BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 5, Number 2, December 1977

AN n-CUBE-FILLING CURVE

BY

RONG-SONG JIH

Abstract. We give a concrete n-cube-filling curve for $n \geq 2$.

1. **Introduction.** Ever since Peano constructed the first curve which fills $[0, 1] \times [0, 1]$, various square-filling-curves have been presented by Hilbert, Moore-Schoenflies, Lebesgue, Schoenberg and Munkres etc. Schoenberg's method is the most interesting and simplest among them. It is also known that there exist n-cube-filling curves (e.g. see [1, p. 105] for the proof of existence). In this article we modify Schoenberg's formulas to construct a rather concrete example of such a curve, i.e. we give a continuous surjection from $[0, 1]$ to $\Pi^n[0, 1]$.

2. **Generalized Schoenberg's method.** Let $n \geq 2$ be a fixed integer and ϕ the even continuous function on R with period 2 which satisfies

$$\begin{aligned} \phi(t) &= k && \text{if } \frac{2k}{2n-1} \leq t \leq \frac{2k+1}{2n-1}, \\ & && \text{where } k \in \{0, 1, 2, \cdots, n-1\}, \\ &= (2n-1)t - k - 1 && \text{if } \frac{2k+1}{2n-1} \leq t \leq \frac{2k+2}{2n-1}, \\ & && \text{where } k \in \{0, 1, 2, \cdots, n-2\}. \end{aligned}$$

Define $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ by $\alpha_j(t) = \sum_{i=1}^{\infty} (\phi[(2n-1)^{ni-n+j-1} t]/n^i)$, $j = 1, 2, \cdots, n$. The inequalities $0 \leq \phi(t) \leq n-1$ imply $0 \leq \alpha_j(t) \leq 1$ as well as the uniform convergence of all series, and hence imply the continuity of all α_j. We claim that $\vec{\alpha}$ maps $[0, 1]$ onto $\Pi^n[0, 1]$. Let $(x_1, \cdots, x_n)$ be a point in $\Pi^n[0, 1]$ with each x_j written in the n-ary expansion: $x_j = \sum_{i=1}^{\infty} (x_i^{(j)}/n^i)$, where $x_i^{(j)} \in \{0, 1, 2, \cdots,$

Received by the editors March 23, 1977.

$n-1\}$, $j=1, 2,\cdots, n$. Set $S_0 = 2\sum_{i=1}^{\infty}(S_i/(2n-1)^i)$, where $S_{ni-n+j} = x_i^{(j)}$; then $0 \leq S_0 \leq 1$ and $(2n-1)^r S_0 =$ an even integer $+\, 2\sum_{i=r+1}^{\infty}(S_i/(2n-1)^{i-r})$. If $S_{r+1}=k$, then

$$\begin{aligned}\frac{2k}{2n-1} &\leq 2\sum_{i=r+1}^{\infty}\frac{S_i}{(2n-1)^{i-r}} \\ &\leq \frac{2k}{2n-1} + 2\sum_{i=r+2}^{\infty}\frac{n-1}{(2n-1)^{i-r}} \\ &= \frac{2k+1}{2n-1},\end{aligned}$$

and hence $\phi[(2n-1)^r S_0] = S_{r+1}$ for $r=0, 1, 2,\cdots$. Thus we have

$$\begin{aligned}\alpha_j(S_0) &= \sum_{i=1}^{\infty}\frac{\phi[(2n-1)^{ni-n+j-1}S_0]}{n^i} \\ &= \sum_{i=1}^{\infty}\frac{S_{ni-n+j}}{n^i} \\ &= x_j \quad \text{for } j=1, 2,\cdots, n.\end{aligned}$$

i.e. $\vec{\alpha}$ maps $[0, 1]$ onto $\Pi^n[0, 1]$.

References

1. J. Dugundji, *Topology*, Allyn & Bacon, Boston, Mass., 1964, pp. 104–105.
2. I. J. Schoenberg, *On the Peano curve of Lebesgue*, Bull. Amer. Math. Soc. **44** (1938), 519.

Institute of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 300, R. O. C.