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CONTINUOUS DEGREES

BY

KO-WEI LIH

Abstract. We study the class of continuous degrees. We show it is not a lattice and it possesses a reducible minimal element.

1. **Baire topology and continuity.** The hierarchy of total objects of finite type is defined as follows.

$$T(0) = \omega, \text{ the set of natural numbers };$$

$$T(n+1) = \text{the set of all unary total functions from } T(n) \text{ to } \omega.$$

Because in this paper we only discuss objects of type less than 3, we follow the convention that small latin letters $x, y, z, \cdots$ denote natural numbers or sequences of natural numbers; small greek letters $\alpha, \beta, \gamma, \cdots$ denote type-1 objects; capital latin letters $F, G, H, \cdots$ denote type-2 objects. The study of recursion theory over such objects was initiated by Kleene [4]. Now we pay our attention to a subclass of type-2 objects which are continuous with respect to the Baire topology. We show that these objects do not form a lattice and they possess a reducible minimal element.

The Baire metric bestowed upon $T(1)$ is defined as follows.

$$d(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ (\mu x(\alpha(x) \neq \beta(x)) + 1)^{-1} & \text{if } \alpha \neq \beta. \end{cases}$$

Under this metric d, $T(1)$ is a complete, separable, and totally disconnected space which is usually called the Baire space. If we equip ω with the discrete topology, then every type-2 object becomes a mapping between two topological spaces. Accordingly, it makes sense to talk about type-2 continuous objects.

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Now we introduce a few notations which will be used frequently.

Let s be a finite sequence of numbers, say, $(s_0, s_1, \cdots, s_k)$. Then the interval $[s]$ is defined as

$$[s] = \{\alpha : (\forall i \leq k)\ (\alpha(i) = s_i)\}\ .$$

If S is a set of finite sequences, then $[S]$ means the set $\bigcup\{[s] : s \in S\}$. For any two objects α^n and β^m, $\alpha^n \leq_T \beta^m$ means α^n is recursive in β^m. Let $\mathrm{dg}(\alpha^n)$ denote the degree of α^n; then $\alpha^n \leq_T \beta^m$ is equivalent to $\mathrm{dg}(\alpha^n) \leq \mathrm{dg}(\beta^m)$. When k is a fixed number, $\alpha | k = (\alpha(0), \cdots, \alpha(k-1))$ and $\bar{\alpha}(k) = \langle \alpha(0), \cdots, \alpha(k-1) \rangle$. The jump α' of α is the representing function of the predicate $J(x) \leftrightarrow (\{(x)_0\}^\alpha ((x)_1)$ is defined). Given a number k, ζ_k is always the recursive function $\lambda i(k)_i$. Finally let the object E^2 be defined as follows.

$$E^2(\alpha) = \begin{cases} 1 & \text{if } (\exists x)\ (\alpha(x) = 0)\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

The following proposition characterizes continuous objects in various ways.

THEOREM 1. *The following conditions are equivalent to each other for an object F.*

(a) *F is continuous.*

(b) $(\forall \beta)\ (\exists s)\ (\beta \in [s]$ *and* F *stays constant on* $[s])$.

(c) *There is an α such that $\alpha \leq_T F \leq_T \alpha'$.*

(d) *There is an α such that $F \leq_T \alpha$.*

(e) *E^2 is not recursive in F and a type-1 object.*

Proof. (a) → (b). Given β, let $F(\beta) = y$. Since F is continuous, there is a real number ε such that $(\forall \alpha)(d(\alpha, \beta) < \varepsilon \rightarrow F(\alpha) = y)$. Let $k = \mu x((x+1)^{-1} < \varepsilon)$ and $s = (\beta(0), \beta(1), \cdots, \beta(k-1))$.

(b) → (c). Let $\alpha = \lambda k F(\zeta_k)$. α is obviously recursive in F. Let $R(n, y, z) \leftrightarrow (\forall k)\ (\bar{\zeta}_k(n) = y \rightarrow \alpha(k) = z)$. Then R is recursive in α'. By (b) F stays constant on an initial interval of any β. It is easy to see actually

$$F(\beta) = (\mu x\ (R((x)_0, \bar{\beta}((x)_0), (x)_1)))_1\ .$$

So F is recursive in α'.

(c) $\rightarrow$ (d). This is trivial. This characterization was first obtained by Addison [1].

(d) $\rightarrow$ (e). This is true because E^2 is not recursive in any type-1 object.

(e) $\rightarrow$ (a). This was first established in Grilliot [2].

A degree is said to be a ***continuous degree*** if it contains at least one continuous object.

REMARK. Characterization (c) is of special interest. We actually can establish the following generalization of Shoenfield's Limit Lemma [5, p. 29].

Let $F(\gamma) = \{e\}(\alpha', \gamma)$ *for all* γ. *Then there exists a partial function* $G(s, \gamma)$ *recursive in* α *such that* $\lim_{s\to\infty} G(s, \gamma)$ *exists for all* γ *and* $\lim_{s\to\infty} G(s, \gamma) = F(\gamma)$.

Proof. We adopt the convention that $\{z\}_s(\alpha, x) = y$ means that we can compute $\{z\}(\alpha, x) = y$ in at most s steps and all $z, x, y \leq s$. The relation $\{z\}_s(\alpha, x) = y$ is recursive. Now let

$$\beta(s, x) = \begin{cases} 0 & \text{if } (\exists y \leq s)\,(\{(x)_0\}_s(\alpha, (x)_1) = y), \\ 1 & \text{otherwise}. \end{cases}$$

β is recursive in α and $\alpha'(x) = \lim_{s\to\infty} \beta(s, x)$ for all x.

Define a recursive total function H such that

$$H(s, x, \gamma) = \begin{cases} \gamma(x) & \text{if } 0 \leq x < s, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\gamma(x) = \lim_{s\to\infty} H(s, x, \gamma)$ for all γ.

Finally we set $G(s, \gamma) \simeq \{e\}(\lambda x \beta(s, x), \lambda x H(s, x, \gamma))$.

Since both β and H are total, G is recursive in α. To show $\lim_{s\to\infty} G(s, \gamma) = F(\gamma)$, we prove, for an arbitrary γ, there is an n such that, for all $k \geq n$, $G(k, \gamma)$ is defined and equal to $F(\gamma)$.

There is a number t such that, given γ, for all δ and η,

$$\delta | t = \alpha' | t \;\&\; \eta | t = \gamma | t \longrightarrow \{e\}(\alpha', \gamma) \simeq \{e\}(\delta, \eta).$$

Now fix i, $0 \leq i < t$. We have two possibilities:

(a) $\alpha'(i) = 0$. Then there is an s_i such that $\beta(k, i) = 0$ for all $k \geq s_i$.

(b) $\alpha'(i) = 1$. Then $\beta(s, i) = 1$ for all s.

We let m be 0 if Case (a) does not appear for any i; and let m be the largest number among those s_i's if Case (a) does appear. Then $\lambda x\beta(k, x) | t = \alpha' | t$ for all $k \geq m$.

On the other hand, it is obvious $\lambda x H(k, x, \tau) | t = \tau | t$ for all $k \geq t$. So let $n = \max\{m, t\}$. $k \geq n \rightarrow$

$$\begin{aligned} G(k, \tau) &\simeq \{e\}(\lambda x\beta(k, x), \lambda x H(k, x, \tau)) \\ &\simeq \{e\}(\alpha', \tau) \\ &\simeq F(\tau). \end{aligned}$$

2. **Upper-semi-lattice of continuous degrees.** For a continuous F, the value $F(\alpha)$ is determined by a finite initial segment of α. However, how long that segment should be is usually not computable from F and α. This fact hinders, though it does not totally prohibit, the generalization of techniques used in dealing with type-1 degrees directly to continuous degrees. We are going to handle a couple of situations to illustrate some of the modifications needed to achieve "lift-up" results.

By the usual normal form theorem, we have a recursive total predicate T and a recursive total function U such that $\{e\}(\alpha) \simeq U(\mu y T(\bar{\alpha}(y), e))$. Let

$$\psi_e(z) = \begin{cases} U(lh(z)) + 1 & \text{if } T(z, e), \\ 0 & \text{otherwise}. \end{cases}$$

This leads to the following

LEMMA 2. *For any e, the recursive function ψ_e satisfies the condition that, if $\{e\}(\alpha)$ is defined, we have*

(a) $(\exists x)(\psi_e(\bar{\alpha}(x)) > 0)$;

(b) $(\forall x)(\psi_e(\bar{\alpha}(x)) > 0 \rightarrow \psi_e(\bar{\alpha}(x)) = \{e\}(\alpha) + 1)$.

It follows that, if $\{e\}(\alpha)$ is defined, then $\{e\}(\alpha) = \psi_e(\bar{\alpha}(x))$, where $x = \mu y(\psi_e(\bar{\alpha}(y)) > 0)$. Therefore how long an initial segment of α is needed to give the value of $\{e\}(\alpha)$ is determined recursively. Thus, when the list $\bar{a}$ contains only variables of types 0 and 1, $\{e\}(\bar{a})$ can be computed in finitely many steps if $\{e\}(\bar{a})$ is defined. This finiteness is essential in establishing the next theorem.

THEOREM 3. *If* $\alpha_0 \leq_T \alpha_1 \leq_T \cdots$, *each* α_i *has* 0 *and* 1 *as function values, and* α *is any given type*-1 *object, then there exists a type*-1 *object* β *such that*

(a) $(\forall F)(F \leq_T \alpha \ \&\ F \leq_T \beta \rightarrow (\exists i)(F \leq_T \alpha_i))$;

(b) $(\forall i)(\alpha_i \leq_T \beta)$.

Proof. Construct β as $\bigcup_{i \in \omega} \theta_i$ where θ_i's are partial functions with recursive and coinfinite domains.

Stage 0. Let θ_0 be the empty function.

Stage $n+1$. Assume that θ_n is constructed. Let $\theta_{n+1} = \theta_n$ unless $n = \langle a, b \rangle$. Now suppose $n = \langle a, b \rangle$. We have two cases.

Case 1. $\lambda r \{a\}(\alpha, r)$ is not total.

Let σ_n be the empty function.

Case 2. $\lambda r \{a\}(\alpha, r)$ is total. We have two subcases.

(A) There exists a partial function τ satisfying the following: (i) τ has 0 and 1 as function values; (ii) dom (τ) is finite; (iii) τ extends σ_n, and there exists $\delta \in T(1)$ such that $\{b\}(\tau, \delta)$ is defined and different from $\{a\}(\alpha, \delta)$.

Let $\sigma_n = \tau$.

(B) Otherwise.

Let σ_n be the empty function. Finally set

$$\theta_{n+1} = \theta_n \cup \sigma_n \cup \{(\langle n, j \rangle, \alpha_n(j)) : \langle n, j \rangle \notin \text{dom}(\theta_n) \cup \text{dom}(\sigma_n)\}.$$

Now we are going to show that the construction really works.

Claim 1. If $n = \langle a, b \rangle$ and $\lambda r \{a\}(\alpha, r) = \lambda r \{b\}(\beta, r) =$ a total object, then $\lambda r \{a\}(\alpha, r) \leq_T \alpha_{n-1}$.

Now assume both $\lambda r \{a\}(\alpha, r)$ and $\lambda r \{b\}(\beta, r)$ are total and equal. To compute $\{a\}(\alpha, \delta)$ for any given δ, search for τ satisfying the following:

(i) τ has 0 and 1 as function values;

(ii) dom (τ) is finite;

(iii) τ extends θ_n, and $\{b\}(\tau, \delta)$ is defined. Then $\{b\}(\tau, \delta) = \{b\}(\beta, \delta) = \{a\}(\alpha, \delta)$ by the assumption. Such a τ exists because $\lambda r \{b\}(\beta, r)$ is total and the computation of $\{b\}(\beta, \delta)$ uses finitely many values of β. Then the computation of $\{b\}(\tau, \delta)$ can be carried out in finitely many steps. It remains to show that τ can be found recursively from α_{n-1}. It suffices to show that the set $\{\langle \text{dom}(\tau) \rangle :$

τ is a partial function having 0 and 1 as function values, dom (τ) is finite, and τ extends $\theta_n\}$ is recursive in α_{n-1}. Note that, for all but finitely many $\langle i, j\rangle$,

$$\theta_n(\langle i, j\rangle) = \begin{cases} \alpha_i(j) & \text{if } i < n, \\ \text{undefined} & \text{otherwise}. \end{cases}$$

Since $\alpha_0 \leq_T \alpha_1 \leq_T \cdots \leq_T \alpha_{n-1}$, the set in question is recursive in α_{n-1}.

Claim 2. $(\forall i)(\alpha_i \leq_T \beta)$.

This is true because, for any i, $\alpha_i(j) = \beta(\langle i, j\rangle)$ for all sufficiently large j.

Each type-1 degree contains a function having only 0 and 1 as function values. So we can rephrase the above theorem in terms of degrees.

THEOREM 4. *Given type-*1 *degrees* $\boldsymbol{a}, \boldsymbol{a}_0, \boldsymbol{a}_1, \cdots$ *such that* $\boldsymbol{a}_0 \leq \boldsymbol{a}_1 \leq \cdots$; *then there exists a type-*1 *degree* $\boldsymbol{b}$ *such that*

(i) $(\forall$ *type-*2 $c)(\boldsymbol{c} \leq \boldsymbol{a}\ \&\ \boldsymbol{c} \leq \boldsymbol{b} \rightarrow (\exists\ i)(\boldsymbol{c} \leq \boldsymbol{a}_i))$;

(ii) $(\forall i)(\boldsymbol{a}_i \leq \boldsymbol{b})$.

THEOREM 5. *No strictly ascending sequence of type-*1 *degrees has a type-*2 *degree as the least upper bound.*

Proof. Suppose $\boldsymbol{a}_0 < \boldsymbol{a}_1 < \cdots$ is a strictly ascending sequence of type-1 degrees. Let $\boldsymbol{a}$ be a type-1 upper bound of this sequence. Let $\boldsymbol{b}$ be as in Theorem 4. Suppose the type-2 degree $\boldsymbol{c}$ were the least upper bound. Then $\boldsymbol{c} \leq \boldsymbol{a}$ and $\boldsymbol{c} \leq \boldsymbol{b}$. So $(\exists i)(\boldsymbol{c} \leq \boldsymbol{a}_i < \boldsymbol{a}_{i+1})$. $\boldsymbol{c}$ could not be an upper bound.

THEOREM 6. *There exists a pair of type-*1 *degrees which has no type-*2 *degree as the greatest lower bound.*

Proof. Let $\boldsymbol{a}_i$ be the i-fold jump of $\boldsymbol{0}$ in the previous theorem. Then the pair $\boldsymbol{a}$ and $\boldsymbol{b}$ is such a pair.

An immediate consequence of Theorem 6 is

THEOREM 7. *Continuous degrees do not form a lattice.*

REMARK. Theorems 5 and 7 answer two questions raised at the end of Hinman [3].

3. **A reducible minimal degree.** Our next task is to construct a type-1 degree which is minimal among type-2 degrees.

A *string* is a finite initial segment of a representing function, i.e., a function with only 0 and 1 as function values. Let St be the set of strings. We use variables $\sigma, \tau, \rho, \cdots$ to range over St. If $i = 0$ or 1, $\sigma^{(i)}$ is the string obtained from σ by adding i at the end. Two strings σ and τ are *incompatible* if there is an x such that $\sigma(x)$ and $\tau(x)$ are defined and different.

A *tree* is a recursive total function Tr from St to St such that, for all σ, $\mathrm{Tr}(\sigma^{(0)})$ and $\mathrm{Tr}(\sigma^{(1)})$ are incompatible extensions of $\mathrm{Tr}(\sigma)$. A tree Tr is a *subtree* of a tree Tr′ if the range of Tr is included in the range of Tr′.

A type-1 object β is a *branch* of a tree Tr if (the range of Tr) $\cap \{\sigma : \beta$ extends $\sigma\}$ is an infinite set, or equivalently $(\forall n)(\exists \sigma)(lh(\sigma) = n$ & β extends $\mathrm{Tr}(\sigma))$.

A tree Tr is called *n-constant* if whenever $\{n\}(\mathrm{Tr}(\tau), \alpha)$ and $\{n\}(\mathrm{Tr}(\sigma), \alpha)$ are defined, they are equal.

LEMMA 8. *If a tree* Tr *is n-constant,* $\lambda\alpha\{n\}(\beta, \alpha)$ *is total, and* β *is a branch of* Tr, *then* $\lambda\alpha\{n\}(\beta, \alpha)$ *is recursive.*

Proof. To compute $\{n\}(\beta, \alpha)$ for a given α look for any σ such that $\{n\}(\mathrm{Tr}(\sigma), \alpha)$ is defined. The computation of $\{n\}(\beta, \alpha)$ uses finitely many values of β. So $(\exists \tau)(\beta$ extends τ & $\{n\}(\tau, \alpha)$ is defined). Since β is a branch of Tr, $(\exists \rho)(\mathrm{Tr}(\rho)$ extends $\tau)$. Now $\{n\}(\mathrm{Tr}(\rho), \alpha) = \{n\}(\tau, \alpha) = \{n\}(\beta, \alpha)$. But by the n-constancy of Tr, $\{n\}(\mathrm{Tr}(\rho), \alpha) = \{n\}(\mathrm{Tr}(\sigma), \alpha)$. Since σ certainly can be found recursively and the computation of $\{n\}(\mathrm{Tr}(\sigma), \alpha)$ halts in finitely many steps, $\lambda\alpha\{n\}(\beta, \alpha)$ is recursive.

A tree Tr is *n-splitting* if $(\forall \sigma)(\exists \alpha)(\{n\}(\mathrm{Tr}(\sigma^{(0)}), \alpha)$ and $\{n\}(\mathrm{Tr}(\sigma^{(1)}), \alpha)$ are defined and different).

LEMMA 9. *If a tree* Tr *is n-splitting,* $\lambda\alpha\{n\}(\beta, \alpha)$ *is total, and* β *is a branch of* Tr, *then* $\beta \leq_T \lambda\alpha\{n\}(\beta, \alpha)$.

Proof. Define recursively from $\lambda\alpha\{n\}(\beta, \alpha)$ a sequence of strings $\sigma_0, \sigma_1, \cdots$ such that σ_{j+1} extends σ_j, $lh(\sigma_j) = j$, and β extends $\mathrm{Tr}(\sigma_j)$ for all j. To start out let σ_0 be the empty string. Assuming that σ_j is determined with the desired properties, try to define σ_{j+1}. Look for a sequence number k such that $\{n\}(\mathrm{Tr}(\sigma_j^{(0)}), \zeta_k)$ and $\{n\}(\mathrm{Tr}(\sigma_j^{(1)}), \zeta_k)$ are defined and different by means of a recursive search. Since Tr is n-splitting, there is an α with $\{n\}(\mathrm{Tr}(\sigma_j^{(0)}), \alpha)$ and $\{n\}(\mathrm{Tr}(\sigma_j^{(1)}), \alpha)$ defined and different. But both computations use only some finite initial segment of α. So the above search will terminate. Since β is a branch of Tr, β extends either $\mathrm{Tr}(\sigma_j^{(0)})$ or $\mathrm{Tr}(\sigma_j^{(1)})$. Thus either $\{n\}(\mathrm{Tr}(\sigma_j^{(0)}), \zeta_k)$ or $\{n\}(\mathrm{Tr}(\sigma_j^{(1)}), \zeta_k)$ is equal to $\{n\}(\beta, \zeta_k)$. Let σ_{j+1} be $\sigma_j^{(0)}$ if $\{n\}(\mathrm{Tr}(\sigma_j^{(0)}), \zeta_k) = \{n\}(\beta, \zeta_k)$; otherwise let σ_{j+1} be $\sigma_j^{(1)}$. By the inductive nature of the construction and the definition of a tree, we know that β extends $\mathrm{Tr}(\sigma_j)$ and $lh(\mathrm{Tr}(\sigma_j)) \geq j$; hence $\beta(j) = \mathrm{Tr}(\sigma_{j+1})(j)$. Consequently, $\beta \leq_T \lambda\alpha\{n\}(\beta, \alpha)$.

LEMMA 10. *If* Tr′ *is a tree and n is a given number, then* Tr′ *has a recursive subtree* Tr *such that* Tr *is either n-constant or n-splitting.*

Proof. We have two cases.

Case 1. $(\exists \sigma_0)(\forall \tau$ extending $\sigma_0)(\forall \rho$ extending $\sigma_0)$ $(\forall k)(\neg(\{n\}$ $(\mathrm{Tr}'(\tau), \zeta_k)$ and $\{n\}(\mathrm{Tr}'(\rho), \zeta_k)$ are defined and different.)).

Let Tr be $\mathrm{Tr}(\tau) = \mathrm{Tr}'(\sigma_0 * \tau)$ for such a fixed σ_0, where $*$ is the recursive concatenation operation on strings. Tr is n-constant. Assuming otherwise, $(\exists \alpha)(\exists \tau)(\exists \rho)$ $(\{n\}(\mathrm{Tr}(\tau), \alpha)$ and $\{n\}(\mathrm{Tr}(\rho), \alpha)$ are defined and different.). Trim α down to some ζ_k such that $\{n\}(\mathrm{Tr}(\tau), \alpha) = \{n\}(\mathrm{Tr}(\tau), \zeta_k)$ and $\{n\}(\mathrm{Tr}(\rho), \alpha) = \{n\}(\mathrm{Tr}(\rho), \zeta_k)$. But $\mathrm{Tr}(\tau) = \mathrm{Tr}'(\sigma_0 * \tau)$ and $\mathrm{Tr}(\rho) = \mathrm{Tr}'(\sigma_0 * \rho)$. By our case assumption $\{n\}(\mathrm{Tr}'(\sigma_0 * \tau), \zeta_k) = \{n\}(\mathrm{Tr}'(\sigma_0 * \rho), \zeta_k)$, a contradiction.

Case 2. $(\forall \sigma_0)(\exists \tau$ extending $\sigma_0)(\exists \rho$ extending $\sigma_0)$ $(\exists k)(\{n\}$ $(\mathrm{Tr}'(\tau), \zeta_k)$ and $\{n\}(\mathrm{Tr}'(\rho), \zeta_k)$ are defined and different.).

Define $\mathrm{Tr}(\sigma)$ by induction on $lh(\sigma)$. Let $\mathrm{Tr}(0) = \mathrm{Tr}'(0)$. Assume that $\mathrm{Tr}(\sigma)$ is defined and in the range of Tr′ for $lh(\sigma) \leq j$. Given any σ with length j, we want to define $\mathrm{Tr}(\sigma^{(0)})$ and $\mathrm{Tr}(\sigma^{(1)})$. Recursively find τ, ρ, and k so that $\{n\}(\mathrm{Tr}'(\tau), \zeta_k)$ and $\{n\}(\mathrm{Tr}'(\rho), \zeta_k)$ are defined and different, and τ and ρ extend σ. Let $\mathrm{Tr}(\sigma^{(0)}) = \mathrm{Tr}'(\tau)$ and $\mathrm{Tr}(\sigma^{(1)}) = \mathrm{Tr}'(\rho)$. Now $\mathrm{Tr}(\sigma) = \mathrm{Tr}'(\sigma_0)$

for some σ_0. So $\mathrm{Tr}(\sigma^{(0)})$ and $\mathrm{Tr}(\sigma^{(1)})$ are incompatible extensions of $\mathrm{Tr}'(\sigma_0) = \mathrm{Tr}(\sigma)$ and Tr is a recursive subtree of Tr′. Tr is n-splitting by the construction.

REMARK. The case assumptions can be decided from $\mathbf{0}''$, the double jump of the recursive degree $\mathbf{0}$.

THEOREM 11. *There exists a type*-1 *degree* $\boldsymbol{m}$ *such that*

(i) $\mathbf{0} < \boldsymbol{m} < \mathbf{0}''$;

(ii) $(\forall$ *type*-2 $\boldsymbol{a})(\boldsymbol{a} < \boldsymbol{m} \rightarrow \boldsymbol{a} = \mathbf{0})$.

Proof. We will construct a sequence of recursive trees $\mathrm{Tr}_0, \mathrm{Tr}_1, \cdots$ such that Tr_{j+1} is a subtree of Tr_j and every branch β of Tr_{j+1} satisfies the following condition C_j:

$$j = 2n, \qquad C_{2n}: \quad \beta \neq \lambda x\{n\}(x);$$

$$j = 2n+1, \qquad C_{2n+1}: \quad \lambda\alpha\{n\}(\beta, \alpha) \text{ is total} \rightarrow \lambda\alpha\{n\}(\beta, \alpha) \text{ is recursive or } \beta \leq_T \lambda\alpha\{n\}(\beta, \alpha).$$

Stage 0. Let $\mathrm{Tr}_0(\sigma) = \sigma$ for all σ.

Stage $2n+1$. Try to meet condition C_{2n}. Find k such that $\mathrm{Tr}_{2n}(0)(k)$ and $\mathrm{Tr}_{2n}(1)(k)$ are defined and different. Let

$$\mathrm{Tr}_{2n+1}(\sigma) = \begin{cases} \mathrm{Tr}_{2n}((0) * \sigma) & \text{if } \{n\}(k) \text{ is defined and } = \mathrm{Tr}_{2n}(1)(k), \\ \mathrm{Tr}_{2n}((1) * \sigma) & \text{otherwise}. \end{cases}$$

Thus if β is a branch of Tr_{2n+1}, then $\beta(k)$ is not equal to $\{n\}(k)$.

Stage $2n+2$. Let Tr_{2n+2} be a recursive subtree of Tr_{2n+1} such that Tr_{2n+2} is either n-splitting or n-constant. Such a Tr_{2n+2} exists by the previous lemma.

As in the ordinary minimal degree construction, there is a branch β belonging to all these trees $\mathrm{Tr}_0, \mathrm{Tr}_1, \cdots$. Let $\boldsymbol{m} = \mathrm{dg}(\beta)$.

REMARK. The above theorem does show that there is a minimal element in the set of continuous degrees. However, the existential status of a minimal continuous degree which contains no type-1 objects is yet to be determined.

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DUKE UNIVERSITY, DURHAM, NORTH CAROLINA 27706, U. S. A.

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI, TAIWAN 115