BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 33, Number 4, December 2005

STRONG LAWS OF LARGE NUMBERS FOR SEQUENCES OF BLOCKWISE AND PAIRWISE m-DEPENDENT RANDOM VARIABLES

BY

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Abstract. In this paper, we introduce the notions of pairwise m-dependence and blockwise and pairwise m-dependence of random variables $\{X_n, n \geq 1\}$. For a sequence of blockwise and pairwise m-dependent random variables $\{X_n, n \geq 1\}$, we provide conditions for $\frac{\sum_{j=1}^{n}(X_j - EX_j)}{n^{1/r}} \to 0$ a.s. as $n \to \infty$ $(1 \leq r < 2)$. We also establish the strong law of large numbers for sequences of pairwise m-dependent random variables.

1. Introduction. Etemadi (1981) extended the classical strong law of large numbers for independent identically distributed random variables to the case where the random variables are pairwise independent identically distributed, i.e., if $\{X_n, n \geq 1\}$ is a sequence of pairwise independent identically distributed random variables with $E|X_1| < \infty$, then

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} X_j}{n} = EX_1 \quad \text{a.s.} \tag{1}$$

Later, Choi and Sung (1985) proved that if $\{X_n, n \geq 1\}$ are pairwise independent and are stochastically dominated by a random variable X with

Received by the editors January 17, 2005.

AMS 2000 Subject Classification: 60F15.

Key words and phrases: Pairwise m-dependence, blockwise and pairwise m-dependence, strong law of large numbers, almost sure convergence, stochastically dominated.

$E(|X|^r(\log^+|X|)^2) < \infty$ $(1 < r < 2)$, then

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n}(X_j - EX_j)}{n^{\frac{1}{r}}} = 0 \quad \text{a.s.} \tag{2}$$

Móricz (1987) introduced the concept blockwise m-dependence of a sequence of random variables and extended the classical Kolmogorov strong law of large numbers to the blockwise m-dependence case. Namely, he proved that if $\{X_n, n \geq 1\}$ is a sequence of random variables with $EX_n = 0$, $EX_n^2 < \infty$ for all n and $\{X_n, n \geq 1\}$ is m-dependent for large k, i.e., $\{X_n, 2^k \leq n \leq i\}$ and $\{X_n, j \leq n < 2^{k+1}\}$ are independent whenever $j - i > m$, then the condition

$$\sum_{j=1}^{\infty} \frac{EX_j^2}{j^2} < \infty \tag{3}$$

implies

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} X_j}{n} = 0 \quad \text{a.s.} \tag{4}$$

In the current work, we extend the result of Choi and Sung (1985) to the blockwise and pairwise m-dependence case and the result of Etemadi (1981) to the pairwise m-dependence case.

Throughout this paper, the symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance, and logarithms are to the base 2. For $x > 0$, the symbol $[x]$ denotes the greatest integer in x.

2. Preliminaries. In this section, notation, technical definitions, and lemmas needed in connection with the main results will be presented.

Let m be a fixed nonnegative integer. We say that a collection $\{X_j, 1 \leq j \leq n\}$ of n random variables is *pairwise m-dependent* if either $n < m + 1$ or $n \geq m + 1$ and X_i and X_j are independent whenever $j - i > m$. A sequence

$\{X_n, n \geq 1\}$ of random variables is said to be *pairwise m-dependent* if X_i and X_j are independent whenever $j - i > m$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *blockwise and pairwise m-dependent* if for each p, the collection $\{X_j, 2^{p-1} < j \leq 2^p\}$ is pairwise m-dependent.

Random variables $\{X_n, n \geq 1\}$ are said to be *stochastically dominated* by a random variable X if for some constant $D < \infty$,

$$P\{|X_n| > t\} \leq DP\{|DX| > t\}, \quad t \geq 0, \quad n \geq 1. \tag{5}$$

This condition is, of course, automatic with $X = X_1$ and $D = 1$ if the $\{X_n, n \geq 1\}$ are identically distributed. It follows from Lemma 5.2.2 of Taylor ((1978), p.123) or Lemma 3 of Wei and Taylor (1978) that stochastic dominance can be accomplished by the sequence of random variables having a bounded absolute r^{th} moment ($r > 0$). Specifically, if $\sup_{n \geq 1} E|X_n|^r < \infty$ for some $r > 0$, then there exists a random variable X with $E|X|^p < \infty$ for all $0 < p < r$ such that (5) holds with $D = 1$. The proviso that $r > 1$ in Lemma 5.2.2 of Taylor ((1978), p.123) or Lemma 3 of Wei and Taylor (1978) is not needed as was pointed out by Adler, Rosalsky, and Taylor (1992).

Lemma 1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables. If*

$$\sum_{n=1}^{\infty} E|X_n|^p < \infty \quad \textit{for some} \quad p > 0, \tag{6}$$

then

$$X_n \to 0 \quad \text{a.s. as} \quad n \to \infty. \tag{7}$$

Proof. For arbitrary $\epsilon > 0$ and all $k \geq 1$,

$$\begin{aligned} P\{\sup_{n \geq k} |X_n| > \epsilon\} &\leq \sum_{n \geq k} P\{|X_n| > \epsilon\} \\ &\leq \frac{1}{\epsilon^p} \sum_{n \geq k} E|X_n|^p \text{ (by Markov's inequality)} \end{aligned}$$

$$\to \quad 0 \quad \text{as } k \to \infty \text{ (by (6))}$$

thereby proving (7).

Lemma 2. If $\{X_j, 1 \le j \le n\}$ is a collection of pairwise m-dependent mean 0 random variables, then

$$E(\sum_{j=1}^{n} X_j)^2 \le (m+1) \sum_{j=1}^{n} EX_j^2. \tag{8}$$

Proof. If $n \le m+1$, then (8) is trivial. Suppose that $n > m+1$. Then

$$\begin{aligned}
& E(\sum_{j=1}^{n} X_j)^2 \\
= \; & E\Big(\sum_{k=1}^{m+1} \sum_{i\ge 0, i(m+1)\le n-k} X_{i(m+1)+k}\Big)^2 \\
\le \; & (m+1) \sum_{k=1}^{m+1} E(\sum_{i\ge 0, i(m+1)\le n-k} X_{i(m+1)+k})^2 \text{ (by Hölder's inequality)} \qquad (9) \\
= \; & (m+1) \sum_{k=1}^{m+1} \sum_{i\ge 0, i(m+1)\le n-k} EX_{i(m+1)+k}^2 \\
= \; & (m+1) \sum_{j=1}^{n} EX_j^2.
\end{aligned}$$

The following lemma can be obtained by using a method similar to that used in the proof of Lemma 2 and the Rademacher-Menshov inequality.

Lemma 3. *If $\{X_j, 1 \le j \le n\}$ is a collection of pairwise m-dependent mean* 0 *random variables, then*

$$E(\max_{k\le n} |\sum_{j=1}^{k} X_j|)^2 \le (m+1)(\log 2n)^2 \sum_{j=1}^{n} EX_j^2. \tag{10}$$

3. Main Results. With the preliminaries accounted for, the main results may be established. Theorem 1 establishes Marcinkiewicz-Zygmund type strong law of large numbers for sequences of blockwise and pairwise m-dependent random variables. This theorem in the pairwise independence case with $1 < r < 2$ is the result of Choi and Sung (1985).

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise and pairwise m-dependent random variables. Suppose that $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X. If*

$$E(|X|^r(\log^+|X|)^2) < \infty \quad (1 \leq r < 2), \tag{11}$$

then

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n}(X_j - EX_j)}{n^{\frac{1}{r}}} = 0 \qquad a.s. \tag{12}$$

Proof. Set

$$Y_n = X_n I(|X_n| \leq n^{\frac{1}{r}}), \quad n \geq 1$$

and

$$\gamma_p = \frac{1}{2^{\frac{p}{r}}} \max_{k \leq 2^p} \Big| \sum_{j=2^{p-1}+1}^{k} (Y_j - EY_j) \Big|, \ p \geq 1.$$

Note at the outset that,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{\frac{2}{r}}} EY_n^2 &\leq \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{\frac{2}{r}}} \int_0^{n^{\frac{1}{r}}} 2xP\{|X_n| > x\}dx \\ &\leq 2D \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{\frac{2}{r}}} \sum_{j=1}^{n} \int_{(j-1)^{\frac{1}{r}}}^{j^{\frac{1}{r}}} xP\{|DX| > x\}dx \\ &= C \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{\log^2 n}{n^{\frac{2}{r}}} \int_{(j-1)^{\frac{1}{r}}}^{j^{\frac{1}{r}}} xP\{|DX| > x\}dx \qquad (13) \end{aligned}$$

$$\begin{aligned}
&\le C\sum_{j=1}^{\infty}\frac{\log^2 j}{j^{\frac{2}{r}-1}}\int_{(j-1)^{\frac{1}{r}}}^{j^{\frac{1}{r}}} xP\{|DX|>x\}dx\\
&\qquad (\text{since } \textstyle\sum_{n=j}^{\infty}\frac{\log^2 n}{n^{\frac{2}{r}}}=O(\frac{\log^2 j}{j^{\frac{2}{r}-1}}))\\
&\le C\sum_{j=1}^{\infty}\log^2 jP\{|DX|^r>j\}\\
&\le CE(|X|^r(\log^+|X|)^2)<\infty.
\end{aligned}$$

It follows from Lemma 3 that for all $p\ge 1$,

$$\begin{aligned}
E\gamma_p^2 &\le (m+1)\frac{\log^2 2^p}{2^{\frac{2p}{r}}}\sum_{j=2^{p-1}+1}^{2^p}E(Y_j-EY_j)^2\\
&\le C\sum_{j=2^{p-1}+1}^{2^p}\frac{EY_j^2}{j^{\frac{2}{r}}}\log^2 j,
\end{aligned}$$

whence $\sum_{p=1}^{\infty}E\gamma_p^2<\infty$ by (13). Then by Lemma 1,

$$\gamma_p\to 0 \text{ a.s. as } p\to\infty. \tag{14}$$

By the Toeplitz lemma (see, e.g., Loève (1977), p.250),

$$2^{-\frac{n}{r}}\sum_{j=1}^{n}2^{\frac{j}{r}}\gamma_j\to 0 \text{ a.s. as } n\to\infty. \tag{15}$$

Note that for $2^{p-1}<n\le 2^p$,

$$\Big|\frac{\sum_{j=1}^{n}(Y_j-EY_j)}{n^{\frac{1}{r}}}\Big|\le\Big|2^{\frac{-p+1}{r}}(Y_1-EY_1)\Big|+2\Big|2^{\frac{-p}{r}}\sum_{j=1}^{p}2^{\frac{j}{r}}\gamma_j\Big|$$

and so (15) ensures that

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}(Y_j-EY_j)}{n^{\frac{1}{r}}}=0 \text{ a.s.} \tag{16}$$

Next,

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_n \neq Y_n) &= \sum_{n=1}^{\infty} P(|X_n| > n^{\frac{1}{r}}) \\ &\leq D\sum_{n=1}^{\infty} P(|DX| > n^{\frac{1}{r}}) \\ &\leq CE|X|^r < \infty \text{ (by (11)).}\end{aligned} \tag{17}$$

By (16), (17), and the Borel-Cantelli lemma we get

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n}(X_j - EY_j)}{n^{\frac{1}{r}}} = 0 \text{ a.s.} \tag{18}$$

Finally,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-\frac{1}{r}} E(X_n - Y_n) &\leq \sum_{n=1}^{\infty} n^{-\frac{1}{r}} \int_{n^{\frac{1}{r}}}^{\infty} P\{|X_n| > x\}dx \\ &\leq D\sum_{n=1}^{\infty} n^{-\frac{1}{r}} \sum_{i=n}^{\infty} \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} P\{|DX| > x\}dx \\ &= C\sum_{i=1}^{\infty}\sum_{n=1}^{i} n^{-\frac{1}{r}} \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} P\{|DX| > x\}dx \\ &\leq C\sum_{i=1}^{\infty} i^{\frac{r-1}{r}} \log i \int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} P\{|DX| > x\}dx \\ &\leq C\sum_{i=1}^{\infty} \log i P\{|DX|^r > i\} \\ &\leq CE(|X|^r \log^+ |X|) < \infty \text{ (by (11)).}\end{aligned}$$

Then by the Kronecker lemma,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} E(X_j - Y_j)}{n^{\frac{1}{r}}} = 0. \tag{19}$$

The conclusion (12) follows from (18) and (19).

When the $\{X_n, n \geq 1\}$ is a sequence of pairwise m-dependent identically distributed random variables and $r = 1$, we can obtain (12) under the weaker condition that $E|X_1| < \infty$. The following theorem reduces to a result of Etemadi (1981) when the $\{X_n, n \geq 1\}$ are pairwise independent.

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise m-dependent identically distributed random variables. If $E|X_1| < \infty$, then*

$$\frac{\sum_{j=1}^{n} X_j}{n} \to EX_1 \ a.s. \ as \ n \to \infty. \tag{20}$$

Proof. Since $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ satisfy the hypotheses of the theorem and $X_j = X_j^+ - X_j^-$, without loss of generality we can assume that $X_j \geq 0$. Set

$$\begin{aligned} Y_j &= X_j I(X_j \leq j), \ j \geq 1, \\ S_n^* &= \sum_{j=1}^{n} Y_j, \ n \geq 1. \end{aligned}$$

We get the following conclusion which is given by Etemadi (1981) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} EY_n^2 \leq CEX_1 < \infty. \tag{21}$$

For $\alpha > 1$ and for all $p \geq 1$, set

$$\sigma_p = \frac{1}{[\alpha^p]} \sum_{j \in I_p} (Y_j - EY_j),$$

where $I_p = \{[\alpha^p]\}$ if $[\alpha^p] = [\alpha^{p-1}]$ and $I_p = ([\alpha^{p-1}], [\alpha^p]]$ if $[\alpha^p] > [\alpha^{p-1}]$. It follows from Lemma 2 that for all $p \geq 1$,

$$\begin{aligned} E\sigma_p^2 &\leq (m+1) \frac{1}{[\alpha^p]^2} \sum_{j \in I_p} E(Y_j - EY_j)^2 \\ &\leq 2(m+1) \sum_{j \in I_p} \frac{EY_j^2}{j^2} \end{aligned}$$

whence $\sum_{p=1}^{\infty} E\sigma_p^2 < \infty$ by (21). By using a method similar to that used in the proof of (18) in Theorem 1, we obtain

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{k_n} X_j - ES_{k_n}^*}{k_n} = 0 \text{ a.s.,} \tag{22}$$

where $k_n = [\alpha^n]$. The rest of the argument follows as in Theorem 1 of Etemadi (1981).

Acknowledgments.

The author is grateful to Professor Andrew Rosalsky (University of Florida, USA) for many helpful and important remarks and for the references [1] and [4]. The author also wishes to thank Professor Ferenc Móricz (University of Szeged, Hungary) for the reference [6].

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