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A NOTE ON A PAPER OF WONG

BY

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Abstract. In a recent paper in this journal, J. S. W. Wong [1] studied the oscillatory behavior of solutions of a second order nonlinear differential equation. In this paper a counterexample to a key lemma used by Wong is constructed. In addition, the error in Wong's proof is indicated, and a correct version of the lemma is suggested.

In a recent paper Wong [1] discussed the oscillatory nature of solutions of the equation

$$x'' + a(t) f(x) = 0, \tag{1}$$

where it is assumed that a is locally integrable and f is continuous. He was particularly interested in obtaining oscillation results for (1) under the assumption

(A_0) There exists a sequence $\{T_n\} \to \infty$ such that

$$\int_{T_n}^{t} a(s)\, ds \geq 0 \qquad \text{for all} \quad t \geq T_n,$$

rather than under the stronger conditions

(A_1)′ For all large T

$$\liminf_{t\to\infty} \int_{T}^{t} a(s)\, ds \geq 0,$$

or

(A_1) For all large T

$$\liminf_{t\to\infty} \int_{T}^{t} a(s)\, ds > 0.$$

In so doing, Wong attempted to show [1, Lemma 2] that under very mild conditions on f, namely, $xf(x) > 0$ for $x \neq 0$ and

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$f'(x) \geq 0$ for all x, condition (A_0) guaranteed that any nonoscillatory solution $x(t)$ of (1) satisfies $x(t)\,x'(t) > 0$ for all large t. We will construct an example satisfying the hypotheses of Lemma 2 in [1], but for which the conclusion fails.

Consider the equation

$$x'' + a(t)\,x = 0\,, \qquad t > 1\,, \tag{2}$$

where $a(t) = [\sin(\ln t) + \cos(\ln t)]/t^2\,[2 + \sin(\ln t)]$. Now $x(t) = 2 + \sin(\ln t)$ is a nonoscillatory solution of (2) and $x(t)\,x'(t) = [2 + \sin(\ln t)]\,[\cos(\ln t)]/t \not> 0$ for all large t. Since $f(x) \equiv x$ clearly satisfies the hypotheses of Wong's lemma, it remains only to show that (A_0) is satisfied.

Let $\{T_n\} = \{\exp(2n\pi)\}$. For convenience we adopt the following notation

$$\theta_n(\alpha) = \exp(2n\pi + \alpha)$$

and observe that $\theta_n(0) = T_n$. The only intervals where $a(t)$ is nonpositive for $t \geq T_n$ occur when $3\pi/4 \leq \alpha \leq 7\pi/4$. We will show that

$$\int_{\theta_n(0)}^{\theta_n(\pi/2)} a(s)\,ds + \int_{\theta_n(3\pi/4)}^{\theta_n(7\pi/4)} a(s)\,ds > 0\,,$$

from which it follows that

$$\int_{T_n}^{t} a(s)\,ds \geq 0 \qquad \text{for all} \quad t \geq T_n\,.$$

If $0 \leq \alpha \leq \pi/2$, then $\exp(2n\pi) \leq t \leq \exp(2n\pi + \pi/2)$, so $\sin(\ln t) + \cos(\ln t) \geq 1$ and $a(t) \geq 1/3t^2$ there. Thus

$$\begin{aligned} I_1 &= \int_{\theta_n(0)}^{\theta_n(\pi/2)} a(s)\,ds \\ &\geq -(1/3)\,\{\exp[-(2n\pi + \pi/2)] - \exp[-(2n\pi)]\} \\ &= [\exp(\pi/2) - 1]/3\exp(2n\pi + \pi/2)\,. \end{aligned}$$

Now if $3\pi/4 \leq \alpha \leq 7\pi/4$, then $\exp(2n\pi + 3\pi/4) \leq t \leq \exp(2n\pi + 7\pi/4)$ so $\sin(\ln t) + \cos(\ln t) \geq -2^{1/2}$ and $a(t) \geq -2^{1/2}/t^2$. Hence

$$\begin{aligned} I_2 &= \int_{\theta_n(3\pi/4)}^{\theta_n(7\pi/4)} a(s)\,ds \\ &\geq 2^{1/2}\,\{\exp[-(2n\pi + 7\pi/4)] - \exp[-(2n\pi + 3\pi/4)]\} \\ &= 2^{1/2}\,[1 - \exp\pi]/\exp(2n\pi + 7\pi/4)\,. \end{aligned}$$

We then have

$$\begin{aligned} I_1 + I_2 &= \{[\exp(\pi/2) - 1]\exp(5\pi/4) \\ &\qquad + 2^{1/2}[1 - \exp\pi](3)\}/3\exp(2n\pi + 7\pi/4) \\ &= [\exp(\pi/2) - 1]\exp(\pi/2)\{\exp(3\pi/4) \\ &\qquad - 2^{1/2}[1 + \exp(-\pi/2)](3)\}/3\exp(2n\pi + 7\pi/4). \end{aligned}$$

Since $\exp(3\pi/4) > 7$ and $1 + \exp(-\pi/2) < 1.3 < 2^{1/2}$, we have $I_1 + I_2 > 0$ so (A_0) holds.

Due to the above example, the results in [1] which depend on Lemma 2, namely, Theorem 1, Corollaries 1, 2, 5, and 6, and parts (a) and (b) of both Theorems 3 and 5 are not valid. It should be pointed out, however, that Wong's lemma is correct when condition (A_0) is replaced by $(A_1)'$. (The error in the proof of Lemma 2 in [1] occurs in assuming that the zeros of $x'(t)$ occur along the sequence $\{T_n\}$ in (A_0).) Finally, we note that the above example can be modified to include nonlinear equations of the type

$$x'' + b(t)\,|x|^r \operatorname{sgn} x = 0, \qquad 0 < r < 1,$$

by letting $b(t) = [\sin(\ln t) + \cos(\ln t)]/t^2\,[2 + \sin(\ln t)]^r$ and observing that in calculating I_1, $b(t) \geq 1/3^r t^2 > 1/3t^2$.

Reference

1. J. S. W. Wong, *Oscillation theorems for second order nonlinear differential equations*, Bull. Inst. Math. Acad. Sinica **3** (1975), 283-309.

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