*

$$\operatorname*{Lim\,inf}_{n\to\infty} \frac{\max_{1\le k\le n} S_k}{n^{1/2}/(\log n)^{\delta}} = \begin{cases} 0 & \text{for } \delta \le 1 \\ \infty & \text{for } \delta > 1 \end{cases} \qquad \text{a.s.}$$

An easy consequence of this is about the limiting behavior of the stopping times $T_c = \inf\{n \ge 1 : S_n > c\}$, as $c \to \infty$, which is

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stated after the proof of the theorem.

2. Proof of the theorem

LEMMA. *Let B_t denote the standard Brownian motion. Let $R_t=\max_{0\le s\le t} B_s$. Let δ be a position number less than or equal to* 1. *For each positive integer n, let $t_n=cn^{\alpha n}$, where c and α are positive numbers with $\alpha>2\delta$. Then*

$$\liminf_{n\to\infty}\frac{R_{t_n}}{t_n^{1/2}/(\log t_n)^\delta}=0 \quad \text{a.s.}$$

Proof. We know that the distribution of R_t is given by

$$P\left\{\frac{R_t}{t^{1/2}}\le x\right\}=P\left\{\frac{|B_t|}{t^{1/2}}\le x\right\}=P\{|B_1|\le x\}$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-u^2/2}\,du.$$

Let $U_n=\max_{t_{n-1}\le s\le t_n}(B_s-B_{t_{n-1}})$. Then the U_n's are independent and for any $\varepsilon>0$,

$$P\left\{U_n<\varepsilon\frac{t_n^{1/2}}{(\log t_n)^\delta}\right\}\ge P\left\{\frac{R_{t_n-t_{n-1}}}{(t_n-t_{n-1})^{1/2}}<\varepsilon\frac{1}{(\log t_n)^\delta}\right\}\ge\frac{K}{(\log t_n)^\delta}$$

for some constant K.

It follows from the choice of t_n and the Borel-Cantelli lemma that

$$\liminf_{n\to\infty}\frac{U_n}{t_n^{1/2}/(\log t_n)^\delta}=0 \quad \text{a.s.}$$

By the usual law of the iterated logarithm and the choice of t_n,

$$\limsup_{n\to\infty}\frac{R_{t_{n-1}}}{t_n^{1/2}/(\log t_n)^\delta}$$
$$=\limsup_{n\to\infty}\frac{R_{t_{n-1}}}{(t_{n-1}\log\log t_{n-1})^{1/2}}\;\frac{(t_{n-1}\log\log t_{n-1})^{1/2}}{t_n^{1/2}/(\log t_n)^\delta}=0 \quad \text{a.s.}$$

Therefore, the lemma follows from the inequality $R_{t_n}\le R_{t_{n-1}}+U_n$.

With the help of the lemma, we are going to prove one part of the theorem. By the Skorokhod embedding, (see [6]), there is a sequence of nonnegative, independent and identically distributed random variables $\{T_n\}$ with $ET_1=1$ such that the distribution of

$S_1, S_2, \cdots$ is the same as that of $B_{T_1}, B_{T_1+T_2}, \cdots$.

Let $\varepsilon > 0$, $t_n = (1+\varepsilon)n^{\alpha n}$,

$$\Omega_1 = \left\{R_{t_n} < \varepsilon \frac{t_n^{1/2}}{(\log t_n)^\delta} \text{ i. o.}\right\},$$

$$\Omega_2 = \left\{\sum_{j=1}^{n} T_j < (1+\varepsilon)n \text{ for all large } n\right\},$$

$$\Omega_0 = \Omega_1 \cap \Omega_2.$$

We know from the law of large numbers and the lemma that $P(\Omega_0) = 1$. Thus, for $\omega \in \Omega_0$ and n large,

$$\max_{1\le k\le n^{\alpha n}} B_{T^k} \le R_{t_n}, \text{ where } T^k = T_1 + \cdots + T_k.$$

Hence

$$\max_{1\le k\le n^{\alpha n}} S_k < \varepsilon \frac{t_n^{1/2}}{(\log t_n)^\delta}$$
$$< \varepsilon(1+\varepsilon)^{1/2} \frac{(n^{\alpha n})^{1/2}}{(\log n^{\alpha n})^\delta} \quad \text{i. o.}$$

This proves

$$\liminf_{n\to\infty} \frac{\max_{1\le k\le n} S_k}{n^{1/2}/(\log n)^\delta} = 0 \qquad \text{if } \delta \le 1.$$

Next, we consider the case $\delta > 1$. We know from [5] that

$$\sup_x \left| P\left\{\frac{\max_{1\le k\le n} S_k}{n^{1/2}} \le x\right\} - P\{|B_1| \le x\} \right| < \frac{K}{n^{1/2-1/p}}$$

for some constant K. Let C be any given positive number. Let $n_k = 2^k$. Then

$$\sum_k P\left\{\max_{1\le j\le n} S_j < C\frac{n^{1/2}}{(\log n)^\delta} \text{ for some } n \in [n_k, n_{k+1})\right\}$$
$$\le \sum_k P\left\{\max_{1\le j\le n_k} S_j < C\frac{n_{k+1}^{1/2}}{(\log n_{k+1})^\delta}\right\}$$
$$= \sum_k P\left\{\frac{\max_{1\le j\le n_k} S_j}{n_k^{1/2}} < C\frac{n_{k+1}^{1/2}}{n_k^{1/2}(\log n_{k+1})^\delta}\right\}$$
$$\le \sum_k \left(P\left\{|N(0,1)| < C\frac{n_{k+1}^{1/2}}{n_k^{1/2}(\log n_{k+1})^\delta}\right\} + \frac{K}{n_k^{1/2-1/p}}\right)$$
$$\le K_2 \sum_k \frac{n_{k+1}^{1/2}}{n_k^{1/2}(\log n_{k+1})^\delta} + K_1 \le 2K_2 \sum_k \frac{1}{((k+1)\log 2)^\delta} + K_1 < \infty,$$

where K_1, K_2 are some constants. It follows then from the Borel-

Cantelli lemma that, with probability one,

$$\max_{1\le j\le n} S_j \ge C\frac{n^{1/2}}{(\log n)^{\delta}} \text{ for large } n.$$

This completes the proof of the theorem.

An easy consequence of the theorem is

COROLLARY. Let $\{S_n\}$ be defined as above, and $T_c = \inf\{n\ge 1, S_n > c\}$, $c > 0$. Then

$$\limsup_{c\to\infty} \frac{T_c}{c^2(\log c)^{2\delta}} = \begin{cases} \infty & \text{as } \delta \le 1 \\ 0 & \text{as } \delta > 1 \end{cases} \quad \text{a. s.}$$

and

$$\liminf_{c\to\infty} \frac{T_c}{c^2/2\log\log c} = 1 \quad \text{a. s.}$$

REFERENCES

1. K.L. Chung, *A course on probability theory*, 2nd Edition, Academic Press, New York, 1974.

2. ———, *On the maximum partial sums of sequences of independent random variables*, Trans. Amer. Math. Soc. **64** (1948), 205–233.

3. P. Erdös and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. **52** (1946), 292–302.

4. N.C. Jain and W.E. Pruitt, *The other law of the iterated logarithm*. Ann. Probability **3** (1975), 1046–1049.

5. V.B. Nevzorov, *The distribution of the maximal sum of independent summands*, Dokl. Akad. Nauk SSSR **208** (1973), 43–45.

6. A.B. Skorokhod, *Research on the theory of random process*, Kiew, 1961.

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