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LIMITING BEHAVIOR OF $\max_{j\le n} S_j j^{-\alpha}$ AND THE FIRST PASSAGE TIMES IN A RANDOM WALK WITH POSITIVE DRIFT

BY

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Abstract. Let $(X_n, n\geq 1)$ be a sequence of independent, identically distributed random variables with $EX_1 = 1$, $E(X_1-1)^2=\sigma^2<\infty$, and $S_n = X_1+\cdots+X_n$, and let $0\leq\alpha<1$. We prove that $n^{\alpha-1/2}(\max_{j\le n} S_j j^{-\alpha} - S_n n^{-\alpha})\to 0$ a.e. and $n^{\alpha-1/2}(S_n n^{-\alpha} - \inf_{j\ge n} S_j j^{-\alpha})\to 0$ a.e. From these results, we derive the law of iterated logarithm for $\max_{j\le n} S_j j^{-\alpha}$, $\inf_{j\ge n} S_j j^{-\alpha}$, T_c and N_c, where $T_c = \inf\{n\geq 1: S_n > cn^\alpha\}$ and $N_c=\inf\{n\geq 1: S_n\leq cn^\alpha\}$ for $c>0$.

1. **Introduction.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n, n\geq 1)$ be a sequence of independent random variables with $EX_n = 1$ for every n. Put $S_n = X_1+\cdots+X_n$. Since $EX_n = 1$, therefore $\max_{j\le n} S_j$ and $\inf_{j\ge n} S_j$ should be close to S_n in some sense. In §2, we will prove some limiting theorems about the differences $\max_{j\le n} S_j j^{-\alpha} - S_n n^{-\alpha}$ and $\inf_{j\ge n} S_j j^{-\alpha} - S_n n^{-\alpha}$ for $0\leq\alpha<1$ and apply them to obtain convergence in distribution theorems and law of iterated logarithm for $\max_{j\le n} S_j j^{-\alpha}$ and $\inf_{j\ge n} S_j j^{-\alpha}$. A central limit theorem for $\max_{j\le n} S_j j^{-\alpha}$ has been recently obtained by Teicher [10]. Our approach is different.

For $c>0$ and $0\leq\alpha<1$, define the first passage time $T_c = \inf\{n\geq 1: S_n > cn^\alpha\}$. A central limit theorem for T_c as $c\to\infty$ has been given by Siegmund [8]. His result follows easily from the results about $\max_{j\le n} S_j j^{-\alpha} - S_n n^{-\alpha}$.

In §3, based on the results of $\max_{j\le n} S_j j^{-\alpha}$, we obtain the law of iterated logarithm for T_c, which in case of $\alpha=0$ has been obtained by Vervaat [11]. If $N_c = \sup\{n\geq 1: S_n\leq cn^\alpha\}$ and $U_c = \sum_1^\infty I_{[S_n\leq cn^\alpha]}$, similar results hold for N_c and U_c also.

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2. **Limiting behavior of** $\max_{j\le n} S_j j^{-\alpha}$ **and** $\inf_{j\ge n} S_j j^{-\alpha}$. Let $(X_n, n\ge 1)$ be a sequence of independent random variables, $EX_n=1$ and $S_n = X_1+\cdots+X_n$. Since $EX_n>0$, therefore $\max_{j\le n} S_j - S_n$ and $S_n - \inf_{j\ge n} S_j$ should be small in comparison with S_n when n is large and their limiting behavior could be obtained from those of S_n.

THEOREM 2.1. *Let* $(X_n, n\ge 1)$ *be a sequence of independent, identically distributed random variables,* $EX_1=1$ *and let* $(b_n, n\ge 1)$ *be a sequence of positive numbers such that* $b_n\to\infty$ *as* $n\to\infty$. *Then as* $n\to\infty$,

(2.1) $$b_n^{-1}\left(\max_{j\le n} S_j - S_n\right) \xrightarrow{P} 0,$$

(2.2) $$b_n^{-1}\left(S_n - \inf_{j\ge n} S_j\right) \xrightarrow{P} 0.$$

Proof. For $\varepsilon>0$, by the i. i. d. property and the strong law of large numbers, as $n\to\infty$,

$$P\left[\max_{j\le n} S_j - S_n > b_n\varepsilon\right] = P\left[\max_{j\le n} -(S_n-S_j) > b_n\varepsilon\right]$$
$$= P\left[\max_{j\le n} -S_{n-j} > b_n\varepsilon\right] = P\left[\max_{j\le n-1} -S_j > b_n\varepsilon\right]\to 0,$$

yielding (2.1). Similarly for (2.2).

In the next theorem we shall assume only the independence (without common distribution) and the proof becomes slightly harder.

THEOREM 2.2. *Let* $(X_n, n\ge 1)$ *be a sequence of independent random variables with* $EX_n=1$, $\sup_{n\ge 1} E|X_n-1|^p\le C<\infty$ *for some* $1<p\le 2$, *and for some distribution function* F *let*

(2.3) $$n^{-1/p}(S_n-n)\xrightarrow{d} F.$$

Then for $0\le\alpha<1$ *and* $n\to\infty$,

(2.4) $$n^{-1/p}\left(\max_{1\le j\le n} S_j j^{-\alpha} - S_n n^{-\alpha}\right)\xrightarrow{P} 0,$$

(2.5) $$n^{-1/p}\left(S_n n^{-\alpha} - \inf_{j\ge n} S_j j^{-\alpha}\right)\xrightarrow{P} 0.$$

Proof. Put $Y_n = X_n - 1$, $W_n = Y_1+\cdots+Y_n$, and $W_{j,n}=W_{n+j}-W_j$. Then

(2.6) $$S_j j^{-\alpha} - S_n n^{-\alpha} = W_j j^{-\alpha} - W_n n^{-\alpha} - n^{1-\alpha} + j^{1-\alpha}.$$

Let $\varepsilon > 0$, $1 > \beta > 0$ and $\gamma = (1 - \beta^{1-\alpha})/2$. For $\beta n \geq j \geq 1$,

$$\begin{aligned} S_j j^{-\alpha} - S_n n^{-\alpha} &\leq W_j j^{-\alpha} - W_n n^{-\alpha} - 2\gamma n^{1-\alpha} \\ &\leq (W_j - \gamma j)/j^{\alpha} - (W_n + \gamma n)/n^{\alpha}. \end{aligned}$$

Since $\sup_{n\geq 1} E|X_n|^p < \infty$, by a strong law of large numbers of Loève [7, p. 241], $W_n/n \to 0$ a. e. as $n \to \infty$, and therefore $\max_{1\leq j\leq \beta n} (S_j j^{-\alpha} - S_n n^{-\alpha}) \to -\infty$ a. e.,

(2.7) $$P\Big[\max_{1\leq j\leq \beta n} (S_j j^{-\alpha} - S_n n^{-\alpha}) > \varepsilon n^{1/p-\alpha}\Big] \longrightarrow 0 \quad \text{as } n \to \infty.$$

For $\beta n < j \leq n$,

$$\begin{aligned} S_j j^{-\alpha} - S_n n^{-\alpha} &= W_n(j^{-\alpha} - n^{-\alpha}) - W_{j,n-j} j^{-\alpha} - n^{1-\alpha} + j^{1-\alpha} \\ &\leq W_n(j^{-\alpha} - n^{-\alpha}) + |W_{j,n-j}| j^{-\alpha}. \end{aligned}$$

Hence

(2.8) $$\begin{aligned} &P\Big[\max_{\beta n < j \leq n} (S_j j^{-\alpha} - S_n n^{-\alpha}) > 2\varepsilon n^{1/p-\alpha}\Big] \\ &\quad \leq P\Big[\max_{\beta n < j \leq n} W_n(j^{-\alpha} - n^{-\alpha}) > \varepsilon n^{1/p-\alpha}\Big] \\ &\qquad + P\Big[\max_{\beta n < j \leq n} |W_{j,n-j}| j^{-\alpha} > \varepsilon n^{1/p-\alpha}\Big] \\ &\quad \leq P[W_n > \varepsilon n^{1/p}/(\beta^{-\alpha} - 1)] \\ &\qquad + P\Big[\max_{\beta n < j \leq n} |W_{j,n-j}| > \varepsilon \beta^{\alpha} n^{1/p}\Big] \\ &\quad = \mathrm{I}_n + \mathrm{II}_n, \end{aligned}$$

say. By (2.3),

$$\mathrm{I}_n \xrightarrow{n\to\infty} 1 - F(\varepsilon/(\beta^{-\alpha} - 1)) \xrightarrow{\beta\to 1} 0,$$

and by Doob's inequality [2, p. 314] and the inequality of Marcinkiewicz and Zygmund [5 or 6], for some constant A,

$$\begin{aligned} \mathrm{II}_n &\leq (\varepsilon \beta^{\alpha})^{-p} n^{-1} E|W_{[\beta n]+1, n-[\beta n]-1}|^p \\ &\leq A(\varepsilon \beta^{\alpha})^{-p} n^{-1} E\Big(\sum_{j=[\beta n]+1}^{n} Y_j^2\Big)^{p/2} \\ &\leq A(\varepsilon \beta^{\alpha})^{-p} n^{-1} E \sum_{j=[\beta n]+1}^{n} |Y_j|^p \leq AC(\varepsilon \beta^{\alpha})^{-p} (1 - \beta) \longrightarrow 0 \end{aligned}$$

as $\beta \to 1$. Hence as $n \to \infty$,

(2.9) $$P\Big[\max_{\beta n < j \leq n} (S_j j^{-\alpha} - S_n n^{-\alpha}) > 2\varepsilon n^{1/p-\alpha}\Big] \longrightarrow 0,$$

yielding (2.4) by reason of (2.7).

The proof of (2.5) is similar.

If the random variables X_n in Theorem 2.2 are independent and identically distributed, the convergence in probability can be improved to a. e. convergence as follows:

THEOREM 2.3. *Let* $(X_n, n \geq 1)$ *be a sequence of independent, identically distributed random variables with* $EX_1 = 1$ *and* $E|X_1|^p < \infty$ *for some* $1 \leq p \leq 2$. *Then for* $0 \leq \alpha < 1$, *as* $n \to \infty$,

$$n^{\alpha - 1/p}\left(\max_{1 \leq j \leq n} S_j j^{-\alpha} - S_n n^{-\alpha}\right) \longrightarrow 0 \quad a.e., \tag{2.10}$$

$$n^{\alpha - 1/p}\left(S_n n^{-\alpha} - \inf_{j \geq n} S_j j^{-\alpha}\right) \longrightarrow 0 \quad a.e. \tag{2.11}$$

To prove Theorem 2.3, we need the following lemma of [1].

LEMMA 2.4. *Let* $(Y_n, n \geq 1)$ *be a sequence of independent, identically distributed random variables with* $EY_1 = 0$, $E|Y_1|^p < \infty$ *for some* $p \geq 1$, $W_n = Y_1 + \cdots + Y_n$ *and* $W_{j,n} = W_{j+n} - W_j$. *Then for* $1 \leq p < 2$, *as* $n \to \infty$,

$$n^{-1/p} \max_{1 \leq j \leq n} |W_{n,j}| \longrightarrow 0 \quad a.e., \tag{2.12}$$

$$n^{-1/p} \max_{1 \leq j \leq n-1} |W_{n-j,j}| \longrightarrow 0 \quad a.e., \tag{2.13}$$

and for $p \geq 2$ *and* $0 < \beta < \min(1, 2/p)$, *as* $n \to \infty$,

$$n^{-1/p} \max_{1 \leq j \leq n^\beta} |W_{n,j}| \longrightarrow 0 \quad a.e., \tag{2.14}$$

$$n^{-1/p} \max_{1 \leq j \leq n^\beta} |W_{n-j,j}| \longrightarrow 0 \quad a.e. \tag{2.15}$$

Proof. (2.12) and (2.14) have been established in [1] and their proofs hold for (2.13) and (2.15) also. Actually (2.13) follows easily by noting that, as $n \to \infty$,

$$\begin{aligned}\max_{1 \leq j \leq n-1} |W_{n-j,j}| &\leq \max_{1 \leq j \leq n-1} |W_{n-j}| + |W_n| \\ &= \max_{1 \leq j \leq n-1} |W_j| + |W_n| = o(n^{1/p}) \quad \text{a. s.}\end{aligned}$$

by the Marcinkiewicz-Zygmund strong law of large numbers [5 or 7, p. 242] and therefore (2.15) is a simple consequence of (2.14) as follows: if $m = [n^\beta] + 1$, then

$$W_{n-j,j} = W_{n-m,m} - W_{n-m,m-j},$$

$$\max_{1\le j\le m} |W_{n-j,j}| \le |W_{n-m,m}| + \max_{1\le j\le m} |W_{n-m,m-j}| = o(n^{1/p}) \quad \text{a.e.}$$

by (2.13) for $\beta<\alpha<\min(1, 2/p)$ and $1\le j\le n^{\beta}$, $m+j\le 2m\le (n-m)^{\alpha}$ for all large n.

Proof of Theorem 2.3. We shall prove (2.11) only and omit the proof of (2.10), which is similar. Put $Y_n = X_n - 1$, $W_n = Y_1 + \cdots + Y_n$ and $W_{j,n} = W_{n+j} - W_j$. Then

$$S_n n^{-\alpha} - S_j j^{-\alpha} = W_n n^{-\alpha} - W_j j^{-\alpha} - j^{1-\alpha} + n^{1-\alpha}. \tag{2.16}$$

For $1<p\le 2$, let $1>\beta>\max(\alpha, 1/p)$ and for $p=1$, let $\beta=1$. Put $m=[n^{\beta}]$. Then for $j\ge n+m$ and n large,

$$\begin{aligned} j^{1-\alpha} - n^{1-\alpha} &= j^{1-\alpha}\left\{1 - \left(1 - \frac{j-n}{j}\right)^{1-\alpha}\right\} \\ &\ge (1-\alpha)\, j^{1-\alpha} \frac{m}{n+m} \ge \frac{1-\alpha}{2} j^{1-\alpha} n^{\beta-1} \ge \frac{1-\alpha}{2} j^{\beta-\alpha}. \end{aligned}$$

Hence for n large,

$$\sup_{j\ge n+m} (S_n n^{-\alpha} - S_j j^{-\alpha}) \le |W_n|\, n^{-\alpha} + \left(|W_j|\, j^{-\beta} - \frac{1-\alpha}{2}\right) j^{\beta-\alpha}.$$

By the Marcinkiewicz-Zygmund strong law of large numbers [5 or 7, p. 242], $W_n n^{-1/p} \to 0$ a.e. and therefore

$$\lim_{n\to\infty} \sup_{j\ge n+m} (S_n n^{-\alpha} - S_j j^{-\alpha})^{+} = 0 \quad \text{a.e.} \tag{2.17}$$

For $n\le j\le n+m$ and all large n,

$$\begin{aligned} n^{-\alpha} - j^{-\alpha} &\le N^{-\alpha}\left\{1 - \left(1 + \frac{m}{n}\right)^{-\alpha}\right\} \\ &\le 2\alpha\, n^{-\alpha+\beta-1} && \text{if } 1<p\le 2, \\ &\le n^{-\alpha}(1-2^{-\alpha}) \le 2\alpha\, n^{-\alpha+\beta-1} && \text{if } p=1. \end{aligned}$$

Choose $r = 1/p + 1 - \beta$. Then for $n\le j\le n+m$,

$$S_n n^{-\alpha} - S_j j^{-\alpha} \le -W_{n,j-n} n^{-\alpha} + W_j(n^{-\alpha} - j^{-\alpha}). \tag{2.18}$$

Since

$$\begin{aligned} \max_{n\le j<n+m} |W_j|\,(n^{-\alpha} - j^{-\alpha}) &\le 2\alpha\, n^{-\alpha+\beta-1} \max_{n\le j<n+m} |W_j| \\ &\le 2^{1+r} \alpha\, n^{1/p-\alpha} \max_{n\le j<n+m} |W_j j^{-r}| = o(n^{1/p-\alpha}) \quad \text{a.e.} \end{aligned} \tag{2.19}$$

by the Marcinkiewicz-Zygmund strong law of large numbers [5], and

$$\begin{aligned}(2.20)\qquad &\max_{n\le j<n+m} |W_{n,j-n}|\, n^{-\alpha} \\ &= n^{1/p-\alpha}\Big(n^{-1/p}\max_{n\le j<n+m}|W_{n,j-n}|\Big) = o(n^{1/p-\alpha}) \quad \text{a.e.}\end{aligned}$$

by (2.12) and (2.14).

From (2.18)–(2.20), as $n\to\infty$,

$$(2.21)\qquad \max_{n\le j<n+m} n^{\alpha-1/p}(S_n n^{-\alpha} - S_j j^{-\alpha}) \longrightarrow 0 \quad \text{a.e.},$$

yielding (2.11) by noting (2.17).

From Theorems 2.1–2.3, we can easily obtain the limiting theorems for $\max_{j\le n} S_j j^{-\alpha}$ and $\inf_{j\ge n} S_j j^{-\alpha}$ from the corresponding theorems for $S_n n^{-\alpha}$. To clarify the point, we mention the following:

COROLLARY 2.5. *Let* $(X_n, n\ge 1)$ *be a sequence of independent random variables with* $EX_n = 1$ *and* $\sup_{n\ge1} E|X_n|^p < \infty$ *for some* $1<p\le 2$.

(i) *If for some distribution function* F

$$n^{-1/p}(S_n - n) \xrightarrow{\text{d}} F,$$

then for $0\le\alpha<1$, *as* $n\to\infty$,

$$(2.22)\qquad n^{\alpha-1/p}\Big(\max_{1\le j\le n} S_j j^{-\alpha} - n^{1-\alpha}\Big) \xrightarrow{\text{d}} F,$$

$$(2.23)\qquad n^{\alpha-1/p}\Big(\inf_{n\le j} S_j j^{-\alpha} - n^{1-\alpha}\Big) \xrightarrow{\text{d}} F.$$

(ii) *If the random variables* X_n *have the same distribution and* $E(X_1-1)^2 = \sigma^2 < \infty$, *then for* $0\le\alpha<1$, *as* $n\to\infty$,

$$(2.24)\qquad \overline{\underline{\lim}}\Big(\max_{1\le j\le n} S_j j^{-\alpha} - n^{1-\alpha}\Big)\Big/(2n^{1-2\alpha}\log\log n)^{1/2} = \pm\sigma \quad \textit{a.e.},$$

$$(2.25)\qquad \overline{\underline{\lim}}\Big(\inf_{j\ge n} S_j j^{-\alpha} - n^{1-\alpha}\Big)\Big/(2n^{1-2\alpha}\log\log n)^{1/2} = \pm\sigma \quad \textit{a.e.},$$

and with probability one, the sets of all limit points of the sequences $(\max_{1\le j\le n} S_j j^{-\alpha} - n^{1-\alpha})(2n^{1-2\alpha}\log\log n)^{-1/2}$ *and* $(\inf_{j\ge n} S_j j^{-\alpha} - n^{1-\alpha})\cdot(2n^{1-2\alpha}\log\log n)^{-1/2}$ *coincide with the interval* $[-\sigma, \sigma]$.

Proof. (i) follows immediately from Theorem 2.2 and (ii) follows from Theorem 2.3 and the iterated logarithm theorems of Hartman-Wintner [4] and Strassen [9].

REMARK. When $p=2$ and F is the distribution of an $N(0, \sigma)$ random variable, (2.22) has been established by Teicher [10] by using the method of stopping times.

3. **Limiting behavior of first passage times.** Let $(X_n, n\geq 1)$ be a sequence of independent random variables with $EX_n = 1$ and $S_n = X_1 + \cdots + X_n$. For $c>0$ and $0\leq\alpha<1$, define $T_c = \inf\{n\geq 1: S_n > cn^{\alpha}\}$, $N_c = \sup\{n\geq 1 : S_n \leq cn^{\alpha}\}$, and $U_c = \sum_1^{\infty} I_{[S_n\leq cn^{\alpha}]}$. In 1968 Siegmund [8] proved that if $E(X_n-1)^2=\sigma^2<\infty$ for each $n\geq 1$ and $(S_n, n\geq 1)$ obeys the central limit theorem, then T_c obeys the central limit theorem, and in 1972 Vervaat [11] proved that if $\alpha=0$, $E(X_1-1)^2<\infty$ and the random variables are identically distributed, then T_c obeys the law of iterated logarithm. The results of §2 yield central limit theorems and iterated logarithm theorems for T_c, N_c and U_c as $c\to\infty$ (and therefore new proofs of the theorems of Siegmund and Vervaat).

THEOREM 3.1. *Let $(X_n, n\geq 1)$ be a sequence of independent random variables with $EX_n=1$, $E(X_n-1)^2=\sigma^2<\infty$ and*

$$n^{-1/2}(S_n - n) \xrightarrow{d} N(0, \sigma)\,. \tag{3.1}$$

Then as $c\to\infty$,

$$(1-\alpha)\, c^{-1/2(1-\alpha)}(T_c - c^{1/(1-\alpha)}) \xrightarrow{d} N(0, \sigma)\,, \tag{3.2}$$

$$(1-\alpha)\, c^{-1/2(1-\alpha)}(N_c - c^{1/(1-\alpha)}) \xrightarrow{d} N(0, \sigma)\,, \tag{3.3}$$

$$(1-\alpha)\, c^{-1/2(1-\alpha)}(U_c - c^{1/(1-\alpha)}) \xrightarrow{d} N(0, \sigma)\,. \tag{3.4}$$

Proof. (3.2) is due to Siegmund [8] and can be proved in the same manner as (3.3). To prove (3.3), for $c>0$ and $-\infty<x<\infty$, let $n = n_x = c^{1/(1-\alpha)} + \sigma x(1-\alpha)^{-1}\, c^{1/2(1-\alpha)}$. Then as $c\to\infty$, $n \sim c^{1/(1-\alpha)}(c-n^{1-\alpha})/(\sigma n^{1/2-\alpha}) \sim -x$ and

$$\begin{aligned}
&P[(1-\alpha)\, c^{1/2(1-\alpha)}(N_c - c^{1/(1-\alpha)}) \geq \sigma x] \\
&\quad = P[N_c \geq n] = P\Big[\inf_{j\geq n} S_j\, j^{-\alpha} \leq c\Big] \\
&\quad = P\Big[\Big(\inf_{j\geq n} S_j\, j^{-\alpha} - n^{1-\alpha}\Big)\Big/(\sigma\, n^{1/2-\alpha}) \leq (c - n^{1-\alpha})/(\sigma\, n^{1/2-\alpha})\Big] \\
&\quad = P\Big[\Big(\inf_{j\geq n} S_j\, j^{-\alpha} - n^{1-\alpha}\Big)\Big/(\sigma\, n^{1/2-\alpha}) \leq -x(1+o(1))\Big] \longrightarrow \Phi(-x) \\
&\quad = 1 - \Phi(x)\,,
\end{aligned}$$

by Corollary 2.5 (i), where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-t^2/2}\,dt$.

Since $T_c - 1 \leq U_c \leq N_c$, (3.2) and (3.3) imply (3.4).

For the law of iterated logarithm for T_c, N_c and U_c, we need the following lemma, part (i) of which is due to Gut [3].

LEMMA 3.2. *Let $(X_n, n \geq 1)$ be a sequence of independent, identically distributed random variables with $EX_1 = 1$. Then as $c \to \infty$,*

(3.5) (i) $c^{-1/(1-\alpha)}\, T_c \longrightarrow 1$ *a.e.,* $c^{-1/(1-\alpha)}\, N_c \longrightarrow 1$ *a.e.*

(ii) *If $EX_1^2 < \infty$, then as $c \to \infty$,*

(3.6) $T_c^{-1/2}(S_{T_c} - cT_c^{\alpha}) \longrightarrow 0$ *a.e.,*

(3.7) $N_c^{-1/2}(cN_c^{\alpha} - S_{N_c}) \longrightarrow 0$ *a.e.*

Proof. The proof of (i) can be found in Theorem 3.3 of [3]. For (3.6), from the definition

$$0 \leq (S_{T_c} - cT_c^{\alpha})\, T_c^{-1/2} = (S_{T_c} - 1 + X_{T_c} - cT_c^{\alpha})\, T_c^{-1/2}$$
$$\leq X_{T_c}\, T_c^{-1/2} \longrightarrow 0 \quad \text{a.e.,}$$

since $EX_1^2 < \infty$. Similarly for (3.7).

THEOREM 3.3. *Let $(X_n, n \geq 1)$ be a sequence of independent, identically distributed random variables with $EX_1 = 1$ and $E(X_1 - 1)^2 = \sigma^2 < \infty$. Then as $c \to \infty$,*

(3.8) $\overline{\underline{\lim}}\, (1-\alpha)\,(T_c - c^{1/(1-\alpha)})/(2c^{1/(1-\alpha)} \log\log c)^{1/2} = \pm\,\sigma$ *a.e.,*

(3.9) $\overline{\underline{\lim}}\, (1-\alpha)\,(N_c - c^{1/(1-\alpha)})/(2c^{1/(1-\alpha)} \log\log c)^{1/2} = \pm\,\sigma$ *a.e.,*

(3.10) $\overline{\underline{\lim}}\, (1-\alpha)\,(U_c - c^{1/(1-\alpha)})/(2c^{1/(1-\alpha)} \log\log c)^{1/2} = \pm\,\sigma$ *a.e.*

Proof. By the Hartman-Wintner law of iterated logarithm [4], as $n \to \infty$,

$$\overline{\underline{\lim}}\, \frac{n - S_n}{(2n \log\log n)^{1/2}} = \pm\,\sigma \quad \text{a.e.}$$

Since $T_c \to \infty$ a.e. as $c \to \infty$

$$\overline{\lim}\, \frac{T_c - S_{T_c}}{(2T_c \log\log T_c)^{1/2}} \leq \sigma \quad \text{a.e.}$$

and by the definition of T_c and (3.6)

(3.11) $\overline{\lim}\, (T_c - cT_c^{\alpha})/(2T_c \log\log T_c)^{1/2} \leq \sigma$ a.e.

As $c\to\infty$, by (3.5)

$$
\begin{aligned}
T_c - cT_c^{\alpha} &= T_c\{1-(1+c^{1/(\alpha-1)}T_c-1)^{\alpha-1}\} \\
&= T_c(1-\alpha)(c^{1/(\alpha-1)}T_c-1)(1+o(1)) \qquad (3.12)\\
&= (1-\alpha)(T_c-c^{1/(1-\alpha)})(1+o(1)) \quad \text{a.e.}
\end{aligned}
$$

From (3.11) and (3.12),

$$(3.13)\qquad \overline{\lim}\,(T_c-c^{1/(1-\alpha)})/(2c^{1/(1-\alpha)}\log\log c)^{1/2}\le \sigma/(1-\alpha) \quad \text{a.e.}$$

For $0<\xi<1$, put $n=[c^{1/(1-\alpha)}+\xi\sigma(2c^{1/(1-\alpha)}\log\log c)^{1/2}/(1-\alpha)]$. Then as $c\to\infty$,

$$c^{1/(1-\alpha)} = n-\xi\sigma(2n\log\log n)^{1/2}(1-\alpha)^{-1}(1+o(1)),$$

$$
\begin{aligned}
C &= n^{1-\alpha}\{1-\xi\sigma(2\log\log n)^{1/2}n^{-1/2}(1-\alpha)^{-1}(1+o(1))\}^{1-\alpha}\\
&= n^{1-\alpha}\{1-\xi\sigma(2\log\log n)^{1/2}n^{-1/2}(1+o(1))\}\\
&= n^{1-\alpha}-\xi\sigma(2n^{1-2\alpha}\log\log n)^{1/2}(1+o(1)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&P[T_c-c^{1/(1-\alpha)}>(1-\alpha)^{-1}\xi\sigma(2c^{1/(1-\alpha)}\log\log c)^{1/2}\ \text{i.o.}]\\
&\quad = P[T_c>n\ \text{i.o.}] = P\Big[\max_{1\le j\le n} S_j j^{-\alpha}\le c\ \text{i.o.}\Big]\\
&\quad = P\Big[\max_{1\le j\le n} S_j j^{-\alpha} \qquad (3.14)\\
&\qquad\qquad \le n^{1-\alpha}-\xi\sigma(2n^{1-2\alpha}\log\log n)^{1/2}(1+o(1))\ \text{i.o.}\Big]\\
&\quad = 1
\end{aligned}
$$

by (2.24), and (3.8) follows immediately from (3.13) and (3.14).

Similarly we can prove (3.9). Since $T_c-1\le U_c\le N_c$, (3.10) follows from (3.8) and (3.9) at once.

Theorem 3.3 yields a rather suprising result:

COROLLARY 3.4. *Under the same hypothesis as Theorem* 3.3, *as* $c\to\infty$,

$$(3.15)\qquad \overline{\underline{\lim}}\,(S_{T_c}-T_c)/(2T_c\log\log T_c)^{1/2}=\pm\sigma \quad a.e.,$$

$$(3.16)\qquad \overline{\underline{\lim}}\,(S_{N_c}-N_c)/(2N_c\log\log N_c)^{1/2}=\pm\sigma \quad a.e.$$

Proof. By (3.6) and (3.12), with probability one

$$
\begin{aligned}
S_T-T_c &= S_{T_c}-cT_c^{\alpha}+cT_c^{\alpha}-T_c\\
&= o(T_c^{1/2})+(1-\alpha)(T_c-c^{1/(1-\alpha)})(1+o(1)).
\end{aligned}
$$

Hence by (3.5) with probability one as $c\to\infty$,

$$\overline{\lim}\, (S_{T_c} - T_c)/(2T_c \log\log T_c)^{1/2}$$
$$= \overline{\lim}\, (1-\alpha)(T_c - c^{1/(1-\alpha)})/(2c^{1/(1-\alpha)} \log\log c)^{1/2} = \pm\,\sigma,$$

yielding (3.15). Similarly for (3.16).

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