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A DECOMPOSITION OF THE FOURIER TRANSFORM ON THE SPACE OF 2×2 HERMITIAN MATRICES

BY

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Abstract. Let X be the space of 2×2 Hermitian matrices. $L^2(X)$ is decomposed into the direct sum of subspaces $L^2(X, \Lambda)$ of covariant functions with respect to the irreducible representation Λ of $SU(2)$. We prove that $L^2(X, \Lambda)$ is isomorphic to either the subspace of symmetric functions of $L^2(R^2)$ or the subspace of antisymmetric functions of $L^2(R^2)$. Since the Fourier transform on $L^2(X)$ leaves $L^2(X, \Lambda)$ invariant, it is expressed on each summand with only spherical Bessel functions in the kernel.

1. Introduction. Let X be the space of 2×2 Hermitian matrices. Since the compact group $SU(2)$ acts on X by conjugation, we decompose $L^2(X)$ into the direct sum of subspaces $L^2(X, \Lambda)$ each of which consists of functions covariant with respect to the irreducible representation Λ of $SU(2)$. Since X is isomorphic to R^4, this gives a decomposition of $L^2(R^4)$ which is not the direct sum of the tensor products of radial functions and the solid spherical harmonics on S^3 as given in [1] and [3].

In this paper we prove that $L^2(X, \Lambda)$ is isomorphic to either the subspace of symmetric functions of $L^2(R^2)$ or the subspace of antisymmetric functions of $L^2(R^2)$; i.e. there is a unitary map between them. Furthermore, since the Fourier transform on $L^2(X)$ leaves $L^2(X, \Lambda)$ invariant, under the unitary equivalence the Fourier transform on each summand is calculated and is seen to involve only spherical Bessel functions which are elementary functions.

The symmetric or antisymmetric functions of $L^2(R^2)$ can be viewed as functions on the diagonals of 2×2 Hermitian matrices. In a paper in preparation, such decompositions of $L^2(X)$ and its Fourier transform are used in finding out all irreducible representations in a degenerate principal series of $U(2, 2)$. The author wishes

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to thank Professor Kenneth I. Gross for many valuable discussions on these topics.

2. **The group $SU(2)$ and the space X of 2×2 Hermitian matrices.** In the first part of this section we give a realization of all irreducible representations of $SU(2)$ as well as some properties of these representations. In the second part of this section we discuss the geometry of $SU(2)$, and of the space X.

Let $SU(2)$ be the subgroup of $GL(2, C)$ which consists of all 2×2 unitary matrices with determinant equal to 1; i.e.

$$SU(2) = U(2) \cap SL(2, C)$$
$$= \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \in C^{2\times 2} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

$SU(2)$ is clearly a compact group; hence all its irreducible representations are finite-dimensional. We shall give a realization of these representations and some facts to be used later. For details see [1, 4].

For $l = 0, 1/2, 1, 1\ 1/2, 2, \cdots$, let $\mathscr{H}_l$ be the $(2l+1)$-dimensional vector space which consists of all polynomials in one complex variable of degree $\leq 2l$. Define the inner product on $\mathscr{H}_l$ by

$$(2.1) \qquad \begin{aligned} (z^j, z^j) &= j!(2l-j)!, \\ (z^j, z^m) &= 0 \qquad \text{if } j \neq m. \end{aligned}$$

With respect to this inner product, $\mathscr{H}_l$ has the orthonormal base

$$(2.2) \qquad \mathscr{E}_l = \{1/\sqrt{(2l)!},\ z/\sqrt{(2l-1)!}, \cdots, z^j/\sqrt{(2l-k)!k!}, \cdots, z^{2l-1}/\sqrt{(2l-1)!},\ z^{2l}/\sqrt{(2l)!}\}.$$

For any $u = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \in SU(2)$ and $\phi \in \mathscr{H}_l$, define a unitary operator $\Lambda_l(u)$ on $\mathscr{H}_l$ by

$$(2.3) \qquad (\Lambda_l(u)\phi)(z) = (\beta z + \bar{\alpha})^{2l}\phi((\alpha z - \bar{\beta})/(\beta z + \bar{\alpha})).$$

Then $\Lambda_l: u \to \Lambda_l(u)$ is a representation of $SU(2)$.

THEOREM (2.4). *$\{\Lambda_l\}_{l=0,1/2,1,\ldots}$ is the complete system of irreducible representations $SU(2)$; i.e. every irreducible representation of*

$SU(2)$ is equivalent to one Λ_l for some $l = 0, 1/2, 1, \cdots$. In other words

$$SU(2)^{\wedge} = \{\Lambda_l\}_{l=0,1/2,1,\cdots}.$$

For any $\phi \in \mathcal{H}_l$ we shall write the coordinates of ϕ, with respect to the orthonormal base $\mathcal{E}_l$ given in (2.2), as a $(2l+1)$-dimensional column vector. Consequently, $\mathcal{H}_l \simeq \boldsymbol{C}^{2l+1}$. As a slight departure from the usual notation, E_j denotes the $(j+1)$th unit vector in $\boldsymbol{C}^{2l+1}$. Relative to $\mathcal{E}_l$, Λ_l can be represented by a $(2l+1) \times (2l+1)$ matrix. In the rest of this article, Λ_l is realized as a matrix representation on $\boldsymbol{C}^{2l+1}$.

$$(2.5) \qquad \boldsymbol{u}(\theta_1,\theta_2) = \begin{bmatrix} \cos\theta_1 & (\sin\theta_1)e^{i\theta_2} \\ -(\sin\theta_1)e^{-i\theta_2} & \cos\theta_1 \end{bmatrix} \text{ for } \begin{matrix} 0 < \theta_1 < \pi/2, \\ 0 \leq \theta_2 \leq 2\pi. \end{matrix}$$

$$(2.6) \qquad \boldsymbol{u}_p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let

$$(2.7) \qquad \boldsymbol{T} = \left\{ \boldsymbol{u}(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \middle|\, 0 \leq \theta \leq 2\pi \right\}$$

be the *maximal abelian subgroup*, or *maximal torus* of $SU(2)$.

LEMMA (2.8). *For any integer l the $(l+1, l+1)$-entry $\Lambda_{ll}(\boldsymbol{u}(\theta_1,\theta_2))$ of $\Lambda_l(\boldsymbol{u}(\theta_1,\theta_2))$ is equal to $P_l(\cos 2\theta_1)$, where P_l is the Legendre polynomial of degree l.*

THEOREM (2.9). *$\Lambda_l|_{\boldsymbol{T}}$ has a fixed vector if and only if l is an integer. If l is an integer, there is up to scalar multiples only one fixed vector, namely E_l, in $\boldsymbol{C}^{2l+1}$ ($z^l/l!$ in $\mathcal{H}_l$ correspondingly).*

Now we introduce coordinates and hence measures on the vector space X of 2×2 Hermitian matrices. For any $x \in X$ we write

$$x = \begin{bmatrix} x_1 + x_4 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_1 - x_4 \end{bmatrix}, \qquad x_i \in \boldsymbol{R},$$

and define the inner product by

$$(x|y) = \operatorname{tr}(xy), \qquad x, y \in X.$$

Then X is isomorphic to the Euclidean space $\boldsymbol{R}^4$. Let $dx = dx_1 dx_2 \cdot dx_3 dx_4$ be the Lebesgue measure on X.

Let Ω be the set of all real 2×2 diagonal matrices,

$$\Omega = \left\{ \omega = (\omega_1, \omega_2) = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \mid \omega_1, \omega_2 \in \boldsymbol{R} \right\}; \tag{2.10}$$

then for any $x \in X$, there exists a $u \in SU(2)$ such that $u^{-1}xu = \omega \in \Omega$. Ω is homeomorphic to $\boldsymbol{R}^2$. We let $d\omega = d\omega_1 d\omega_2$ be the Lebesgue measure on Ω. Since $u(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ commutes with all $\omega \in \Omega$ and

$$u_p^{-1} \omega u_p = u_p^{-1}(\omega_1, \omega_2) u_p = (\omega_2, \omega_1),$$

where u_p is given in (2.6), one can prove that for ω such that $\omega_1 > \omega_2$, the map Φ: $(\omega, \dot{u}) \to u\omega u^{-1}$ is continuous and one-to-one. Let

$$\Omega_0 = \{\omega = (\omega_1, \omega_2) \in \Omega \mid \omega_1 \geq \omega_2\}. \tag{2.11}$$

Since the set of ω for which $\omega_1 = \omega_2$ is of measure 0, we shall say that Φ identifies $\omega_0 \times SU(2)/T$ with X. This gives a new coordinate system on X, namely $(\omega_1, \omega_2, \theta_1, \theta_2)$. The coordinate transformation is given by

$$\begin{aligned} x_1 &= \frac{1}{2}(\omega_1 + \omega_2), \\ x_2 &= \frac{1}{2}(\omega_1 - \omega_2) \sin 2\theta_1 \cos \theta_2, \\ x_3 &= \frac{1}{2}(\omega_1 - \omega_2) \sin 2\theta_1 \sin \theta_2, \\ x_4 &= \frac{1}{2}(\omega_1 - \omega_2) \cos 2\theta_1. \end{aligned} \tag{2.12}$$

Thus one may think of $(\omega_1, \omega_2, \theta_1, \theta_2)$ as spherical coordinates of X. Define the measure $d\dot{u} = d\dot{u}(\theta_1, \theta_2) = \frac{1}{4} \sin 2\theta_1 \, d\theta_1 \, d\theta_2$ on $SU(2)/T$.

LEMMA (2.13). (i) $\int_{SU(2)/T} d\dot{u} = \pi/2$,

(ii) *for* $f \in L^1(X)$,

$$(2.14)\qquad \int_X f(x)dx = \int_{\Omega_0}\int_{SU(2)/T} f(u\omega u^{-1}) \cdot \frac{1}{4}(\omega_1-\omega_2)^2 \sin 2\theta_1\, d\theta_1 d\theta_2 d\omega_1 d\omega_2$$
$$= \int_{\Omega_0}\int_{SU(2)/T} f(u\omega u^{-1})(\omega_1-\omega_2)^2\, d\dot{u}d\omega.$$

3. **A decomposition of the Fourier transform on $L^2(X)$.** In this section we first decompose $L^2(X)$ according to the irreducible representations of $SU(2)$ into the direct sum of spaces of covariant functions. Then we prove that $L^2(X, \Lambda)$ is isomorphic to the subspace of symmetric functions of $L^2(R^2)$ or the subspace of antisymmetric functions of $L^2(R^2)$. Finally we use Legendre polynomials and Bessel functions to compute the restriction of the Fourier transform on $L^2(X)$ to each invariant subspace. We first introduce some terminology and notation.

For any $\Lambda \in SU(2)^\wedge$ and $u \in SU(2)$, let $\bar{\Lambda}(u)$, $\Lambda_{(j)}(u)$ and $\Lambda_{(i,j)}(u)$ denote the conjugate matrix, the jth column and the (i, j)th entry of $\Lambda(u)$ respectively. A function f in $L^2(X, C^{d_\Lambda})$, d_Λ being the degree of Λ, is said to be *Λ-covariant* if and only if

$$f(uxu^{-1}) = \bar{\Lambda}(u)f(x) \qquad \text{for all } u\in SU(2).$$

Let $L^2(X, \Lambda)$ be the subspace of $L^2(X, C^{d_\Lambda})$ which consists of all Λ-covariant functions. $L^2(X, \Lambda)$ is also a Hilbert space. Consider the Hilbert space $L^2(X)^\sim = \sum_{\Lambda\in SU(2)^\wedge} \oplus d_\Lambda L^2(X, \Lambda)$. Let du be normalized Haar measure on $SU(2)$.

THEOREM (3.1). *The generalized Peter-Weyl transform $\mathcal{G}$ of $C(X) \cap L^2(X)$ into $\sum_{\lambda\in SU(2)^\wedge} \oplus L^2(X, \Lambda)$ given by*

$$(3.2)\qquad (\mathcal{G}f)_j(\Lambda)(x) = \int_{SU(2)} f(u^{-1}xu)\bar{\Lambda}_j(u)\, du, \qquad j = 1, 2, \cdots, d_\Lambda,$$

extends uniquely to a unitary transformation of $L^2(X)$ onto $L^2(X)^\sim$. Where we have occasion to do so, we omit the subindex j.

See [2] for the proof.

By the above theorem we identify $L^2(X)$ with $\sum\oplus d_\Lambda L^2(X, \Lambda)$; we use F to denote $\mathcal{G}F\mathcal{G}^{-1}$. Since dx is an invariant measure with respect to conjugation by $SU(2)$,

$$(L(u)f)(x) = f(u^{-1}xu) \qquad \text{for } f \in L^2(X)$$

defines a unitary representation L of $SU(2)$ on $L^2(X)$. Therefore any bounded operator of $L^2(X)$ which commutes with all $L(u)$, $u \in SU(2)$, must leave $L^2(X, \Lambda)$ invariant. By simple computation $FL(u) = L(u)F$ for all $u \in SU(2)$. Let F_Λ be the restriction of F to $L^2(X, \Lambda)$. By a straightforward verification,

PROPOSITION (3.3). *For* $f_\Lambda \in C_C(X) \cap L^2(X, \Lambda)$,

$$(F_\Lambda f_\Lambda)(x) = \frac{1}{\pi^2}\int_X e^{i(x|y)} f_\Lambda(y)\, dy. \tag{3.4}$$

In summary, $F = \sum_{\Lambda \in SU(2)^\wedge} \oplus\, d_\Lambda F_\Lambda$, where F_Λ (the Fourier transform on $L^2(X, \Lambda)$ is given by the usual formula, namely (3.4). Note in particular $(F_\Lambda^{-1} f_\Lambda)(x) = (F_\Lambda f_\Lambda)(-x)$.

In view of the decomposition of $L^2(X)$ and from the result of the preceding section, we have

$$F \simeq \sum_{l=0}^{\cdot} \oplus\, (2l+1) F_{\Lambda_l}, \tag{3.5}$$

where l ranges through 0, 1/2, 1, 1 1/2, 2,$\cdots$.

Let us determine which of the vector spaces $L^2(X, \Lambda_l)$ are nonzero. For $f \in L^2(X, \Lambda_l)$ let f_j, $j = 0, 1, \cdots, 2l$, be the component functions of f, i.e.

$$f(x) = \begin{pmatrix} f_0(x) \\ \vdots \\ f_j(x) \\ \vdots \\ f_{2l}(x) \end{pmatrix} = \sum_{j=0}^{2l} f_j(x) E_j,$$

where f_j are scalar-valued functions. Recall that if $f \in L^2(X, \Lambda_l)$ then $f(uxu^{-1}) = \bar{\Lambda}_l(u) f(x)$ for all $u \in SU(2)$. Thus functions in $L^2(X, \Lambda_l)$ are determined by their values on Ω_0. On the other hand, for any x choose a $u \in SU(2)$ such that $uxu^{-1} = x$; then

$$f(x) = f(uxu^{-1}) = \bar{\Lambda}_l(u) f(x) \qquad \text{for } f \in L^2(X, \Lambda_l). \tag{3.6}$$

Hence $f(x)$ is a vector fixed by $\bar{\Lambda}_l(u)$.

Let S_x be the stability subgroup of x under conjugation by $SU(2)$. So if $f \in L^2(X, \Lambda_l)$ then almost everywhere $f(x)$ must be a vector fixed by $\bar{\Lambda}_l(u)$ for $u \in S_x$. For any $\omega \in \Omega$ except $\omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or $\omega = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, the stability subgroup $S_\omega = \boldsymbol{T}$. Therefore by Theorem (2.9) $L^2(X, \Lambda_l) = \{0\}$ for $l = 1/2, 3/2, \cdots$. On the other hand, when l is an integer, by Theorem (2.9) $f(\omega) = \bar{\Lambda}_l(u(\theta))f(\omega)$ for all $u(\theta) \in \boldsymbol{T}$, which is equivalent to the condition

$$f(\omega) = f_l(\omega)E_l \in CE_l \qquad \text{for all } \omega \in \Omega. \tag{3.7}$$

Moreover, the function $f_l|_\Omega$ satisfies the symmetry property

$$f_l(\omega') = (-1)^l f(\omega), \tag{3.8}$$

where for $\omega = (\omega_1, \omega_2)$ we have set $\omega' = (\omega_2, \omega_1)$. Indeed (3.8) follows from (3.6) and the fact that

$$\omega' = u_p \omega u_p^{-1}, \qquad u_p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

From now on, l is an integer.

Let $L_l^2(\Omega)$ be the space of all square integrable functions ϕ on Ω such that $\phi(\omega') = (-1)^{l+1}\phi(\omega)$. Thus, when l is odd $L_l^2(\Omega)$ is the Hilbert space of all "symmetric" functions in $L^2(\Omega)$, and when l is even $L_l^2(\Omega)$ is the Hilbert space of all "antisymmetric" functions in $L^2(\Omega)$.

THEOREM (3.9). *For $f \in L^2(X, \Lambda_l)$, define the map Φ by*

$$(\Phi f)(\omega) = \frac{\sqrt{\pi}}{2}(\omega_1 - \omega_2)f_l(\omega); \tag{3.10}$$

then Φ is a unitary map of $L^2(X, \Lambda_l)$ onto $L_l^2(\Omega)$ with Φ^{-1} given by

$$(\Phi^{-1}\phi)(x) = \frac{2}{\sqrt{\pi}}(\omega_1 - \omega_2)^{-1}\cdot\phi(\omega)\bar{\Lambda}_l(u)E_l \tag{3.11}$$

for $x = u\omega u^{-1}$ and $\omega_1 \neq \omega_2$. Moreover, the Fourier transform $F_l = \Phi F_{\Lambda_l}\Phi^{-1}$ on $L_l^2(\Omega)$ is given by

$$\begin{aligned} (F_l\phi)(\omega) &= \pi^{-1/2}\int_{\Omega_0}((\omega_1 - \omega_2)(\eta_1 - \eta_2))^{1/2}e^{(i/2)(\omega_1+\omega_2)(\eta_1+\eta_2)} \\ &\qquad \cdot J_{l+1/2}((\omega_1 - \omega_2)(\eta_1 - \eta_2))\phi(\eta)\,d\eta \\ &= (2\pi)^{-1}\int_\Omega e^{i(\eta|\omega)}\sum_{r=0}^{l}\frac{C_r}{(\eta_1 - \eta_2)^r(\omega_1 - \omega_2)^r}\phi(\eta)\,d\eta, \end{aligned} \tag{3.12}$$

(3.13) $$(F_l^{-1}\phi)(\omega) = -(F_l\phi)(-\omega),$$

where $J_{l+(1/2)}$ *is the spherical Bessel function of order* $l+1/2$ *and*

$$C_r = \frac{i^{r-1}(l+r)!}{r!(l-r)!}, \qquad r = 0, 1, \cdots, l.$$

Proof. The fact that Φ is a well-defined bijective map is a straightforward consequence of the defining relations (3.10), (3.11) and (3.8) of the spaces involved. Let us prove that Φ is unitary. Recall that $\Omega_0 = \{\omega \in \Omega \mid \omega_1 \geq \omega_2\}$, and define

(3.14) $$\Omega_0' = \{\omega' \mid \omega \in \Omega_0\} = \{\omega \in \Omega \mid \omega_1 \leq \omega_2\}.$$

Then $\Omega = \omega_0 \cup \Omega_0'$, and $\Omega_0 \cap \Omega_0' = \{\omega \in \Omega \mid \omega_1 = \omega_2\}$, which is of measure zero in Ω. Now

$$\begin{aligned}
\|f\|^2 &= \int_X (f(x) \mid f(x))\, dx \\
&= \int_{\Omega_0}\int_{SU(2)/T} (f(u\omega u^{-1}) \mid f(u\omega u^{-1}))(\omega_1-\omega_2)^2\, d\dot{u}\, d\omega \\
&= \int_{\Omega_0}\int_{SU(2)/T} (f(\omega) \mid f)\omega \mid (\omega_1-\omega_2)^2\, d\dot{u}\, d\omega \\
&= \int_{\Omega_0} (\pi/4)\,|f_l(\omega)|^2(\omega_1-\omega_2)^2\, d\omega + \int_{\Omega_0} (\pi/4)\,|f_l(\omega)|^2(\omega_1-\omega_2)^2\, d\omega \\
&= \int_{\Omega_0} (\pi/4)\,|f_l(\omega)|^2(\omega_1-\omega_2)^2\, d\omega + \int_{\Omega_0'} (\pi/4)\,|f_l(\omega)|^2(\omega_1-\omega_2)^2\, d\omega \\
&= \int_{\Omega} (\pi/4)\,|f_l(\omega)|^2(\omega_1-\omega_2)^2\, d\omega \\
&= \int_{\Omega} |(\Phi f)(\omega)|^2\, d\omega.
\end{aligned}$$

Therefore Φ is a unitary map of $L^2(X, \Lambda_l)$ onto $L_l^2(\Omega)$.

We next compute $F_l = \Phi F_{\Lambda_l}\Phi^{-1}$. Since

$$\begin{aligned}
(F_{\Lambda_l}f)_l(\omega) &= \pi^{-2}\int_X e^{i(y\mid\omega)} f_l(y)\, dy \\
&= \pi^{-2}\int_{\Omega_0}\int_{SU(2)/T} e^{i(v\eta v^{-1}\mid\omega)} \bar{\Lambda}_{ll}(v) f_l(\eta)(\eta_1-\eta_2)^2\, d\dot{v}\, d\eta.
\end{aligned}$$

Hence by (3.10)

$$\begin{aligned}
&(\Phi F_{\Lambda_l}f)(\omega) \\
&= \pi^{-2}\int_{\Omega_0} ((\omega_1-\omega_2)(\eta_1-\eta_2)\int_{SU(2)/T} e^{i(v\eta v^{-1}\mid\omega)}\bar{\Lambda}_{ll}(v)\, d\dot{v})(\Phi f)(\eta)\, d\eta,
\end{aligned}$$

so

(3.15) $$(F_l\phi)(\omega) = \pi^{-1}\int_{\Omega_0} J_l(\omega, \eta)\phi(\eta)\, d\eta,$$

where

$$J_l(\omega, \eta) = \pi^{-1}(\omega_1 - \omega_2)(\eta_1 - \eta_2)\int_{SU(2)/T} e_i^{(v\eta v^{-1}|\omega)}\bar{A}_{ll}(v)\, d\dot{v}.$$

Similarly, we obtain

$$(F_l^{-1}\phi)(\omega) = \pi^{-1}\int_{\Omega_0} -J_l(-\omega, \eta)\phi(\eta)\, d\eta = -(F_l\phi)(-\omega).$$

Let

$$v = \begin{bmatrix} \cos\theta_1 & (\sin\theta_1)e^{i\theta_2} \\ -(\sin\theta_1)e^{-i\theta_2} & \cos\theta_1 \end{bmatrix}.$$

Then by (2.8) $\bar{A}(v) = P_l(\cos 2\theta_1)$, where P_l is the Legendre polynomial of degree l. By (2.14) $d\dot{v} = \frac{1}{4}\sin 2\theta_1\, d\theta_1\, d\theta_2$.

$$J_l(\omega, \eta) = (4\pi)^{-1}\int_0^{\pi/2}\int_0^{2\pi}(\omega_1 - \omega_2)(\eta_1 - \eta_2)e^{i[(\eta|\omega)\cos^2\theta_1 + (\eta'|\omega)\sin^2\theta_1]} \cdot P_l(\cos 2\theta_1)\sin 2\theta_1\, d\theta_2\, d\theta_1$$

$$= \beta e^{i\alpha}\int_0^{\pi/2} e^{i\beta\cos 2\theta_1}P_l(\cos 2\theta_1)\sin 2\theta_1\, d\theta_1,$$

where $\alpha = \frac{1}{2}[(\eta|\omega) + (\eta'|\omega)]$, and $\beta = \frac{1}{2}[(\eta|\omega) - (\eta'|\omega)] = \frac{1}{2}(\omega_1 - \omega_2)(\eta_1 - \eta_2)$. This integral can be interpreted as the Fourier transform of a Legendre polynomial. By a classical result, this is a Bessel function. Specifically,

$$J_{n+1/2}(z) = (-i)^n\left(\frac{z}{2\pi}\right)^{1/2}\int_0^{\pi} e^{iz\cos\theta}P_n(\cos\theta)\sin\theta\, d\theta$$

(see [5, p. 50]). Thus

(3.16) $$J_l(\omega, \eta) = \tfrac{1}{2}i^l(2\beta\pi)^{1/2}e^{i\alpha}J_{l+1/2}(\beta).$$

It is is also known [5, p. 53, (1)] that

(3.17) $$J_{l+1/2}(\beta) = \frac{1}{\sqrt{2\pi\beta}}\left(e^{i\beta}\sum_{r=0}^{l}\frac{i^{r-l-1}(l+r)!}{r!(l-r)!(2\beta)^r} + e^{-i\beta}\sum_{r=0}^{l}\frac{(-i)^{r-l-1}(l+r)!}{r!(l-r)!(2\beta)^r}\right)$$

$$= (2\pi\beta)^{-1/2}\left(e^{i\beta}\sum_{r=0}^{l}\frac{i^{-l}C_r}{(2\beta)^r} + e^{-i\beta}\sum_{r=0}^{l}\frac{(-1)^{r-l-1}i^{-l}C_r}{(2\beta)^r}\right).$$

Since $\alpha+\beta=(\eta|\omega)$ and $\alpha-\beta=(\eta'|\omega)$, by (3.16) and (3.17) we have

$$J_l(\omega,\eta)=\frac{1}{2}\left(e^{i(\alpha+\beta)}\sum_{r=0}^{l}\frac{C_r}{(2\beta)^r}+e^{i(\alpha-\beta)}\sum_{r=0}^{l}\frac{(-1)^{r-l-1}C_r}{(2\beta)^r}\right)$$

$$=\frac{1}{2}\left(e^{i(\eta|\omega)}\sum_{r=0}^{l}\frac{C^r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\right.$$

$$\left.+e^{i(\eta'|\omega)}\sum_{r=0}^{l}\frac{(-1)^{r-l-1}C_r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\right),$$

$$(F_l\phi)(\omega)=(2\pi)^{-1}\int_{\Omega_0}e^{i(\eta|\omega)}\sum_{r=0}^{l}\frac{C_r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\phi(\eta)\,d\eta$$

$$+(2\pi)^{-1}\int_{\Omega_0}e^{i(\eta'|\omega)}\sum_{r=0}^{l}\frac{(-1)^{r-l-1}C_r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\phi(\eta)\,d\eta.$$

Recall that $\phi\in L_l^2(\Omega)$; hence $\phi(\eta)=(-1)^{l+1}(\eta')$. Therefore

$$\int_{\Omega_0}e^{i(\eta'|\omega)}\sum_{r=0}^{l}\frac{(-1)^{r-l-1}C_r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\phi(\eta)\,d\eta$$

$$=\int_{\Omega_0}e^{i(\eta'|\omega)}\sum_{r=0}^{l}\frac{(-1)^{r-l-1}C_r}{(\eta_2-\eta_1)^r(\omega_1-\omega_2)^r(-1)^r}(-1)^{l+1}\phi(\eta')\,d\eta$$

$$=\int_{\Omega_0'}e^{i(\eta|\omega)}\sum_{r=0}^{l}\frac{C_r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\phi(\eta)\,d\eta.$$

By change of variables $(\eta_1,\eta_2)\to(\eta_2,\eta_1)$; hence

$$(F_l\phi)(\omega)=(2\pi)^{-1}\int_{\Omega}e^{i(\eta|\omega)}\sum_{r=0}^{l}\frac{C_r}{(\eta_1-\eta_2)^r(\omega_1-\omega_2)^r}\phi(\eta)\,d\eta.$$

Thus the proposition is proved.

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