BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 33, Number 4, December 2005

PRODUCTS OF RANDOM VARIABLES WITH LOGISTIC KERNEL

BY

SARALEES NADARAJAH AND ARJUN K. GUPTA

Abstract. The distribution of the product XY of random variables X and Y is of interest in many applied problems. The existing literature deals with cases when X and Y arise from the same family of distributions. In this paper, we study the distribution of XY when X and Y arise from different but similar distributions by deriving various explicit expressions for its characteristic and distribution functions. We take X to come from the logistic family and Y to be from one of normal, t, Laplace, logistic or Bessel families.

1. Introduction. For given random variables X and Y, the distribution of the product XY is of interest in problems in biological and physical sciences, econometrics, and classification. As an example in Physics, Sornette (1998) mentions:

> "... To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold $x_c \ldots$ and found a stretched exponential truncating the power-law pdf beyond x_c. Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables ..."

Received by the editors November 8, 2004 and in revised form January 19, 2005.

The distribution of XY has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, Abu-Salih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). There is relatively little work of the above kind when X and Y belong to different families. In the applications mentioned above, it is quite possible that X and Y could arise from different but similar distributions.

In this note, we study the distribution of XY when X and Y are independent random variables with X having the logistic pdf given by

$$f(x) = \frac{\lambda \exp\{-\lambda(x-\theta)\}}{[1+\exp\{-\lambda(x-\theta)\}]^2} \tag{1}$$

for $-\infty < x < \infty$, $-\infty < \theta < \infty$ and $\lambda > 0$. We take Y to come from one of normal, t, Laplace, logistic or Bessel families with its pdf denoted by $g(\cdot)$. For each combination of X and Y, we study the distribution of XY by deriving its characteristic function (cf) and the corresponding distribution function (df) of $\mid XY \mid$. Using the representation of (1) as the mixture of Laplace pdfs:

$$f(x) = \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{-2}{k} \frac{\lambda(k+1)}{2} \exp\{-\lambda(k+1) \mid x-\theta \mid\},$$

one can express the cf of XY as

$$\begin{aligned} \phi(t) &= E[\exp(itXY)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(itxy) f(x) g(y) dx dy \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(itxy\right)\Big[\sum_{k=0}^{\infty}\frac{2}{k+1}\binom{-2}{k}\frac{\lambda(k+1)}{2} \\
&\qquad \exp\left\{-\lambda(k+1)\mid x-\theta\mid\right\}\Big] g(y)dxdy \\
(2)\quad &= \sum_{k=0}^{\infty}\frac{2}{k+1}\binom{-2}{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(itxy\right)\frac{\lambda(k+1)}{2} \\
&\qquad \exp\left\{-\lambda(k+1)\mid x-\theta\mid\right\} g(y)dxdy \\
&= \sum_{k=0}^{\infty}\frac{2}{k+1}\binom{-2}{k} E\left[\exp\left(itX_kY\right)\right] \\
&= \sum_{k=0}^{\infty}\frac{2}{k+1}\binom{-2}{k}\phi_k(t),
\end{aligned}
$$

where X_k are Laplace random variables with parameters $(k+1)\lambda$ and $\phi_k(\cdot)$ denotes the cf of the product X_kY. Thus, deriving $\phi(\cdot)$ amounts to deriving expressions for $\phi_k(\cdot)$. Various explicit expressions for $\phi_k(\cdot)$ are derived in Sections 2–6. We also provide – without details of derivation – expressions for the corresponding df of $\mid XY \mid$ obtained from $\phi(\cdot)$ by the inversion theorem. The calculations involve the Bessel function of the first kind defined by

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma\left(\nu+1\right)}\sum_{k=0}^{\infty}\frac{1}{(\nu+1)_k k!}\left(-\frac{x^2}{4}\right)^k,$$

the Bessel function of the second kind defined by

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

with $Y_0(\cdot)$ interpreted as the limit

$$Y_0(x) = \lim_{\nu\to 0} Y_\nu(x),$$

the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp\left(-t^2\right)dt,$$

the integral cosine defined by

$$\mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt,$$

the integral sine defined by

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt,$$

the modified Bessel function of the third kind defined by

$$K_\nu(x) = \frac{\sqrt{\pi} x^\nu}{2^\nu \Gamma(\nu + 1/2)} \int_1^\infty \left(t^2 - 1\right)^{\nu - 1/2} \exp(-xt)\, dt,$$

the Struve function defined by

$$H_\nu(x) = \frac{2x^{\nu+1}}{\sqrt{\pi} 2^{\nu+1} \Gamma(\nu + 3/2)} \sum_{k=0}^\infty \frac{1}{(3/2)_k (\nu + 3/2)_k} \left(-\frac{x^2}{4}\right)^k,$$

the hypergeometric functions defined by

$$E(a, b, c; x) = \sum_{k=0}^\infty \frac{1}{(a)_k (b)_k (c)_k} \frac{x^k}{k!}$$

and

$$D(a; b, c, d; x) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k (c)_k (d)_k} \frac{x^k}{k!}$$

and, the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\right) = \frac{1}{2\pi i} \int_L \frac{x^{-t} \Gamma(b_1 + t) \cdots \Gamma(b_m + t) \Gamma(1 - a_1 - t) \cdots \Gamma(1 - a_n - t)}{\Gamma(a_{n+1} + t) \cdots \Gamma(a_p + t) \Gamma(1 - b_{m+1} - t) \cdots \Gamma(1 - b_q - t)} dt,$$

where $(e)_k = e(e+1) \cdots (e+k-1)$ denotes the ascending factorial and L denotes an integration path (see Section 9.3 in Gradshteyn and Ryzhik (2000) for a description of this path). We also need the following important lemmas.

Lemma 1. (Equation (2.3.15.4), Prudnikov et al. (1986, volume 1)) *For* $p > 0$,

$$\int_0^\infty \exp\left(-px^2 - qx\right) dx = \frac{\sqrt{\pi}}{2\sqrt{p}} \exp\left(\frac{q^2}{4p}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right).$$

Lemma 2. (Equation (2.3.8.4), Prudnikov et al. (1986, volume 1)) *For* $y > 0$ *and* $z > 0$,

$$\int_0^\infty \frac{\exp(i\lambda x)}{\left(x^2 + z^2\right)(y \pm ix)} dx = \frac{\pi}{2} \exp(\pm \lambda z)(y \pm z)^{-1}.$$

Lemma 3. (Equation (2.3.7.13), Prudnikov et al. (1986, volume 1)) *For* $p > 0$ *and* $z > 0$,

$$\int_0^\infty \frac{\exp(-px)}{x^2 + z^2} dx = \frac{1}{z} \left[\sin(pz) \operatorname{ci}(pz) - \cos(pz) \operatorname{si}(pz)\right].$$

Lemma 4. (Equation (2.16.3.16), Prudnikov et al. (1986, volume 2)) *For* $c > 0$, $z > 0$ *and* $\nu > -1/2$,

$$\int_0^\infty \frac{x^\nu}{x^2 + z^2} K_\nu(cx) dx = \frac{\pi^2 z^{\nu-1}}{4\cos(\nu\pi)} \left\{H_{-\nu}(cz) - Y_{-\nu}(cz)\right\}.$$

Further properties of the above special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2. Normal g. If g is the normal pdf given by

$$g(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$$

then $\phi_k(\cdot)$ in (2) can be expressed as

$$\begin{aligned}\phi_k(t) &= \frac{(k+1)\lambda}{2}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{itxy-\frac{(y-\mu)^2}{2\sigma^2}\right\} \\ &\qquad \exp\{-(k+1)\lambda \mid x-\theta \mid\}\,dydx \\ &= \frac{(k+1)\lambda}{2}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{\frac{2\sigma^2itxy-y^2+2\mu y-\mu^2}{2\sigma^2}\right\} \\ &\qquad \exp\{-(k+1)\lambda \mid x-\theta \mid\}\,dydx \qquad (3) \\ &= \frac{(k+1)\lambda}{2}\int_{-\infty}^{\infty}\exp\left(\mu itx-\frac{\sigma^2t^2x^2}{2}\right)\exp\{-(k+1)\lambda \mid x-\theta \mid\}\,dx.\end{aligned}$$

After substituting $z = x - \theta$, (3) can be further reduced as

$$\begin{aligned}\phi_k(t) &= \frac{(k+1)\lambda}{2}\exp\left(\mu it\theta-\frac{\sigma^2t^2\theta^2}{2}\right) \\ &\quad \times\left[\int_0^{\infty}\exp\left[-\frac{\sigma^2t^2z^2}{2}-\left\{\sigma^2t^2\theta-\mu it+(k+1)\lambda\right\}z\right]dz\right. \qquad (4) \\ &\quad \left.+\int_0^{\infty}\exp\left[-\frac{\sigma^2t^2z^2}{2}+\left\{\sigma^2t^2\theta-\mu it+(k+1)\lambda\right\}z\right]dz\right].\end{aligned}$$

The two integrals in (4) can be calculated by direct application of Lemma 1. One notes that (4) reduces to

$$\begin{aligned}&\phi_k(t) \\ &= \frac{(k+1)\lambda}{4}\exp\left(\mu it\theta-\frac{\sigma^2t^2\theta^2}{2}\right)\sqrt{\frac{2\pi}{\sigma^2t^2}}\exp\left[\frac{\left\{\sigma^2t^2\theta-\mu it+(k+1)\lambda\right\}^2}{2\sigma^2t^2}\right] \\ &\quad \times\left[\operatorname{erfc}\left(\frac{\sigma^2t^2\theta-\mu it+(k+1)\lambda}{\sqrt{2}\sigma t}\right)+\operatorname{erfc}\left(-\frac{\sigma^2t^2\theta-\mu it+(k+1)\lambda}{\sqrt{2}\sigma t}\right)\right] \qquad (5) \\ &= \frac{(k+1)\lambda}{4}\sqrt{\frac{2\pi}{\sigma^2t^2}}\exp\left\{\theta(k+1)\lambda+\frac{(k+1)^2\lambda^2}{2\sigma^2t^2}+\frac{\mu^2}{2\sigma^2}-\frac{i(k+1)\lambda\mu}{\sigma^2t}\right\} \\ &\quad \times\left[\operatorname{erfc}\left(\frac{\sigma^2t^2\theta-\mu it+(k+1)\lambda}{\sqrt{2}\sigma t}\right)+\operatorname{erfc}\left(-\frac{\sigma^2t^2\theta-\mu it+(k+1)\lambda}{\sqrt{2}\sigma t}\right)\right].\end{aligned}$$

If $\theta = 0$ and $\mu = 0$ then the inversion theorem can be applied to show that the corresponding df of $Z =\mid XY \mid$ is

$$
\begin{aligned}
F(z) &= \frac{\sqrt{2}\lambda z}{\sigma} \sum_{k=0}^{\infty} \binom{-2}{k} \left\{ \frac{3C}{\sqrt{\pi}} D\left(\frac{1}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{\lambda^2 (k+1)^2 z^2}{8\sigma^2} \right) \right. \\
&\qquad \left. + \frac{\lambda (k+1) z}{\sqrt{2}\sigma} D\left(1; 2, \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 (k+1)^2 z^2}{8\sigma^2} \right) \right\},
\end{aligned}
$$

where C denotes Euler's constant.

3. Student's *t g*. If g is the Student's t pdf given by

$$
g(y) = \frac{1}{\sqrt{\nu} B\left(\nu/2, 1/2\right)} \left(1 + \frac{y^2}{\nu} \right)^{-(1+\nu)/2}
$$

then $\phi_k(\cdot)$ in (2) can be reduced as

$$
\begin{aligned}
\phi_k(t) &= \frac{(k+1)\lambda \nu^{\nu/2}}{2B\left(\nu/2, 1/2\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{ itxy - (k+1)\lambda \mid x - \theta \mid \right\} \\
&\qquad \left(y^2 + \nu \right)^{-(1+\nu)/2} dx dy \\
&= \frac{(k+1)\lambda \nu^{\nu/2}}{2B\left(\nu/2, 1/2\right)} \int_{-\infty}^{\infty} \left(y^2 + \nu \right)^{-(1+\nu)/2} \\
&\qquad \left[\frac{\exp\left(it\theta y\right)}{(k+1)\lambda + ity} + \frac{\exp\left(it\theta y\right)}{(k+1)\lambda - ity} \right] dy. \qquad (6)
\end{aligned}
$$

The integral in (6) cannot be calculated in its general form. However, in the particular case $\nu = 1$, one can reduce (6) as

$$
\begin{aligned}
\phi_k(t) &= \frac{(k+1)\lambda}{2\pi t} \left[\int_{-\infty}^{\infty} \frac{\exp\left(it\theta y\right)}{\left(y^2+1\right)\left\{(k+1)\lambda/t + iy\right\}} dy \right. \\
&\qquad \left. + \int_{-\infty}^{\infty} \frac{\exp\left(it\theta y\right)}{\left(y^2+1\right)\left\{(k+1)\lambda/t - iy\right\}} dy \right] \qquad (7) \\
&= \frac{(k+1)\lambda}{2} \left[\frac{\exp\left(t\theta\right)}{(k+1)\lambda + t} + \frac{\exp\left(-t\theta\right)}{(k+1)\lambda - t} \right],
\end{aligned}
$$

where the last step follows by direct application of Lemma 2. Nevertheless, one can obtain expressions for the df of $Z = \mid XY \mid$ by inverting (6) for $\theta = 0$ and all integer values of ν. In particular,

$$F(z) = I(\nu) + \frac{\lambda z}{\pi\sqrt{\nu}} \sum_{l=0}^{\infty} \binom{-2}{l} \sum_{k=1}^{(\nu-1)/2} \frac{1}{\Gamma(k+\frac{1}{2})} G_{13}^{31} \left(\frac{\lambda^2 (l+1)^2 z^2}{4\nu} \,\middle|\, \begin{matrix} 1-k \\ 0, 0, \frac{1}{2} \end{matrix} \right)$$

if ν is an odd integer, and

$$F(z) = \frac{\lambda z}{\pi\sqrt{\nu}} \sum_{l=0}^{\infty} \binom{-2}{l} \sum_{k=1}^{\nu/2} \frac{1}{\Gamma(k)} G_{13}^{31} \left(\frac{\lambda^2 (l+1)^2 z^2}{4\nu} \,\middle|\, \begin{matrix} \frac{3}{2}-k \\ 0, 0, \frac{1}{2} \end{matrix} \right)$$

if ν is an even integer, where $I(\cdot)$ denotes the integral

$$I(a) = \frac{4\lambda}{\pi} \int_0^{\infty} \arctan\left(\frac{z}{\sqrt{a}y}\right) \frac{\exp(-\lambda y)}{\{1+\exp(-\lambda y)\}^2} dy.$$

Furthermore, using special properties of the Meijer G-function (see, for example, Chapter 8 of Prudnikov et al. (1986, volume 3)), simpler forms of the above can be obtained. For instance, if $\nu = 2, 4, \ldots, 10$ then one can express

$$F(z) = 2 \sum_{l=0}^{\infty} \frac{1}{l+1} \binom{-2}{l} F_{\nu,l}(z),$$

where

$$\begin{aligned}
F_{2,l}(z) &= -\pi y \Big\{ Y_0\left(y/\sqrt{2}\right) - H_0\left(y/\sqrt{2}\right) \Big\} \Big/ 4, \\
F_{4,l}(z) &= y \Big\{ -6\pi Y_0(\frac{y}{2}) + 6\pi H_0(\frac{y}{2}) + 2\pi y Y_1(\frac{y}{2}) - 2\pi y H_1(\frac{y}{2}) + 2y \Big\} \Big/ 16, \\
F_{6,l}(z) &= -\sqrt{6} y \Big\{ 90\pi Y_0(\frac{y}{\sqrt{6}}) - 90\pi H_0(\frac{y}{\sqrt{6}}) + 8\sqrt{6}\pi y Y_1(\frac{y}{\sqrt{6}}) \\
&\quad -8\sqrt{6}\pi y H_1(\frac{y}{\sqrt{6}}) - 18\sqrt{6} y - \pi y^2 Y_0(\frac{y}{\sqrt{6}}) + \pi y^2 H_0(\frac{y}{\sqrt{6}}) \Big\} \Big/ 576, \\
F_{8,l}(z) &= -\sqrt{2} y \Big\{ 3360\pi Y_0(\frac{y}{2\sqrt{2}}) - 3360\pi H_0(\frac{y}{2\sqrt{2}}) - 568\sqrt{2}\pi y Y_1(\frac{y}{2\sqrt{2}}) \\
&\quad +568\sqrt{2}\pi y H_1(\frac{y}{2\sqrt{2}}) - 1392\sqrt{2} y - 68\pi y^2 Y_0(\frac{y}{2\sqrt{2}})
\end{aligned}$$

$$
\begin{aligned}
&+68\pi y^2 H_0(\frac{y}{2\sqrt{2}}) + \sqrt{2}\pi y^3 Y_1(\frac{y}{2\sqrt{2}}) - \sqrt{2}\pi y^3 H_1(\frac{y}{2\sqrt{2}}) \\
&+2\sqrt{2}y^3\Big\}\Big/12288, \\
F_{10,l}(z) \;=\; &-\sqrt{10}y\Big\{\pi y^4 Y_0(\frac{y}{\sqrt{10}}) - \pi y^4 H_0(\frac{y}{\sqrt{10}}) - 19500\sqrt{10}y \\
&-7440\sqrt{10}\pi y Y_1(\frac{y}{\sqrt{10}}) + 28\sqrt{10}\pi y^3 Y_1(\frac{y}{\sqrt{10}}) \\
&+7440\sqrt{10}\pi y H_1(\frac{y}{\sqrt{10}}) - 28\sqrt{10}\pi y^3 H_1(\frac{y}{\sqrt{10}}) \\
&+58\sqrt{10}y^3 - 94500\pi H_0(\frac{y}{\sqrt{10}}) + 94500\pi Y_0(\frac{y}{\sqrt{10}}) \\
&+2580\pi y^2 H_0(\frac{y}{\sqrt{10}}) - 2580\pi y^2 Y_0(\frac{y}{\sqrt{10}})\Big\}\Big/768000,
\end{aligned}
$$

where $y = \lambda(l+1)z$.

4. Laplace g. Suppose g is the Laplace pdf given by

$$g(y) = \frac{\mu}{2}\exp\left(-\mu \mid y - \phi \mid\right). \tag{8}$$

Note that $\phi_k(\cdot)$ in (2) can be expressed as

$$
\begin{aligned}
\phi_k(t) &= \frac{(k+1)\lambda}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{itxy - (k+1)\lambda \mid x - \theta \mid\right\} g(y)dxdy \\
&= \frac{(k+1)\lambda}{2}\int_{-\infty}^{\infty}\left[\frac{\exp(it\theta y)}{(k+1)\lambda + ity} + \frac{\exp(it\theta y)}{(k+1)\lambda - ity}\right]g(y)dy \qquad (9)\\
&= \frac{(k+1)\lambda}{2}\int_{-\infty}^{\infty}\frac{\exp\{it\theta y\}}{(k+1)^2\lambda^2 + t^2y^2}g(y)dy.
\end{aligned}
$$

For the form of $g(\cdot)$ given by (8), the integral in (9) cannot be calculated in its general form. However, in the particular case $\phi = 0$, one can reduce (9) by application of Lemma 3. In particular, one obtains

$$
\begin{aligned}
\phi_k(t) &= \frac{(k+1)\lambda\mu}{4t^2}\left[\int_0^{\infty}\frac{\exp(-\mu y + it\theta y)}{(k+1)^2\lambda^2/t^2 + y^2}dy + \int_0^{\infty}\frac{\exp(-\mu y - it\theta y)}{(k+1)^2\lambda^2/t^2 + y^2}dy\right] \\
&= \frac{\mu}{4\mid t \mid}\left[\sin\left(\frac{(k+1)\lambda(\mu - it\theta)}{\mid t \mid}\right)\text{ci}\left(\frac{(k+1)\lambda(\mu - it\theta)}{\mid t \mid}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
(10) \qquad & -\cos\left(\frac{(k+1)\lambda(\mu-it\theta)}{\mid t\mid}\right)\mathrm{si}\left(\frac{(k+1)\lambda(\mu-it\theta)}{\mid t\mid}\right)\\
& +\sin\left(\frac{(k+1)\lambda(\mu+it\theta)}{\mid t\mid}\right)\mathrm{ci}\left(\frac{(k+1)\lambda(\mu+it\theta)}{\mid t\mid}\right)\\
& \left.-\cos\left(\frac{(k+1)\lambda(\mu+it\theta)}{\mid t\mid}\right)\mathrm{si}\left(\frac{(k+1)\lambda(\mu+it\theta)}{\mid t\mid}\right)\right],
\end{aligned}
$$

after applying Lemma 3 to each of the two integrals. If in addition $\theta = 0$ then, by inverting (10), one can obtain the df of $Z =\mid XY\mid$ as

$$
F(z) = 1 - 4\lambda\sum_{k=0}^{\infty}\binom{-2}{k}\sqrt{\frac{\mu z}{(k+1)\lambda}}K_1\left(2\sqrt{\lambda\mu(k+1)z}\right).
$$

5. Logistic *g*. Suppose g is the logistic pdf given by

$$
g(y) = \frac{\mu\exp\{-\mu(y-\phi)\}}{[1+\exp\{-\mu(y-\phi)\}]^2}.
$$

Note that this can be reexpressed as the mixture of Laplace pdfs:

$$
g(y) = \sum_{l=0}^{\infty}\frac{2}{l+1}\binom{-2}{l}\frac{\mu(l+1)}{2}\exp\{-\mu(l+1)\mid y-\phi\mid\}.
$$

Thus, using the result for the Laplace distribution given by (10), one obtains

$$
\begin{aligned}
\phi_k(t) \;=\; & \frac{\mu}{2\mid t\mid}\sum_{l=0}^{\infty}\binom{-2}{l}\left[\sin\left(\frac{(k+1)\lambda\{(l+1)\mu-it\theta\}}{\mid t\mid}\right)\right.\\
& \qquad\times\mathrm{ci}\left(\frac{(k+1)\lambda\{(l+1)\mu-it\theta\}}{\mid t\mid}\right)\\
(11) \qquad & -\cos\left(\frac{(k+1)\lambda\{(l+1)\mu-it\theta\}}{\mid t\mid}\right)\mathrm{si}\left(\frac{(k+1)\lambda\{(l+1)\mu-it\theta\}}{\mid t\mid}\right)\\
& +\sin\left(\frac{(k+1)\lambda\{(l+1)\mu+it\theta\}}{\mid t\mid}\right)\mathrm{ci}\left(\frac{(k+1)\lambda\{(l+1)\mu+it\theta\}}{\mid t\mid}\right)\\
& \left.-\cos\left(\frac{(k+1)\lambda\{(l+1)\mu+it\theta\}}{\mid t\mid}\right)\mathrm{si}\left(\frac{(k+1)\lambda\{(l+1)\mu+it\theta\}}{\mid t\mid}\right)\right]
\end{aligned}
$$

for the particular case $\phi = 0$. If in addition $\theta = 0$ then, by inverting (11), one can obtain the df of $Z = \mid XY \mid$ as

$$F(z) = 1-8\lambda \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \binom{-2}{k}\binom{-2}{l} \sqrt{\frac{\mu z}{(k+1)(l+1)\lambda}} K_1 \left(2\sqrt{(k+1)(l+1)\lambda\mu z}\right).$$

6. Bessel *g*. Suppose g is the Bessel function pdf given by

$$g(y) = \frac{\mid y \mid^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma\left(m+1/2\right)} K_m \left(\left|\frac{y}{b}\right|\right).$$

For this form, (9) can be expressed as

$$\phi_k(t) = \frac{(k+1)\lambda}{\sqrt{\pi} 2^{m+1} b^{m+1} \Gamma\left(m+\frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{\mid y \mid^m \exp\left(it\theta y\right)}{(k+1)^2\lambda^2 + t^2 y^2} K_m \left(\frac{y}{b}\right) dy. \tag{12}$$

The integral in (12) is difficult to calculate if $\theta \neq 0$. In the case $\theta = 0$, an application of Lemma 4 shows that one can reduce (12) as

$$\begin{aligned}
\phi_k(t) &= \frac{(k+1)\lambda}{\sqrt{\pi} 2^m b^{m+1} t^2 \Gamma\left(m+1/2\right)} \int_{-\infty}^{\infty} \frac{y^m}{(k+1)^2\lambda^2/t^2 + y^2} K_m \left(\frac{y}{b}\right) dy \\
&= \frac{\pi^{3/2}(k+1)^m\lambda^m}{2^{m+2} b^{m+1} \mid t \mid^{m+1} \cos(m\pi)\Gamma\left(m+1/2\right)} \\
&\quad \times \left\{ H_{-m}\left(\frac{(k+1)\lambda}{b \mid t \mid}\right) - Y_{-m}\left(\frac{(k+1)\lambda}{b \mid t \mid}\right)\right\}. \qquad (13)
\end{aligned}$$

Inverting (13), one can obtain the df of $Z = \mid XY \mid$ as

$$\begin{aligned}
&F(z) = \\
&1-\sum_{k=0}^{\infty} \frac{2}{k+1}\binom{-2}{k}\left\{\sqrt{\pi} 2^m b^{m+1} \Gamma\left(m+\frac{1}{2}\right) E\left(\frac{1}{2}, \frac{1}{2}-m, \frac{1}{2}; \frac{(k+1)^2\lambda^2 z^2}{16b^2}\right)\right. \\
&+\frac{\{(k+1)\lambda z\}^{2m+1}}{(2b)^m}\Gamma(-m)\Gamma(-2m-1) E\left(1+m, \frac{3}{2}+m, 1+m; \frac{(k+1)^2\lambda^2 z^2}{16b^2}\right)
\end{aligned}$$

$$+\left(\frac{3C}{2}-1\right)(2b)^m(k+1)\lambda z\Gamma(m)E\left(1-m,\frac{3}{2},1;\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\Bigg\}$$
$$\Bigg/\left\{\sqrt{\pi}2^m b^{m+1}\Gamma\left(m+\frac{1}{2}\right)\right\},$$

where C denotes the Euler's constant. If $-m-1/2$ is an integer then the above can be reduced to simpler forms by using the identity

$$\begin{aligned}
&H_\nu(z)-Y_\nu(z)\\
=\ &\frac{z^{\nu-1}}{(\nu-1/2)!\sqrt{\pi}2^{\nu-1}}\sum_{k=0}^{\nu-1/2}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(-\frac{z^2}{4}\right)^{-k}\\
&+\frac{\sqrt{2}(-1)^{\nu+1/2}}{\sqrt{\pi}\sqrt{z}}\left[\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)+z\right)\sum_{k=0}^{|(2\nu-1)/4|}\frac{(-1)^k\left(2k+\nu-\frac{1}{2}\right)!}{(2k)!\left(-2k+\nu-\frac{1}{2}\right)!(2z)^{2k}}\right.\\
&\left.+\cos\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)+z\right)\sum_{k=0}^{|(2\nu-3)/4|}\frac{(-1)^k(2k+\nu+1/2)!(2z)^{-2k-1}}{(2k+1)!(-2k+\nu-3/2)!}\right]\\
&-\frac{\sqrt{2}(-1)^{\nu+\frac{1}{2}}}{\sqrt{\pi}\sqrt{z}}\left[\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)+z\right)\sum_{k=0}^{[(2|\nu|-1)/4]}\frac{(-1)^k\left(2k+\mid\nu\mid-\frac{1}{2}\right)!}{(2k)!\left(-2k+\mid\nu\mid-\frac{1}{2}\right)!(2z)^{2k}}\right.\\
&\left.+\cos\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)+z\right)\sum_{k=0}^{[(2|\nu|-3)/4]}\frac{(-1)^k(2k+\mid\nu\mid+1/2)!(2z)^{-2k-1}}{(2k+1)!(-2k+\mid\nu\mid-3/2)!}\right],
\end{aligned}$$

where it is assumed that $\nu-1/2$ is an integer. For instance, if $m=3/2,5/2,7/2,9/2,11/2$ then one can express

$$F(z)=\sum_{k=0}^{\infty}\frac{2}{k+1}\binom{-2}{k}F_{m,k}(z),$$

where

$$\begin{aligned}
F_{\frac{3}{2},k}(z)\ =\ &\frac{-1}{8y}\Big\{-8y-4I_0(2y)y-3I_0(2y)y^3C+2I_0(2y)y^3-4J_0(2y)y\\
&-3J_0(y)y^3C+2J_0(2y)y^3+8J_1(2y)+6I_1(2y)y^2C-4I_1(2y)y^2
\end{aligned}$$

$$
\begin{aligned}
&+8J_1(2y)+6J_1(2y)\,y^2C-4J_1(2y)\,y^2\Big\},\\
F_{\frac{5}{2},k}(z) &= -\frac{1}{96y}\Big\{-96y-80I_0(2y)y-45I_0(2y)y^3C+30I_0(2y)y^3-80J_0(2y)y\\
&-45J_0(2y)y^3C+30J_0(2y)\,y^3+128I_1(2y)+72I_1(2y)y^2C\\
&-32I_1(2y)\,y^2+9I_1(2y)\,y^4C-6I_1(2y)\,y^4+128J_1(2y)\\
&+72J_1(2y)\,y^2C-64J_1(2y)y^2-9J_1(2y)y^4C+6J_1(2y)\,y^4\Big\},\\
F_{\frac{7}{2},k}(z) &= \frac{-1}{960y}\Big\{-960y-1056I_0(2y)\,y-495I_0(2y)\,y^3C+298I_0(2y)\,y^3\\
&-15I_0(2y)\,y^5C+10I_0(2y)\,y^5-1056J_0(2y)\,y-495J_0(2y)\,y^3C\\
&+362J_0(2y)\,y^3+15J_0(2y)\,y^5C-10J_0(2y)\,y^5+1536I_1(2y)\\
&+720I_1(2y)\,y^2C-160I_1(2y)\,y^2+150I_1(2y)\,y^4C-100I_1(2y)\,y^4\\
&+1536J_1(2y)+720J_1(2y)\,y^2C-800J_1(2y)\,y^2-150J_1(2y)\,y^4C\\
&+100J_1(2y)\,y^4\Big\},\\
F_{\frac{9}{2},k}(z) &= \frac{-1}{53760y}\Big\{-1120J_0(2y)\,y^5+23626J_0(2y)\,y^3+15434I_0(2y)\,y^3\\
&-512I_1(2y)\,y^2-6954I_1(2y)\,y^2-53248J_1(2y)\,y^2+7466J_1(2y)\,y^4\\
&-70I_1(2y)\,y^6-70J_1(2y)\,y^6-71424I_0(2y)\,y-71424J_0(2y)\,y\\
&-29295I_0(2y)\,y^3-1680I_0(2y)\,y^5C-29295J_0(2y)\,y^3C\\
&+1680J_0(2y)\,y^5C+105I_1(2y)\,y^6C-10815J_1(2y)\,y^4C\\
&+105J_1(2y)\,y^6C+40320I_1(2y)\,y^2C+40320J_1(2y)\,y^2C\\
&+10815I_1(2y)\,y^4C+1120I_0(2y)\,y^5+98304I_1(2y)\\
&-53760y+98304J_1(2y)\Big\},\\
F_{\frac{11}{2},k}(z) &= \frac{-1}{967680y}\Big\{1966080I_1(2y)+1966080J_1(2y)-967680y\\
&+483042J_0(2y)\,y^3-29618J_0(2y)\,y^5+126J_0(2y)\,y^7\\
&+227934I_0(2y)\,y^4C+4536I_1(2y)\,y^6C-227934J_1(2y)\,y^4C\\
&+4536J_1(2y)\,y^6C+725760I_1(2y)\,y^2C+725760J_1(2y)\,y^2C\\
&-547155I_0(2y)\,y^3C-43659I_0(2y)\,y^5C-189I_0(2y)\,y^7C\\
&-547155J_0(2y)\,y^3C+43659J_0(2y)\,y^5C-189J_0(2y)\,y^7C
\end{aligned}
$$

$$+28594 I_0(2y)\, y^5 - 1101312 J_1(2y)\, y^2 + 164244 J_1(2y)\, y^4$$
$$+246498 I_0(2y)\, y^3 + 126 I_0(2y)\, y^7 - 3024 I_1(2y)\, y^6$$
$$-1482240 I_0(2y)\, y - 3024 J_1(2y)\, y^6 - 1482240 J_0(2y)\, y$$
$$+133632 I_1(2y)\, y^2 - 139668 I(2y)\, y^4 \Big\},$$

where $y = \sqrt{(k+1)\lambda z/b}$.

Acknowledgments.

The authors would like to thank the editor and the referee for carefully reading the paper and for their great help in improving the paper.

References

1. M. S. Abu-Salih, *Distributions of the product and the quotient of power-function random variables*, Arab Journal of Mathematics, **4**(1983), 77-90.

2. R. P. Bhargava and C. G. Khatri, *The distribution of product of independent beta random variables with application to multivariate analysis*, Annals of the Institute of Statistical Mathematics, **33**(1981), 287-296.

3. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (sixth edition), San Diego: Academic Press, 2000.

4. H. L. Harter, *On the distribution of Wald's classification statistic*, Annals of Mathematical Statistics, **22**(1951), 58-67.

5. H. J. Malik and R. Trudel, *Probability density function of the product and quotient of two correlated exponential random variables*, Canadian Mathematical Bulletin, **29**(1986), 413-418.

6. S. Nadarajah and S. Kotz, *Skewed distributions generated by the normal kernel*, Statistics and Probability Letters, **65**(2003), 269-277.

7. H. Podolski, *The distribution of a product of n independent random variables with generalized gamma distribution*, Demonstratio Mathematica, **4**(1972), 119-123.

8. A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, *Integrals and Series* (volumes 1, 2 and 3), Amsterdam: Gordon and Breach Science Publishers, 1986.

9. P. N. Rathie and H. G. Rohrer, *The exact distribution of products of independent random variables*, Metron, **45**(1987), 235-245.

10. H. Sakamoto, *On the distributions of the product and the quotient of the independent and uniformly distributed random variables*, Tohoku Mathematical Journal, **49**(1943), 243-260.

11. D. Sornette, *Multiplicative processes and power laws*, Physical Review E, **57**(1998), 4811-4813.

12. M. D. Springer and W. E. Thompson, *The distribution of products of beta, gamma and Gaussian random variables*, SIAM Journal on Applied Mathematics, **18**(1970), 721-737.

13. B. M. Steece, *On the exact distribution for the product of two independent beta-distributed random variables*, Metron, **34**(1976), 187-190.

14. A. Stuart, *Gamma-distributed products of independent random variables*, Biometrika, **49**(1962), 564-565.

15. J. Tang and A. K. Gupta, *On the distribution of the product of independent beta random variables*, Statistics & Probability Letters, **2**(1984), 165-168.

16. C. M. Wallgren, *The distribution of the product of two correlated t variates*, Journal of the American Statistical Association, **75**(1980), 996-1000.

Department of Statistics, University of Nebraska, Lincoln, Nebraska 68583, U.S.A.

E-mail: snadrajah2@unl.edu

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio 43403, U.S.A.