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A NOTE ON THE VON NEUMANN-SION MINIMAX PRINCIPLE

BY

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To the memory of Marston Morse

Abstract. A general form of minimax principle involving two functions with one of them majorizing the other is established. When the two functions coincide, Sion's version of Von Neumann's minimax theorem is obtained. Applications to systems of inequalities, Sup Inf Sup inequality and fixed points are also considered.

1. The purpose of this note is to prove a general form of minimax principle:

THEOREM 1. *Let E be a real Hausdorff topological vector space and F a real vector space, and let X and Y be nonempty convex subsets of E and F respectively, of which X is assumed to be compact. Suppose that f and g are two real-valued functions defined on $X \times Y$ with the following properties:*

(1) $f(x, y) \le g(x, y)$ *for all* $(x, y) \in X \times Y$;

(2) $x \to f(x, y)$ *is lower semicontinuous on X for each y of Y*;

(3) $y \to f(x, y)$ *is quasi-concave on Y for each x of X*;

(4) $x \to g(x, y)$ *is quasi-convex on X for each y of Y; and*

(5) $y \to g(x, y)$ *is upper semicontinuous on convex hulls of finite subsets of Y for each x of X.*

Then $\mathrm{Inf}_{x\in X} \mathrm{Sup}_{y\in Y} f(x, y) \le \mathrm{Sup}_{y\in Y} \mathrm{Inf}_{x\in X}\, g(x, y)$.

We recall that a function f defined on a convex set is said to be quasi-concave (quasi-convex) if $\{x: f(x) > \alpha\}$ $(\{x: f(x) < \alpha\})$ is convex for each real number α. We also note that since any Hausdorff topology which makes a finite-dimensional vector space a topological vector space is necessarily the euclidean topology, we

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adopt the convention of endowing any finite-dimensional subspace of a vector space with the euclidean topology. Therefore condition (5) in Theorem 1 makes sense.

When $f = g$, Theorem 1 reduces to the Von Neumann-Sion minimax theorem [8], for in this case the opposite inequality of that in the conclusion of Theorem 1 holds trivially.

To show the merit of Theorem 1, we also consider in this note its applications to systems of inequalities, Sup Inf Sup inequalities, and fixed points.

The auther wishes to express his sincere thanks to Ky Fan for introducing him to minimax principles, and for many helpful discussions.

2. Our proof of Theorem 1 follows the line of arguments as presented in Ky Fan's proof of the Von Neumann-Sion minimax theorem when both X and Y are compact [5]. For this purpose, we consider in this section an intersection theorem of Ky Fan [5] (see also [7]) with slightly relaxed conditions in the case when only two sets are involved.

THEOREM 2. *Let E, F, X and Y be as in Theorem* 1, *and let A and B be two subsets of $X \times Y$. Suppose that*

(1) $B(x) = \{y \in Y: (x, y) \in B\}$ *is nonempty convex for each x of X and* $B(y) = \{x \in X: (x, y) \in B\}$ *is open in X for each y of Y.*

(2) $A(y) = \{x \in X: (x, y) \in A\}$ *is nonempty convex for each y of Y and* $A(x) = \{y \in Y: (x, y) \in A\}$ *is open in convex hulls of finite subsets of Y.*

Then $A \cap B \neq \emptyset$.

Proof. Since $X = \bigcup_{x \in X} \bigcup_{y \in B(X)} B(y) = \bigcup_{y \in Y} B(y)$, and since X is compact, there are $y_1, \cdots, y_n$ of Y such that $\phi = \{B(y_1), \cdots, B(y_n)\}$ is a finite open covering of X. $\{\varphi_i\}_{i=1}^n$ be a partition of unity of X with respect to ϕ, i. e., each φ_i is non-negative continuous on X, $\sum \varphi_i(x) \equiv 1$, and $\varphi_i(x) > 0 \Rightarrow x \in B(y_i)$. Let H be the convex hull of $\{y_1, \cdots, y_n\}$, and define a mapping $T: x \to H$ by

$$T(x) = \sum_{i=1}^{n} \varphi_i(x)\, y_i, \qquad x \in X.$$

Since $\varphi_i(x) > 0 \Rightarrow x \in B(y_i) \Rightarrow y_i \in B(x)$, and since $B(x)$ is convex, we have

$$T(x) \in B(x) \qquad \text{for all } x \in X.$$

Now, as above, $H = \bigcup_{x \in X} (A(x) \cap H)$. Since, by our assumption, each $A(x) \cap H$ is open in H, and since H is compact, there are $x_1, \cdots, x_m$ of X such that $\psi = \{A(x_i) \cap H\}_{i=1}^{m}$ is a finite open covering of H in H. Let $\{\xi_i\}_{i=1}^{m}$ be a partition of unity of H with respect to ψ, i.e., each ξ_i is non-negative continuous on H, $\sum \xi_i(y) \equiv 1$, $\xi_i(y) > 0 \Rightarrow y \in A(x_i) \cap H$. Let K be the convex hull of $\{x_1, \cdots, x_m\}$ and define a mapping $S: H \to K$ by

$$S(y) = \sum_{i=1}^{m} \xi_i(y)\, x_i, \qquad y \in H.$$

Then

$$S(y) \in A(y) \qquad \text{for all } y \in H.$$

Now define a self mapping U of $K \times H$ into itself by

$$U(x, y) = (S(y), T(x)), \qquad (x, y) \in K \times H.$$

Obviously U is continuous; therefore, by Brouwer's fixed point theorem, there is $(x_0, y_0) \in K \times H$ such that $(x_0, y_0) = (S(y_0), T(x_0))$ or $x_0 = S(y_0)$ and $y_0 = T(x_0)$, which implies that $(x_0, y_0) \in A \cap B$. q. e. d.

3. We are now going to prove Theorem 1. If either

$$\mathrm{Inf}_{x \in X} \mathrm{Sup}_{y \in Y} f(x, y) = -\infty \text{ or } \mathrm{Sup}_{y \in Y} \mathrm{Inf}_{x \in X}\, g(x, y) = +\infty,$$

then the theorem holds. We therefore assume that

$$\mathrm{Inf}_{x \in X} \mathrm{Sup}_{y \in Y} f(x, y) > -\infty \text{ and } \mathrm{Sup}_{y \in Y} \mathrm{Inf}_{x \in X}\, g(x, y) < +\infty.$$

Let λ, μ be any two finite real numbers such that

$$\lambda < \mathrm{Inf}_{x \in X} \mathrm{Sup}_{y \in Y} f(x, y) \text{ and } \mu > \mathrm{Sup}_{y \in Y} \mathrm{Inf}_{x \in X}\, g(x, y).$$

We need only prove that $\mu > \lambda$.

Let $B = \{(x, y) \in X \times Y : f(x, y) > \lambda\}$ and
$A = \{(x, y) \in X \times Y : g(x, y) < \mu\}$.

From the choice of λ and μ, it is obvious that A, B satisfy the conditions of Theorem 2. Therefore there is $(x_0, y_0) \in A \cap B$. Consequently,

$$\lambda < f(x_0, y_0) \leq g(x_0, y_0) < \mu ,$$

which proves Theorem 1.

REMARKS. (1) Since $\mathrm{Sup}_{y\in Y} f(x, y)$ is lower semicontinuous on X, the conclusion of Theorem 1 is actually

$$\mathrm{Min}_{x\in X} \mathrm{Sup}_{y\in Y} f(x, y) \leq \mathrm{Sup}_{y\in Y} \mathrm{Inf}_{x\in X} g(x, y) .$$

(2) If, in Theorem 1, E is a real vector space which contains X as its convex subset, while F is a real Hausdorff topological space with Y its compact convex subset, and if conditions (2) and (5) are replaced by

(2)′ $x \to f(x, y)$ is lower semicontinuous on convex hulls of finite subsets of X for each y of Y;

(5)′ $y \to g(x, y)$ is upper semicontinuous on Y for each x of X, respectively, then the conclusion becomes

$$\mathrm{Inf}_{x\in X} \mathrm{Sup}_{y\in Y} f(x, y) \leq \mathrm{Max}_{y\in Y} \mathrm{Inf}_{x\in X} g(x, y) .$$

4. We consider in this section some applications of Theorem 1.

I. *Systems of inequalities.* The following theorem generalizes a theorem of Ky Fan [4]:

THEOREM 3. *Let X be a nonempty compact convex set in a real Hausdorff topological space. Let $f_1, \cdots, f_n$ be n real-valued lower semicontinuous functions defined on X. Suppose that there are n real-valued convex functions $g_1, \cdots, g_n$ defined on X such that $f_i(x) \leq g_i(x)$, $x \in X$, $i = 1, \cdots, n$ and such that for any n non-negative numbers α_i with $\sum_{i=1}^n \alpha_i = 1$, there is a point $x \in X$ for which $\sum_{i=1}^n \alpha_i g_i(x) \leqq 0$. Then there is $x_0 \in X$ such that*

$$f_i(x_0) \leq 0, \quad i = 1, \cdots, n .$$

Proof. Let $Y = \{y = (y_1, \cdots, y_n) \in R^n : y_i \geqq 0, \sum_{i=1}^n y_i = 1\}$. Define f and g on $X \times Y$ by

$$f(x, y) = \sum_{i=1}^n y_i f_i(x) ;$$

$$g(x, y) = \sum_{i=1}^{n} y_i\, g_i(x)\,.$$

Then the conditions of Theorem 1 are satisfied, consequently there is x_0 such that

$$\mathrm{Sup}_{y\in Y} f(x_0, y) = \mathrm{Sup}_{y\in Y} \sum_{i=1}^{n} y_i f_i(x_0)$$

$$\leq \mathrm{Sup}_{y\in Y} \mathrm{Inf}_{x\in X} \sum_{i=1}^{n} y_i\, g_i(x)\,.$$

But from our assumption, for each y of Y

$$\mathrm{Inf}_{x\in X} \sum_{i=1}^{n} y_i\, g_i(x) \leq 0\,,$$

therefore $\mathrm{Sup}_{y\in Y} \mathrm{Inf}_{x\in X} \sum_{i=1}^{n} y_i\, g_i(x) \leq 0$. Since

$$\mathrm{Sup}_{y\in Y} \sum_{i=1}^{n} y_i f_i(x_0) \leq 0\,,$$

we have necessarily $f_i(x_0) \leq 0$, $\quad i = 1, \cdots, n$. q. e. d.

II. Sup Inf Sup *inequality.*

THEOREM 4. *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space, and let Z be a nonempty convex subset of a real vector space. Suppose that h is a real-valued function defined on $X \times X \times Z$ satisfying the following properties:*

(1) *$x \to h(x, x, z)$ is lower semicontinuous on X for each $z \in Z$;*

(2) *$z \to h(x, x, z)$ is quasi-concave on Z for each $x \in X$;*

(3) *$x \to h(x, y, z)$ is quasi-convex on X for each $(y, z) \in X \times Z$;*

(4) *$y \to h(x, y, z)$ is upper semicontinuous on X for each $(x, z) \in X \times Z$;*

and if nets $\{y_\alpha\}$ and $\{z_\alpha\}$ converge, where $\{y_\alpha\}$ is in X and $\{z_\alpha\}$ is in the convex hull of a finite subset of z, then $\lim_\alpha \sup \{h(x, y_\alpha, z_\alpha) - h(x, y_\alpha, \lim_\alpha z_\alpha)\} \leq 0$. *Then there is x_0 in X such that*

$$h(x_0, x_0, z) \leq \mathrm{Sup}_{z\in Z} \mathrm{Inf}_{x\in X} \mathrm{Max}_{y\in Y} h(x, y, z)$$

for all z of Z.

Proof. Let f and g be functions on $X \times Z$ defined by

$$f(x, z) = h(x, x, z)\;;$$

$$g(x, z) = \mathrm{Max}_{y \in X} h(x, y, z).$$

Obviously, f and g satisfy conditions (1), (2), (3) and (4) of Theorem 1 if we let $Y = Z$. We will now show that condition (5) of Theorem 1 is also satisfied. Let C be the convex hull of a finite subset of Z, and let $\{z_\alpha\}$ be a net in C converging to z such that $g(x, z_\alpha)$ converges. For each α, let y_α be such that $g(x, z_\alpha) = h(x, y_\alpha, z_\alpha)$. Consider a converging subnet of $\{y_\alpha\}$ and denote it again by $\{y_\alpha\}$. We have then

$$\begin{aligned}
\lim_\alpha g(x, z_\alpha) &\leq \lim_\alpha \sup h(x, y_\alpha, z) \\
&\qquad + \lim_\alpha \sup \{h(x, y_\alpha, z_\alpha) - h(x, y_\alpha, z)\} \\
&\leq \lim_\alpha \sup h(x, y_\alpha, z) \\
&\leq h(x, \lim_\alpha y_\alpha, z) \\
&\leq g(x, z),
\end{aligned}$$

which shows that $g(x, z)$ is upper semicontinuous on C. We may then apply Theorem 1 to conclude the proof. q. e. d.

COROLLARY 1. *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space E. Let f be a continuous mapping from X into E', the topological dual of E with strong topology relative to the bilinear canonical pairing $\langle \cdot, \cdot \rangle$ between E' and E. Then there is $x_0 \in X$ such that*

$$\langle f(x_0), z - x_0 \rangle \leq 0$$

for all z of X.

Proof. Apply Theorem 4 with $Z = X$ and with h defined by

$$h(x, y, z) = \langle f(y), z - x \rangle. \qquad \text{q. e. d.}$$

Corollary 1 is due to Browder [2, Theorem 3]. It is useful in nonlinear functional analysis. This kind of variational inequality also follows directly from Ky Fan's minimax inequality [6]. (See also H. Brezis [1].)

COROLLARY 2 [Tychonoff]. *Let X be a nonempty compact convex subset of a real locally convex Hausdorff topological vector space E, and let T be a continuous mapping of X into X. Then T has at least one fixed point.*

Proof. Applying Theorem 4 with $Z = E'$ and with h defined by

$$h(x, y, z) = \langle z, T(y) - x \rangle,$$

we conclude that there is $x_0 \in X$ such that

$$\langle z, T(x_0) - x_0 \rangle \leq \mathrm{Sup}_{z \in E'} \mathrm{Inf}_{x \in X} \mathrm{Max}_{y \in X} \langle z, T\langle z, T(y) - x \rangle \leq 0$$

for all $z \in E'$, which necessarily forces x_0 to be a fixed point of T. q. e. d.

III. *Fixed points.* Our proof of Theorem 1 relies on Brouwer's fixed point theorem; here we illustrate the possible application of Theorem 1 to fixed points by showing that Ky Fan's generalization [3] of Tychonoff's fixed point theorem follows easily from Theorem 1.

THEOREM 5. *Let X be a nonempty compact convex subset of a real Hausdorff locally convex topological vector space E. Let T be a set-valued upper semicontinuous mapping from X into itself such that for each x of X, $T(x)$ is a nonempty closed convex subset of X. Then there is x_0 in X such that $x_0 \in T(x_0)$.*

Proof. For x in X and y in E', the topological dual of E, let

$$f(x, y) = \mathrm{Min}_{z \in T(x)} \langle y, z - x \rangle,$$

$$g(x, y) = \mathrm{Max}_{z \in X} \langle y, z - x \rangle.$$

It is easily checked that f, g satisfy the conditions of Theorem 1; there is, therefore, x_0 of X such that

$$\mathrm{Sup}_{y \in E'} f(x_0, y) \leq \mathrm{Sup}_{y \in E'} \mathrm{Inf}_{x \in X} \{\mathrm{Max}_{z \in X} \langle y, z - x \rangle\}.$$

But for each y, there is x such that

$$\langle y, x \rangle = \mathrm{Max}_{z \in X} \langle y, z \rangle,$$

that is,

$$\mathrm{Inf}_{x \in X} \{\mathrm{Max}_{z \in X} \langle y, z - x \rangle\} \leq 0 \qquad \text{for all } y \in E'.$$

We have then

$$f(x_0, y) \leq 0 \qquad \text{for all } y \in E',$$

that is,

$$\mathrm{Min}_{z\in T(x_0)}\langle y, z-x_0\rangle \leq 0 \quad \text{for all } y\in E',$$

which implies that x_0 necessarily belongs to $T(x_0)$. q.e.d.

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