BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 4, Number 1, June 1976

GOLDIE CONDITIONS IN RINGS WITH INVOLUTION

BY

PJEK-HWEE LEE

Abstract. Let R be a ring with involution and $\bar{K}$ the subring generated by the skew elements. If R is a semiprime Goldie ring, then so is $\bar{K}$. Conversely, if R is *-prime and $\bar{K}$ is Goldie, R must be Goldie too.

This note is in fact a supplement to our recent work [2]. In [1] Lanski investigated the relation between Goldie conditions of a ring with involution and those of the subring generated by its symmetric elements. The analogous theorems for the subrings generated by "pairs" of skew elements or skew traces were proved in [2]. In the present paper, we prove the corresponding results for the subring generated by the skew elements.

Throughout the paper R will denote a ring with involution $*$. Also, $K=\{x\in R\mid x^*=-x\}$ is the set of skew elements of R, and $\bar{K}$ is the subring of R generated by K, and $Z(R)$ stands for the center of R. For $x, y\in R$ let $[x, y]=xy-yx$, and for subsets U and V of R let $[U, V]$ be the additive subgroup of R generated by all $[u, v]$ for $u\in U$ and $v\in V$.

We remind the reader of the fact that $\bar{K}^2$ contains the ideal of R generated by $[\bar{K}^2, \bar{K}^2]$, which enables us to prove in [2] many results concerning $\bar{K}^2$. For the subring $\bar{K}$ we have a weaker property which is the content of

LEMMA 1. *$\bar{K}$ contains the right ideal of R generated by $[\bar{K}, \bar{K}]$.*

Proof. For $k_1, k_2\in K$ and $x\in R$, $[k_1, k_2]x=[k_1k_2, x]+xk_1k_2-k_2k_1x=[k_1k_2, x]+(xk_1k_2-k_2k_1x^*)+k_2k_1(x^*-x)\in\bar{K}$ because $[K^2, R]\subseteq K^2$ [2, Lemma 27]. Hence $[K, K]R\subseteq\bar{K}$. Since $[K, K^n]\subseteq[K, K]K^{n-1}+K[K, K]^{n-2}+\cdots+K^{n-1}[K, K]$, $[K, K^n]R\subseteq\bar{K}$ and hence $[K, \bar{K}]R\subseteq\bar{K}$. Similarly, from $[K^m, \bar{K}]=[K, \bar{K}]K^{m-1}+K[K, \bar{K}]K^{m-2}$

Received by the editors December 14, 1975.

$+\cdots+K^{m-1}[K,\bar{K}]$ we get $[K^m,\bar{K}]R\subseteq\bar{K}$ and consequently $[\bar{K},\bar{K}]R\subseteq\bar{K}$.

LEMMA 2. *If R is semiprime and $c\in[\bar{K},\bar{K}]$ such that $c[\bar{K},\bar{K}]=0$, then $c=0$.*

Proof. $x\in\bar{K}$, we have $cxc=c[x,c]+c^2x=0$. That is, $c\bar{K}c=0$. Since R is semiprime, so is $\bar{K}$ [2, Theorem 49]. Hence $c=0$.

Now we prove the main theorem of this paper.

THEOREM 3. *If R is a semiprime Goldie ring, so is $\bar{K}$.*

Proof. Since the ascending chain condition for right annihilators is inherited by subrings, it suffices to show that $\bar{K}$ contains no infinite direct sum of nonzero right ideals. Suppose that $\{\rho_\alpha\}$ is an infinite collection of right ideals of $\bar{K}$ which forms a direct sum. Denote by ρ the right ideal of R generated by $[\bar{K},\bar{K}]$. Then $\{\rho_\alpha\rho\}$ is a collection of right ideals of R which also forms a direct sum. So $\rho_\alpha\rho=0$ for all but a finite number of α. Set $l(\rho)=\{x\in R\mid x\rho=0\}$. Then $\rho_\alpha\subseteq l(\rho)$ for almost all α. For $a\in\rho_\alpha$ and $b\in\bar{K}$, $ab-ba\in\rho_\alpha\bar{K}+\bar{K}l(\rho)\subseteq\rho_\alpha+l(\rho)\subseteq l(\rho)$. Thus $ab-ba\in[\bar{K},\bar{K}]\cap l(\rho)=0$ by the previous lemma. That means $\rho_\alpha\subseteq Z(\bar{K})$ for almost all α. Being a commutative semiprime subring of the Goldie ring R, $Z(\bar{K})$ is itself a Goldie ring. Therefore $\rho_\alpha=0$ for almost all α.

The converse to Theorem 3 is in general not true even if R is preassumed to be semiprime. For example, let F be a field of characteristic unequal to 2, and A an infinite-dimensional semiprime algebra with identity over F. The involution $*$ on the algebra $R=F\oplus A$ is defined by $(\alpha,a)^*=(\alpha^\sigma,a)$ where σ is an automorphism on F with $\sigma^2=1$ but $\sigma\neq1$. Then $\bar{K}=F$ is certainly a Goldie ring, while R has infinite direct sum of nonzero ideals.

However, the converse does hold in case R is *-prime; that is, if $I=I^*$ and $J=J^*$ are nonzero ideals of R, then $IJ\neq0$. As a matter of fact, it is just an easy consequence of the following two theorems.

THEOREM 4 [2, Theorem 43]. *If R is a semiprime Goldie ring, so is $\bar{K}^2$.*

THEOREM 5 [2, Theorem 63]. *If R is *-prime and if $\bar{K}^2$ is Goldie, then R is Goldie too.*

THEOREM 6. *If R is *-prime and if $\bar{K}$ is Goldie, then R is Goldie too.*

Proof. Since $\bar{K}$ is a semiprime Goldie ring, so is $\bar{K}^2$ by Theorem 5. Hence, R is also a Goldie ring by Theorem 6.

REFERENCES

1. C. Lanski, *Chain conditions in rings with involution,* J. London Math. Soc. **9** (1974), 93-102.

2. P. H. Lee, *On subrings of rings with involution,* Pacific J. Math. **60** (1975).

NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN