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CHERN CLASSES AND KAEHLER METRICS

BY

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To the memory of Marston Morse

Abstract. We prove the following results: (I) an algebraic hypersurface of CP_5 admits a BK-metric if and only if it is linear and (II) a compact complex manifold with vanishing first Chern class admits a BK-metric if and only if it is covered biholomorphically by C^n.

1. **Introduction.** Let M be a compact, connected, complex manifold of n complex dimensions. A Kaehler metric g on M is called a *BK-metric* if the Bochner tensor of g vanishes. The Bochner tensor was first introduced by *S.* Bochner in [1] as a complex version of the Weyl conformal curvature tensor. For a Kaehler manifold M, let N be a trivial circle bundle over a small open subset of M. It is shown in [8] that the Bochner tensor of M is nothing but the fourth order Chern-Moser tensor of N. The latter notion has a deal great to do with the pseudo-conformal geometry [4].

Since BK-metrics are simple Kaehler metrics on complex manifolds closely related to pseudo-conformal geometry, it is basic and interesting to determine the class of complex manifolds which admit such metrics. In a previous paper, the author was able to give a complete answer to this problem for analytic complex surfaces. In this paper, we shall study this problem for higher-dimensional complex manifolds. In particular, we shall prove the following theorems.

THEOREM 1. *An algebraic hypersurface of CP_5 admits a BK-metric if and only if it is linear.*

THEOREM 2. *A compact complex manifold of n complex dimensions is covered biholomorphically by C^n if and only if its first Chern class vanishes and it admits a BK-metric.*

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2. **Lemmas.** R. Thom [7] has made the class of all oriented compact differentiable (not necessarily connected) manifolds into an algebra Ω by identifying two such manifolds X and Y of the same dimension if there exists a bounded W with $\partial W = X - Y$, and by defining addition of manifolds as the (disjoint) union, and multiplication as the topological product, these operations being compatible with the identifications. Thom considers the algebra $\Omega \otimes Q$ and states the theorem:

$\Omega \otimes Q$ is generated by the complex-projective spaces CP_{2k} of $2k$ complex dimensions ($k \geq 0$).

Each manifold X represents an element of $\Omega \otimes Q$ which we denote by (X). In particular, if M is a complex manifold of 4 complex dimensions, the cobordism class (M) of M can be written uniquely in the following form:

$$(M) = aCP_4 + bCP_2 \times CP_2.$$

It is known that the cobordism coefficients a, b for a 4-dimensional complex manifold are integers [5]. The sum of a and b gives the Hirzebruch index.

Let c_i and p_i be the ith Chern class and the ith Pontryagin class of M. If w is a cohomology class in $H^{2n}(M; \mathbf{R})$, $\dim_C M = n$, we shall also regard w as the number obtained from w by taking its value on the fundamental cycle of M.

Let M be a Kaehler manifold with Kaehler metric $g = \frac{1}{2} \sum (\omega^i \otimes \bar{\omega}^i + \bar{\omega}^i \otimes \omega^i)$ and curvature form $\Omega^i_j = \sum R^i_{jk\bar{l}} \omega^k \wedge \bar{\omega}^l$ associated with a unitary frame $\omega^1, \cdots, \omega^n$, where n denotes the complex dimension of M. The Ricci tensor S and the scalar curvature ρ are given respectively by

$$S = \frac{1}{2} \sum (R_{i\bar{j}} \omega^i \otimes \bar{\omega}^j + \bar{R}_{i\bar{j}} \bar{\omega}^i \otimes \omega^j),$$

$$\rho = 2 \sum R_{i\bar{i}},$$

where $R_{i\bar{j}} = 2 \sum R^k_{ik\bar{j}}$. The fundamental 2-form Φ is given by

$$\Phi = \frac{\sqrt{-1}}{2} \sum \omega^i \wedge \bar{\omega}^i,$$

which is a harmonic form. The cohomology class represented by the fundamental 2-form Φ of M is denoted by ω, which is called the *fundamental class* of the Kaehler manifold.

The Bochner tensor B is a tensor field with local components

$$B^i_{jk\bar{l}} = R^i_{jk\bar{l}} - \frac{1}{2(n+2)}(R_{\bar{i}k}\delta_{jl} + R_{\bar{i}j}\delta_{kl} + \delta_{ik}R_{j\bar{l}} + \delta_{ij}R_{k\bar{l}})$$
$$+ \frac{\rho}{4(n+1)(n+2)}(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}).$$

If we define a closed $2k$-form γ_k by

$$\gamma_k = \frac{(-1)^k}{(2\pi\sqrt{-1})^k k!}\sum \delta^{i_1\cdots i_k}_{j_1\cdots j_k}\Omega^{j_1}_{i_1}\wedge\cdots\wedge\Omega^{j_k}_{i_k},$$

then γ_k is closed and the the kth Chern class c_k of M is represented by γ_k. In particular, we have

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi}\sum \Omega^i_i, \tag{2.1}$$

$$\gamma_2 = -\frac{1}{8\pi^2}\sum(\Omega^i_i\wedge\Omega^j_j - \Omega^i_j\wedge\Omega^j_i), \tag{2.2}$$

$$\begin{aligned}\gamma_3 = \frac{-\sqrt{-1}}{48\pi^3}\sum(&\Omega^i_i\wedge\Omega^j_j\wedge\Omega^k_k - 3\Omega^i_i\wedge\Omega^j_k\wedge\Omega^k_j\\ &+ 2\Omega^i_j\wedge\Omega^j_k\wedge\Omega^k_i),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\gamma_4 = \frac{1}{384\pi^4}\sum(&\Omega^i_i\wedge\Omega^j_j\wedge\Omega^k_k\wedge\Omega^l_l - 6\Omega^i_i\wedge\Omega^j_j\wedge\Omega^k_l\wedge\Omega^l_k\\ &+ 8\Omega^i_i\wedge\Omega^j_k\wedge\Omega^k_l\wedge\Omega^l_j + 3\Omega^i_j\wedge\Omega^j_i\wedge\Omega^k_l\wedge\Omega^l_k\\ &- 6\Omega^i_j\wedge\Omega^j_k\wedge\Omega^k_l\wedge\Omega^l_i).\end{aligned} \tag{2.4}$$

For later use we prove the following lemmas.

LEMMA 1. *Let M be a compact complex manifold of 4 complex dimensions. If M admits a BK-metric, then its first cobordism coefficient satisfies*

$$a \geq \frac{31}{1035}(c_1^4 - 3c_2c_1^2 + 3c_3c_1). \tag{1}$$

If the equality holds, either every BK-metric g on M is flat or (M, g) is a locally product manifold of two Kaehler surfaces, one with constant positive holomorphic sectional curvature H, the other with constant negative holomorphic sectional curvature $-H$.

LEMMA 2. *Let M be a compact complex manifold of 4 complex dimensions. If M admits a BK-metric, we have*

$$84525\,a + 13\,b \geq 2107\,c_1^4 - 5723\,c_2 c_1^2 + 3968\,c_3 c_1, \tag{2}$$

$$978075\,a - 201\,b \geq 35861\,c_1^4 - 116829 c_2 c_1^2 + 143964\,c_3 c_1. \tag{3}$$

If the equality of (2) *holds, every BK-metric on M is flat or (M, g) is a constant holomorphic sectional curvature or (M, g) is a locally product manifold mentioned in Lemma* 1.

If the equality of (3) *holds, then every BK-metric on M is flat or (M, g) is a locally product manifold mentioned in Lemma* 1.

Moreover, inequalities (1), (2) *and* (3) *are best possible.*

Proof of Lemmas 1 and 2. We denote by $*$ the Hodge star operator and we put

$$S^{(i)} = \operatorname{trace}(S^i), \quad i = 1, 2, 3, 4.$$

Then $S^{(1)} = \rho/2$. By direct computation we obtain the following:

$$\begin{aligned} *\sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\beta} \wedge \Omega^{\gamma}_{\gamma} \wedge \Omega^{\delta}_{\delta} = & -6S^{(4)} + 4S^{(3)}\rho + 3(S^{(2)})^2 \\ & - \frac{3}{2} S^{(2)}\rho^2 + \frac{1}{16}\rho^4, \end{aligned} \tag{2.5}$$

$$\begin{aligned} *\sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\beta} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\gamma} = & -\frac{1}{16}\|R\|^2\rho^2 \\ & + \frac{1}{4} S^{(2)}\rho^2 + \frac{1}{4}\|R\|^2 S^{(2)} - (S^{(2)})^2 - 4\rho \sum R_{k\bar{l}} R^{\alpha}_{\beta l\bar{k}} R^{\beta}_{\alpha} \\ & + 8\rho \sum R_{k\bar{l}}\, R^{\alpha}_{\beta m\bar{k}}\, R^{\beta}_{\alpha l\bar{m}} + 8\sum R_{i\bar{j}}\, R_{k\bar{i}}\, R^{\alpha}_{\beta j\bar{k}}\, R^{\beta}_{\alpha} \\ & - 16\sum R_{i\bar{j}}\, R_{k\bar{i}}\, R^{\alpha}_{\beta m\bar{k}}\, R^{\beta}_{\alpha j\bar{m}} - 8\sum R_{i\bar{j}}\, R_{k\bar{l}}\, R^{\alpha}_{\beta l\bar{i}}\, R^{\beta}_{\alpha j\bar{k}} \\ & + 8\sum R_{i\bar{j}}\, R^{\alpha}_{\beta j\bar{i}}\, R^{\beta}_{\alpha k\bar{l}}\, R_{l\bar{k}}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} & *\sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\beta} \\ & = \frac{1}{2} S^{(3)}\rho - 6\rho \sum R^{\alpha}_{\beta}\, R^{\beta}_{\gamma i\bar{j}}\, R^{\gamma}_{\alpha j\bar{i}} + 4\rho \sum R^{\alpha}_{\beta i\bar{j}}\, R^{\beta}_{\gamma k\bar{i}}\, R^{\gamma}_{\alpha j\bar{k}} \\ & \quad + 4\rho \sum R^{\alpha}_{\beta i\bar{j}}\, R^{\beta}_{\gamma j\bar{k}}\, R^{\gamma}_{\alpha k\bar{i}} + 24\sum R_{i\bar{j}}\, R^{\alpha}_{\beta j\bar{i}}\, R^{\beta}_{\gamma k\bar{l}}\, R^{\gamma}_{\alpha l\bar{k}} \\ & \quad - 24\sum R_{i\bar{j}}\, R^{\alpha}_{\beta k\bar{i}}\, R^{\beta}_{\gamma m\bar{k}}\, R^{\gamma}_{\alpha j\bar{m}} - 24\sum R_{i\bar{j}}\, R^{\alpha}_{\beta k\bar{i}}\, R^{\beta}_{\gamma j\bar{m}}\, R^{\gamma}_{\alpha m\bar{k}} \\ & \quad + 12\sum R_{i\bar{j}}\, R^{\alpha}_{\beta k\bar{i}}\, R^{\beta}_{\gamma j\bar{k}}\, R^{\gamma}_{\alpha} + 12\sum R_{i\bar{j}}\, R^{\alpha}_{\beta j\bar{k}}\, R^{\beta}_{\gamma k\bar{i}}\, R^{\gamma}_{\alpha} \\ & \quad - 6\sum R_{i\bar{j}}\, R^{\alpha}_{\beta j\bar{i}}\, R^{\beta}_{\gamma}\, R^{\gamma}_{\alpha}, \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&*\sum \Omega^\alpha_\beta \wedge \Omega^\beta_\alpha \wedge \Omega^\gamma_\delta \wedge \Omega^\delta_\gamma \\
&\quad = -\frac{1}{2}\|R\|^2 S^{(2)} + \frac{1}{16}\|R\|^4 - 16\sum R^\alpha_\beta R^\beta_{\alpha l\bar{k}} R^\gamma_{\delta k\bar{l}} R^\delta_\gamma \\
&\quad\quad + 64\sum R^\alpha_\beta R^\beta_{\alpha k\bar{l}} R^\gamma_{\delta m\bar{k}} R^\delta_{\gamma l\bar{m}} - 64\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\alpha k\bar{i}} R^\gamma_{\delta m\bar{k}} R^\delta_{\gamma j\bar{m}} \\
&\quad\quad + 32\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\alpha k\bar{l}} R^\gamma_{\delta j\bar{i}} R^\delta_{\gamma l\bar{k}} - 32\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\alpha k\bar{l}} R^\gamma_{\delta l\bar{i}} R^\delta_{\gamma j\bar{k}},
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&*\sum \Omega^\alpha_\beta \wedge \Omega^\beta_\gamma \wedge \Omega^\gamma_\delta \wedge \Omega^\delta_\alpha \\
&\quad = S^{(4)} - 16\sum R^\alpha_\beta R^\beta_\gamma R^\gamma_{\delta i\bar{j}} R^\delta_{\alpha j\bar{i}} - 8\sum R^\alpha_{\beta i\bar{j}} R^\beta_\gamma R^\gamma_{\delta j\bar{i}} R^\delta_\alpha \\
&\quad + 32\sum R^\alpha_\beta R^\beta_{\gamma k\bar{l}} R^\gamma_{\delta i\bar{k}} R^\delta_{\alpha l\bar{i}} + 32\sum R^\alpha_\beta R^\beta_{\gamma k\bar{l}} R^\gamma_{\delta l\bar{i}} R^\delta_{\alpha i\bar{k}} \\
&\quad + 32\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\gamma j\bar{i}} R^\gamma_{\delta k\bar{l}} R^\delta_{\alpha l\bar{k}} - 64\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\gamma k\bar{i}} R^\gamma_{\delta j\bar{l}} R^\delta_{\alpha l\bar{k}} \\
&\quad - 16\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\gamma k\bar{i}} R^\gamma_{\delta l\bar{k}} R^\delta_{\alpha j\bar{l}} + 16\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\gamma k\bar{l}} R^\gamma_{\delta j\bar{i}} R^\delta_{\alpha l\bar{k}} \\
&\quad - 16\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\gamma j\bar{l}} R^\gamma_{\delta l\bar{k}} R^\delta_{\alpha k\bar{i}},
\end{aligned}
\tag{2.9}
$$

where

$$R^\alpha_\beta = 2\sum R^\alpha_{k\beta\bar{k}} \text{ and } \|R\|^2 = 16\sum R^\alpha_{\beta i\bar{j}} R^\beta_{\alpha j\bar{i}}.$$

If the Kaehler metric g on M is a BK-metric, we have

$$
\begin{aligned}
R^\alpha_{\beta i\bar{j}} &= \frac{1}{12}(R_{\bar{\alpha} i}\delta_{\beta j} + R_{\bar{\alpha}\beta}\delta_{ij} + \delta_{\alpha i}R_{\beta\bar{j}} + \delta_{\alpha\beta}R_{i\bar{j}}) \\
&\quad - \frac{\rho}{120}(\delta_{\alpha i}\delta_{\beta j} + \delta_{\alpha\beta}\delta_{ij}).
\end{aligned}
$$

Substituting this into (2.5)–(2.9) we obtain the following:

$$
\begin{aligned}
&*\sum \Omega^\alpha_\alpha \wedge \Omega^\beta_\beta \wedge \Omega^\gamma_\gamma \wedge \Omega^\delta_\delta \\
&\quad = -6S^{(4)} + 4S^{(3)}\rho + 3(S^{(2)})^2 - \frac{3}{2}S^{(2)}\rho^2 + \frac{1}{16}\rho^4,
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&*\sum \Omega^\alpha_\alpha \wedge \Omega^\beta_\beta \wedge \Omega^\gamma_\delta \wedge \Omega^\delta_\gamma \\
&\quad = -\frac{8}{9}S^{(4)} + \frac{3}{5}S^{(3)}\rho \\
&\quad\quad + \frac{7}{18}(S^{(2)})^2 - \frac{19}{20}S^{(2)}\rho^2 + \frac{13}{1440}\rho^4,
\end{aligned}
\tag{2.11}
$$

$$* \sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\beta}$$

$$(2.12) \qquad = -\frac{1}{6} S^{(4)} + \frac{23}{180} S^{(3)} \rho + \frac{1}{24} (S^{(2)})^2 - \frac{7}{200} S^{(2)} \rho^2 + \frac{41}{28800} \rho^4,$$

$$* \sum \Omega^{\alpha}_{\beta} \wedge \Omega^{\beta}_{\alpha} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\gamma}$$

$$(2.13) \qquad = -\frac{11}{81} S^{(4)} + \frac{73}{810} S^{(3)} \rho + \frac{29}{324} (S^{(2)})^2 - \frac{67}{2025} S^{(2)}) \rho^2 + \frac{179}{129600} \rho^4,$$

$$* \sum \Omega^{\alpha}_{\beta} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\alpha}$$

$$(2.14) \qquad = \frac{1}{162} S^{(4)} + \frac{31}{1620} S^{(3)} \rho - \frac{1}{648} (S^{(2)})^2 - \frac{41}{8100} S^{(2)} \rho^2 + \frac{277}{1296000} \rho^4.$$

From (2.12) and (2.14) we find

$$(2.15) \qquad 207 * \sum \Omega^{\alpha}_{\beta} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\gamma} - 31 * \sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\beta} = \frac{58}{9} S^{(4)} - \frac{29}{18} (S^{(2)})^2 + \frac{67}{1800} S^{(2)} \rho^2 + \frac{1}{9000} \rho^4.$$

From [5], we know that the cobordism coefficients satisfy

$$(2.16) \qquad 5a = -4c_4 + 4c_3 c_1 + 2c_2^2 - 4c_2 c_1^2 + c_1^4,$$

$$(2.17) \qquad 9b = 10c_4 - 10c_3 c_1 - 3c_2^2 + 8c_2 c_1^2 - 2c_1^4.$$

Thus from (2.1), (2.2), (2.3), (2.4) and (2.16) we find

$$(2.18) \qquad \int * \sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\beta} \wedge \Omega^{\gamma}_{\gamma} \wedge \Omega^{\delta}_{\delta} = 16 \pi^4 c_1^4,$$

$$(2.19) \qquad \int * \sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\beta} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\gamma} = 16\pi^4 c_1^4 - 32\pi^4 c_2 c_1^2,$$

$$(2.20) \qquad \int * \sum \Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\beta} = 16\pi^4 c_1^4 - 48\pi^4 c_2 c_1^2 + 48\pi^4 c_3 c_1,$$

$$(2.21) \qquad \int * \sum \Omega^{\alpha}_{\beta} \wedge \Omega^{\beta}_{\gamma} \wedge \Omega^{\gamma}_{\delta} \wedge \Omega^{\delta}_{\alpha} = 80\pi^4 a.$$

Substituting (2.20), (2.21) into (2.15), we get

$$1035a = 31(c_1^4 - 3c_2c_1^2 + 3c_3c_1) + \frac{29}{288\pi^4}\int \{4S^{(4)} - (S^{(2)})^2\} + \frac{1}{144000\pi^4}\int \{335S^{(2)}\rho^2 + S^{(4)}\}. \tag{2.23}$$

On the other hand, we have

$$4S^{(4)} \geq (S^{(2)})^2. \tag{2.24}$$

The equality sign of (2.24) holds if and only if S^2 is proportional to the identity transformation, i.e., the eigenvalues of S have the same absolute value. Combining (2.23) and (2.24) we find

$$1035a \geq 31(c_1^4 - 3c_2c_1^2 + 3c_3c_1), \tag{2.23}$$

where the equality sign holds if and only if, with respect to a suitable frame, the Ricci tensor S of M has the following form:

$$S = \begin{pmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & -\lambda & \\ 0 & & & -\lambda \end{pmatrix}. \tag{2.24}$$

Since the metric g is Bochner-Kaehlerian, this implies that either the metric is flat or (M, g) is a locally product manifold of two Kaehler surfaces, one with constant positive holomorphic sectional curvature H, the other with constant negative holomorphic sectional curvature $-H$ (see, for example, [3]). This proves Lemma 1.

Now we prove Lemma 2. From (2.10)–(2.12) find

$$S^{(4)} = -\frac{1}{25}S^{(2)}\rho^2 + \frac{7}{400}\rho^4 + \frac{20}{3}*\sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\gamma^\gamma \wedge \Omega_\delta^\delta - \frac{117}{2}*\sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\gamma^\delta + 66*\sum \Omega_\alpha^\alpha \wedge \Omega_\gamma^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\beta^\delta, \tag{2.25}$$

$$S^{(3)}\rho = \frac{3}{2}S^{(2)}\rho^2 + \frac{1}{80}\rho^4 + \frac{15}{2}*\sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\gamma^\gamma \wedge \Omega_\delta^\delta - \frac{5}{12}*\sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\gamma^\delta + \frac{5}{9}*\sum \Omega_\alpha^\alpha \wedge \Omega_\gamma^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\beta^\delta, \tag{2.26}$$

$$
\begin{aligned}
(S^{(2)})^2 = \frac{11}{50} S^{(2)}\rho^2 - \frac{1}{400}\rho^4 + \frac{11}{3} * \sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\gamma^\gamma \wedge \Omega_\delta^\delta \\
- 27 * \sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\gamma^\delta \\
+ 12 * \sum \Omega_\alpha^\alpha \wedge \Omega_\gamma^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\beta^\delta.
\end{aligned}
\tag{2.27}
$$

On the other hand, from (2.10)–(2.14) and (2.17) we get

$$
\begin{aligned}
(12\pi)^4 b = \int \Big\{ - 12 S^{(4)} - \frac{2}{5} S^{(3)}\rho + 7(S^{(2)})^2 - \frac{14}{25} S^{(2)}\rho^2 \\
+ \frac{9}{400}\rho^4 \Big\}.
\end{aligned}
\tag{2.28}
$$

Substituting (2,25)–(2,27) into (2,28), we find

$$
\begin{aligned}
96\pi^4 b + 43 \int * \sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\gamma^\gamma \wedge \Omega_\delta^\delta \\
- 405 \int * \sum \Omega_\alpha^\alpha \wedge \Omega_\beta^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\gamma^\delta \\
+ 558 \int * \sum \Omega_\alpha^\alpha \wedge \Omega_\gamma^\beta \wedge \Omega_\delta^\gamma \wedge \Omega_\beta^\delta \\
= \int \Big\{ \frac{21}{20} S^{(2)}\rho^2 - \frac{63}{400}\rho^4 \Big\}.
\end{aligned}
\tag{2.29}
$$

Therefore, by using (2.18)–(2.20) and (2.29) we obtain

$$
\begin{aligned}
3b = - 98 c_1^4 + 432 c_2 c_1^2 - 837 c_3 c_1 \\
+ \frac{1}{12800\pi^4} \int \{420 S^{(2)}\rho^2 - 63\rho^4\}.
\end{aligned}
\tag{2.30}
$$

Thus, from (2.23) and (2.30) we find

$$
\begin{aligned}
84525a + 13b = 2107 c_1^4 - 5723 c_2 c_1^2 + 3968 c_3 c_1 \\
+ \frac{7105}{864\pi^4} \int \{4S^{(4)} - (S^{(2)})^2\} + \frac{287}{13824\pi^4} \int \{16 S^{(2)}\rho^2 - \rho^4\},
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
978075a - 201b = 35861 c_1^4 - 116829 c_2 c_1^2 + 143964 c_3 c_1 \\
+ \frac{3045}{32\pi^4} \int \{4S^{(4)} - (S^{(2)})^2\} + \frac{12621}{12800\pi^4} \int \rho^4.
\end{aligned}
\tag{2.32}
$$

On the other hand, from [3], we have

$$
16 S^{(2)} \geq \rho^2. \tag{2.33}
$$

The equality sign holds if and only if the metric is Einsteinian. Thus from (2.24) and (2.31)–(2.33) we obtain the inequalities (2) and (3). If the equality sign of (2) holds, (2.31) implies that either $\rho = 0$, $4S^{(4)} \equiv (S^{(2)})^2$ or the Bochner-Kaehler metric is Einsteinian. In the first case, either the metric is flat or (M, g) is a locally product manifold mentioned in Lemma 1. In the second case, the metric must have constant holomorphic sectional curvature. If the equality sign of (3) holds, (2.32) implies that $\rho = 0$ and $4S^{(4)} = (S^{(2)})^2$. Thus (M, g) is either flat or the mentioned locally product manifold. (Q. E. D.)

3. **Proof of Theorem** 1. If M is an algebraic hypersurface of CP_5, let $\tilde{h}$ be the generator of $H^2(CP_5; Z)$ corresponding to the divisor class of a hyperplane section CP_4. Then the total Chern class $c(CP_5)$ of CP_5 is given by

$$c(CP_5) = (1 + \tilde{h})^6.$$

Let $j: M \to CP_5$ be the imbedding and ν be the normal bundle of $j(M)$ in CP_5. Denote by d the degree of M. The total Chern class $c(\nu)$ is given by

$$c(\nu) = 1 + dh,$$

where h is the image of $\tilde{h}$ under the homomorphism $j^*: H^2(CP_5; Z) \to H^2(M; Z)$. Since $j^*T(CP_5) = TM \oplus \nu$, we have $j^*c(CP_5) = c(M) \cdot c(\nu)$, where $c(M)$ is the total Chern class of M. Thus we find

$$(1 + h)^6 = \{1 + c_1 + c_2 + c_3 + c_4\}(1 + dh),$$

which implies

$$c_k = \left\{\sum_{i=0}^{k} (-1)^i \binom{6}{k-i} d^i\right\} h^k.$$

From this we find

$$a = \frac{1}{5}(6 - d^4)d, \qquad b = \frac{1}{3}(1 - 2d^2 + d^4)d.$$

On the other hand, we have

$$c_1^4 - 3c_2 c_1^2 + 3c_3 c_1 = (d^3 - 6)(d - 6).$$

Since $207(6 - d^4) < 31(d^3 - 6)(d - 6)$ for $d > 2$, Lemma 1 implies that M admits no BK-metric for $d > 2$. Thus $d = \deg M \leq 2$. If

$d=2$, we have

$$2107c_1^4 - 5723c_2c_1^2 + 3968c_3c_1 = 84525,$$

and

$$84528a + 13b = -169017.$$

Thus from Lemma 2 we see that M admits no BK-metric. If M is linear, then M is CP_4. In this case M admits a BK-metric, namely, the Fubini-Study metric. (Q. E. D.)

4. **Proof of Theorem 2.** Let M be a compact complex manifold of n complex dimensions. Assume that M admits a BK-metric g. Then we have

$$\omega^{n-2}c_1^2 = \int * \Phi^{n-2}\gamma_1^2,$$

and

$$\omega^{n-2}c_2 = \int * \Phi^{n-2}\gamma_2.$$

By a straightforward computation we get

$$\omega^{n-2}c_1^2 = \frac{(n-2)!}{16\pi^2}\int(\rho^2 - 4S^{(2)}) \tag{4.1}$$

and

$$\omega^{n-2}c_2 = \frac{(n-2)!}{32\pi^2}\int(\rho^2 - 8S^{(2)} + \|R\|^2), \tag{4.2}$$

where $S^{(2)} = \sum R_{i\bar{j}}R_{j\bar{i}}$. On the other hand, since the metric is a BK-metric, we have

$$\|R\|^2 = \frac{16}{n+2}S^{(2)} + \frac{2}{(n+1)(n+2)}\rho^2. \tag{4.3}$$

Substituting (4.3) into (4.2) we get

$$\omega^{n-2}c_2 = \frac{(n-2)!}{32\pi^2}\int\left[\frac{n(n+3)}{(n+1)(n+2)}\rho^2 - \frac{8n}{n+2}S^{(2)}\right]. \tag{4.4}$$

From (4.1) and (4.4) we find

$$\omega^{n-2}c_2 - \frac{n}{2(n+1)}\omega^{n-1}c_1^2 = \frac{(n-2)!n}{32(n+1)(n+2)\pi^2}\int(\rho^2 - 8nS^{(2)}). \tag{4.5}$$

Since $\rho^2 \leq 4nS^{(2)}$ and the equality holds if and only if the metric g is Einsteinian, (4.5) implies

$$\omega^{n-2} c_2 \leq \frac{n}{2(n+1)} \omega^{n-1} c_1^2, \tag{4.6}$$

and the equality of (4.6) holds if and only if the metric g is Einsteinian.

Now suppose that the first Chern class c_1 of M vanishes. Then (4.6) gives

$$\omega^{n-2} c_2 \leq 0. \tag{4.7}$$

Since every Einsteinian BK-metric has constant holomorphic sectional curvature, the equality of (4.7) holds when and only when the metric g on M is flat.

Now, by the assumption, the first Chern class of M vanishes. The main result of [9] shows that there exists an Einstein-Kaehler metric $\tilde{g}$ on M such that g and $\tilde{g}$ have the same fundamental class, that is, $\omega = \tilde{\omega}$. Since $\tilde{g}$ is Einsteinian, Theorem 2 of [3] shows that

$$\omega^{n-2} c_2 = \tilde{\omega}^{n-2} c_2 \geq \frac{n}{2(n+1)} \tilde{\omega}^{n-2} c_1^2 = 0.$$

Combining this with (4.7), we find that $\omega^{n-2} c_2 = 0$ and the BK-metric is flat. Consequently, M is covered biholomorphically by C^n.

Conversely, if M is covered biholomorphically by C^n, the canonical flat metric on C^n induces a flat BK-metric. In particular, the first Chern class of M vanishes. (Q. E. D.)

5. **Remarks.** REMARK 1. By using the technique given in the proof of Theorem 2, we have the following.

THEOREM 3. *If a compact complex manifold M admits a BK-metric g and an Einstein-Kaehler $\tilde{g}$ such that they have the same fundamental class, then M is covered biholomorphically by CP^n, C^n or the ball D^n in C^n.*

The converse is also true.

REMARK 2. In view of Theorems 1, 2 and 3, and results in [2] we would like to make the following conjecture.

CONJECTURE. *A compact complex manifold is covered biholomorphically by CP^n (respectively, D^n) if and only if it admits a BK-metric and its first Chern class is positive (respectively, negative).*

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