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ON A SUFFICIENT CONDITION FOR THE CONVERGENCE OF MULTIPLE FOURIER SERIES

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Abstract. It is proved that for multiple Fourier series the condition $\sum_m |c_m|^2 \operatorname{Log} m < \infty$ is sufficient for almost everywhere convergence of the rectangular sums of the corresponding Fourier series. This result is a generalization of Kaczmarz for double Fourier series.

1. In view of the theorem of Carleson and Hunt to the effect that the Fourier series of an L^p function on $[0, 2\pi)$ converges almost everywhere $(p>1)$ (see [2, 4]), the previously interesting theorem of Kolmogorov-Seliverstov-Plessner [1, 7], which says that if $\sum (a_j^2 + b_j^2) \log j < \infty$, then the corresponding Fourier series converges almost everywhere, loses much of its interest. However, in view of the fact that Fefferman [3] has found a continuous function on $T_2 = [0, 2\pi) \times [0, 2\pi)$ whose Fourier series has everywhere divergent rectangular partial sums, the analogous result of Kaczmarz [5] becomes interesting, namely that for double series the conditions $\sum_{i,j=1}^{\infty} c_{ij}^2 \log i \log j < \infty$ and $\sum_{j=1}^{\infty} (c_{j0}^2 + c_{0j}^2) \log j < \infty$ imply almost everywhere convergence of the rectangular partial sums of the corresponding Fourier series. For 1-dimension, $\sum c_j^2 < \infty$ is sufficient. For 2-dimension it is not enough but Kaczmarz's conditions are sufficient. So it is natural to ask: what is the situation for higher dimensions? In this paper we show that the condition $\sum_m |c_m|^2 \operatorname{Log} m < \infty$, where $m = (m_1, m_2, \cdots, m_n)$ is an integer lattice point of R^n and $\operatorname{Log} m = \prod_{j=1}^{\infty} \log |m_j|$ with $\log 0$ and $\log 1$ interpreted as 1, implies almost everywhere convergence of the rectangular partial sums of the corresponding multiple Fourier series $\sum_m c_m e^{im\cdot x}$.

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To judge from an example given for multiple Fourier series by Fefferman [3], or from the interesting improvement of Kaczmarz's theorem in Sjölin [6], it seems that our result cannot be essentially improved.

2. Consider the n-torus $T_n = [0, 2\pi) \times [0, 2\pi) \times \cdots \times [0, 2\pi)$ of points $x = (x_1, x_2, \cdots, x_n)$. For $f \in L^1(T_n)$ let $S_l(x, f)$, $\sigma_l(x, f)$ be respectively the rectangular partial sum and the $(C, 1)$ sum for the Fourier series of f at x, where $l = (l_1, l_2, \cdots, l_n)$ is a non-negative integer lattice point of R^n. We shall make use of the standard equalities

$$S_l(x, f) = \frac{1}{\pi^n} \int_{T_n} f(y)\, D_l(x - y)\, dy\,,$$

$$\sigma_l(x, f) = \frac{1}{\pi^n} \int_{T_n} f(y)\, K_l(x - y)\, dy\,,$$

where $D_l(x) = D_{l_1}(x_1) D_{l_2}(x_2) \cdots D_{l_n}(x_n)$ and $K_l(x) = K_{l_1}(x_1) K_{l_2}(x_2) \cdots K_{l_n}(x_n)$.

Then it is convenient to introduce the following definition.

DEFINITION. Let ν be any strictly increasing function from $\{1, 2, \cdots, k\}$, $1 \le k < n$, into $\{1, 2, \cdots, n\}$ and let ν' be a strictly increasing function from $\{1, 2, \cdots, n-k\}$ into $\{1, 2, \cdots, n\}$ such that $\{\nu_1, \nu_2, \cdots, \nu_k, \nu'_1, \nu'_2, \cdots, \nu'_{n-k}\} = \{1, 2, \cdots, n\}$. Then ν, ν' are called a complementary pair.

For points $x \in R^n$, we designate $x_\nu = (x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_k})$ as points in k space. We also write $dx = dx_\nu\, dx_{\nu'}$.

The following two lemmas are needed:

LEMMA 1. *If* $f \in L^2(T_n)$, *set*

(i) $$S_n^*(x, f) = \sup_l \frac{|S_l(x, f)|}{\sqrt{\operatorname{Log} l}}\,,$$

(ii) $$\sigma_n^*(x, f) = \sup_l |\sigma_l(x, f)|\,.$$

Then

$$\|S_n^*(\cdot, f)\|_2 \le A\, \|f\|_2\,,$$

$$\|\sigma_n^*(\cdot, f)\|_2 \le A\, \|f\|_2\,,$$

where A *is an absolute constant depending only on* n.

Proof. The proof of (i) is essentially the same as in [7, pp 161–162], and (ii) is given in [7, p. 308].

LEMMA 2. *If $f \in L^2(T_n)$ and if ν, ν' are a complementary pair defined as above, set*

$$P^*_{\nu\nu'}(x, f) = \sup_{l_\nu} \int_{T_k} K_\nu(x-y)_\nu \, dy_\nu \sup_{l_{\nu'}} \left\{ \frac{\left| \int_{T_{n-k}} D_{l_{\nu'}}(x-y)_{\nu'} f(y) \, dy_{\nu'} \right|}{\sqrt{\operatorname{Log} l_{\nu'}}} \right\}.$$

Then

$$\|P^*_{\nu\nu'}(x, f)\|_2 \le A \|f\|_2 .$$

Proof. Note that

$$P^*_{\nu\nu'}(x, f) = \pi^{n-k} \sup_{l_\nu} \int_{T_k} K_{l_\nu}(x-y)_\nu S^*_{n-k}(x_{\nu'}, f_{y_\nu}(y_{\nu'})) \, dy_\nu ,$$

where f_{y_ν} is the restriction of f on T_{n-k}. Then

$$P^*_{\nu\nu'}(x, f) = \pi^n \sigma^*_k(x_\nu, S^*_{n-k}(x_{\nu'}, f_{y_\nu})) .$$

Hence

$$\begin{aligned}
\|P^*_{\nu\nu'}(x, f)\|_2^2 &= \int_{T_n} |P^*_{\nu\nu'}(x, f)|^2 \, dx \\
&= \int_{T_{n-k}} dx_{\nu'} \int_{T_k} |P^*_{\nu\nu'}(x, f)|^2 \, dx_\nu \\
&= \pi^n \int_{T_{n-k}} dx_{\nu'} \int_{T_k} |\sigma^*_k(x_\nu, S^*_{n-k}(x_{\nu'}, f_{y_\nu}))|^2 \, dx_\nu \\
&\le A \int_{T_{n-k}} \|S^*_{n-k}(x_{\nu'}, f_{y_\nu})\|^2_{L^2(T_k)} \, dx_{\nu'} \quad \text{(by Lemma 1, ii)} \\
&= A \int_{T_{n-k}} \left\{ \int_{T_k} |S^*_{n-k}(x_{\nu'}, f_{y_\nu})|^2 \, dy_\nu \right\} dx_{\nu'} \\
&= A \int_{T_k} \left\{ \int_{T_{n-k}} |S^*_{n-k}(x_{\nu'}, f_{y_\nu}(\cdot))|^2 \, dx_{\nu'} \right\} dy_\nu \\
&\le A \int_{T_k} \|f_{y_\nu}(\cdot)\|^2_{L^2(T_{n-k})} \, dy_\nu \quad \text{(by Lemma 1, i)} \\
&= A \|f\|_2^2 .
\end{aligned}$$

The proof is complete.

3. Now we can prove our theorem.

THEOREM. *If* $f \in L^2(T_n)$ *and* $\hat{f}_m$ *is the mth Fourier coefficient of* f, *suppose that*

$$\sum_m |\hat{f}_m|^2 \operatorname{Log} m < \infty,$$

where $m = (m_1, m_2, \cdots, m_n)$ *and* $\operatorname{Log} m = \prod_{j=1}^n \log |m_j|$ *with* $\log 0$ *and* $\log 1$ *interpreted as* 1.

Then the rectangular partial sums $S_l(x, f)$ *converge almost everywhere.*

Proof. Let $S^* f(x) = \sup_l |S_l(x, f)|$.

Note that

$$S_l(x, f) = \sum_m \hat{f}_m e^{im\cdot x}$$
$$= \sum_m (\hat{f}_m \sqrt{\operatorname{Log} m}\, e^{im\cdot x})\, b_{|m_1|} b_{|m_2|} \cdots b_{|m_n|},$$

where the summation is taken over all integer lattice points $m = (m_1, m_2, \cdots, m_n)$ such that $|m_j| \le l_j$ and $b_{|m_j|} = 1/\sqrt{\log |m_j|}$ $(j = 1, 2, \cdots, n)$. There is a $g \in L^2(T_n)$ such that $\hat{g}_m = \hat{f}_m \sqrt{\operatorname{Log} m}$ for each m. By n-dimensional Abel partial summation we obtain

$$(*) \qquad \begin{aligned} S_l(x, f) = &\sum_{p=(0,0,\cdots,0)}^{l-1} S_p(x, g)\, \Delta b_{p_1} \Delta b_{p_2} \cdots \Delta b_{p_n} + \cdots \\ &+ \sum_\nu \Big\{ \sum_{d_\nu=(0,0,\cdots,0)}^{l_\nu - 1} S_{(d_\nu, l_{\nu'})}(x, g) \Big(\prod_{j=1}^{k} \Delta b_{d_{\nu_j}} \Big) \Big(\prod_{i=1}^{n-k} b_{l_{\nu'_i}} \Big) \Big\} + \cdots \\ &+ S_l(x, g) \frac{1}{\sqrt{\operatorname{Log} l}}, \end{aligned}$$

where $l-1 = (l_1 - 1, l_2 - 1, \cdots, l_n - 1)$, $l_\nu - 1 = (l_{\nu_1} - 1, l_{\nu_2} - 1, \cdots, l_{\nu_k} - 1)$, ν, ν' is a complementary pair $(d_\nu, l_{\nu'})$ is an integer lattice point $p = (p_1, p_2, \cdots, p_n)$ with $p_{\nu_j} = d_{\nu_j}$, $p_{\nu'_i} = l_{\nu'_i}$ $(i = 1, \cdots, n-k;$ $j = 1, \cdots, k)$; and the summation $\sum_\nu$ is taken over all ν satisfying $1 \le \nu_1 < \nu_2 < \cdots < \nu_k \le n$.

Denote the first term in $(*)$ by F, the last term by L, and the general term by G.

Obviously, we have $|L| \le S_n^*(x, g)$. Another Abel partial summation and the convexity of the sequence $(1/\sqrt{\log j})_{j=2}^{\infty}$ show that

$$|F| \leq A\sigma_n^*(x, g),$$

$$|G| \leq A \sum_{\mu} P_{\mu\mu'}^*(x, g),$$

where the summation is taken over all complementary pairs. Therefore we have

$$|S^* f(x)| \leq A\left\{\sigma_n^*(x, g) + S_n^*(x, g) + \sum_{\mu} P_{\mu\mu'}(x, g)\right\}.$$

By Lemmas 1 and 2, we conclude

$$\|S^* f(x)\|_2 \leq A\,\|g\|_2 = A\left(\sum_m |\hat{f}_m|^2 \operatorname{Log} m\right)^{1/2}.$$

From this inequality and from the density of the trigonometric polynomials in $L^2(T_n)$ it follows that

$$\|\limsup_{l,l'} |S_l(x, f) - S_{l'}(x, f)|\,\|_2 = 0.$$

So the almost everywhere convergence of $S_l(x, f)$ follows, and the proof is complete.

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