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TOPOLOGICAL CLASSIFICATION OF CONFORMAL ACTIONS ON 2-HYPERELLIPTIC RIEMANN SURFACES

BY

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Abstract. A compact Riemann surface X of genus $g > 1$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ for which X/ρ is an orbifold of genus p. Here we classify conformal actions on 2-hyperelliptic Rieman surfaces of genus $g > 9$, up to topological conjugacy and determine which of them can be maximal.

1. Introduction. A Riemann surface X of genus $g > 2$ is said to be *p-hyperelliptic* if and only if X admits a conformal involution ρ_p called *p-hyperelliptic involution*, such that X/ρ_p is an orbifold of genus p. In the particular cases $p = 0, 1$, X are called *hyperelliptic* and *elliptic-hyperelliptic* Riemann surfaces respectively. If $g > 4p+1$ then ρ_p is unique and central in the group $\mathrm{Aut}(X)$ of all automorphisms of X [15] and this is why we restrict ourselves to Riemann surfaces of genus $g > 9$ only.

Hyperelliptic Riemann surfaces and their automorphisms have received a good deal of attention in the literature. In [2] and [11] the authors determined the full groups of conformal automorphisms of such surfaces which

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made possible recently to classify symmetry types of such actions in [4].

The p-hyperelliptic ($p \geq 1$) surfaces at large have been studied in [6]-[10], [12]-[14], where the most attention has been paid to a study of groups of automorphisms of bordered Klein surfaces with the exception of [5] were the pairs of symmetries of compact p-hyperelliptic Riemann surfaces were classified.

The classification of conformal actions up to topological conjugacy is a classical problem which up to now was solved only for surfaces of genera $g = 2, 3$ in [3] and $g = 4$ in [1]. We deal with such classification for p-hyperelliptic Riemann surfaces. In [19] we solved this problem in case $p = 1$ and in this paper we classify the actions on 2-hyperelliptic Riemann surfaces and determine which of them can be maximal.

2. Preliminaries. We shall approach the problem using Riemann uniformization theorem by which each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore a group of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as Λ/Γ for another Fuchsian group Λ. Each Fuchsian group Λ is given a signature $\sigma(\Lambda) = (g; m_1, \dots, m_r)$, where g, m_i are integers verifying $g \geq 0, m_i \geq 2$. The signature determines the presentation of Λ:

generators: $x_1, \dots, x_r, a_1, b_1, \dots, a_g, b_g$,
relations : $x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \dots x_r [a_1, b_1] \dots [a_g, b_g] = 1.$

Such set of generators is called the *canonical set of generators* and often, by abuse of language, the set of *canonical generators*. Geometrically x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1, m_2, \dots, m_r$ are called the *periods* of Λ, g is the genus of the orbit space $\mathcal{H}/\Lambda$ and it is called the *orbit genus*. Fuchsian groups with signatures $(g; -)$ are called *surface groups* and they are characterized among Fuchsian groups as these ones which are torsion free.

The group Λ has associated to it a fundamental region whose area $\mu(\Lambda)$, called the *area of the group*, is:

$$\mu(\Lambda) = 2\pi\left(2g-2+\sum_{i=1}^{r}(1-1/m_i)\right). \tag{1}$$

If Γ is a subgroup of finite index in Λ, then we have the *Riemann-Hurwitz formula* which says that

$$[\Lambda:\Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda).} \tag{2}$$

Moreover if it is a normal subgroup of Λ then its signature can be obtain by the following

Proposition 2.1. *If Γ is a normal subgroup of Λ of finite index N, $x_1,\ldots,x_r$ is the set of canonical elliptic generators of Λ, $[m_1,\ldots,m_r]$ the set of periods of Λ and p_i denotes the order of $\Gamma x_i \in \Lambda/\Gamma$, then the proper periods in $\sigma(\Gamma)$ are*

$$[m_1/p_1,\overset{N/p_1}{\ldots},m_1/p_1,\ldots,m_r/p_r,\overset{N/p_r}{\ldots},m_r/p_r].$$

Let X be a Riemann surface of genus $g\geq 2$, $\mathrm{Hom}^+(X)$ its group of orientation-preserving homeomorphisms and G finite group. We say that G acts on X if there is monomorphism $\varepsilon: G\to \mathrm{Hom}^+(X)$. The action G on X may be constructed by means of a pair of Fuchsian groups Γ, Λ and an epimorphism $\theta:\Lambda\to G$ with kernel Γ. The signature $\sigma(\Lambda) = (s;m_1,\ldots,m_r)$ not only determines the presentation of the group Λ but it also characterizes topologically the branched covering $X\to X/G$, since s is the genus of the surface X/G and m_j are ramification numbers of this covering. We call $(s:m_1,\ldots,m_r)$ *the branching data* of G on X. For the sake of notation simplicity we shall omit s in the case $s=0$ and we shall write briefly $(2^3, 3^2)$ instead of $(2,2,2,3,3)$, etc. The $(2s+r)$-tuple $(\theta(a_1),\ldots,\theta(a_s),\theta(b_1),\ldots,\theta(b_s),\theta(x_1),\ldots,\theta(x_r))$ is called a *generating* $(s;m_1,\ldots,m_r)$*-vector* . We say that $\theta:\Lambda\to G$ and $\theta':\Lambda'\to G'$ give rise to *topologically equivalent actions* if $\varphi\theta=\theta'\psi$ for some isomorphisms $\varphi:G\to G'$ and $\psi:\Lambda\to\Lambda'$. The relation of the equivalence of actions induces an equivalence relation on generating vectors.

The p-hyperelliptic Riemann surface X can be identified with $\mathcal{H}/\Gamma$, where Γ is a Fuchsian surface group with orbit genus g and the group $\langle\rho\rangle$ generated by the p-hyperelliptic involution with Δ/Γ, where Δ is a Fuchsian group with signature $(p;2,\overset{2g+2-4p}{\ldots},2)$. Let G be the group of automorphisms of X. Then $G=\Lambda/\Gamma$ for some Fuchsian group Λ, say with signature $(g';m_1,\ldots,m_r)$. Let $\widetilde{G}=G/\rho$ and let θ and π be the canonical epimorphisms from Λ onto G and G onto $\widetilde{G}$ and $\tilde{\theta}=\pi\theta$. We shall use these notations throughout all the paper. Denote the order of $\tilde{\theta}(x_i)$ by p_i. Then by Proposition 2.1 $m_i/p_i=2$ for $i=1,\ldots,s$, $m_i/p_i=1$ for $i=s+1,\ldots,r$ and $2g+2-4p=\sum_{i=1}^{s}N/p_i$, where $|G|=2N$. Thus the Hurwitz-Riemann formula for (Λ,Δ) gives

$$2p-2+N(2-2g')=N\sum_{i=1}^{r}(1-1/p_i).$$

For $p=2$ the only solutions of the above equation are

(a) $g'=1,\ N=2,\ p_1=p_2=2,p_3=\cdots=p_r=1,$
(b) $g'=0,\ N=2,\ p_1=\cdots=p_6=2,p_7=\cdots=p_r=1,$
(c) $g'=0,\ N=3,\ p_1=\cdots=p_4=3,p_5=\ldots=p_r=1$
(d) $g'=0,\ N=4,\ p_1=p_2=2,p_3=p_4=4,p_5=\ldots=p_r=1,$
(e) $g'=0,\ N=4,\ p_1=\cdots=p_5=2,p_6=\cdots=p_r=1,$
(f) $g'=0,\ N=5,\ p_1=p_2=p_3=5,p_4=\cdots=p_r=1,$
(g) $g'=0,\ N=6,\ p_1=3,p_2=p_3=6,p_4=\cdots=p_r=1,$
(h) $g'=0,\ N=6,\ p_1=p_2=2,p_3=p_4=3,p_5=\ldots=p_r=1,$

(i) $g' = 0,\ N = 8,\quad p_1 = 2, p_2 = p_3 = 8, p_4 = \cdots = p_r = 1,$

(j) $g' = 0,\ N = 8,\quad p_1 = p_2 = p_3 = 4, p_4 = \cdots = p_r = 1,$

(k) $g' = 0,\ N = 8,\quad p_1 = p_2 = p_3 = 2, p_4 = 4, p_5 = \cdots = p_r = 1,$

(l) $g' = 0,\ N = 10,\ p_1 = 2, p_2 = 5, p_3 = 10, p_4 = \cdots = p_r = 1,$

(m) $g' = 0,\ N = 12,\ p_1 = 2, p_2 = p_3 = 6, p_4 = \cdots = p_r = 1,$

(n) $g' = 0,\ N = 12,\ p_1 = 3, p_2 = p_3 = 4, p_4 = \cdots = p_r = 1,$

(o) $g' = 0,\ N = 12,\ p_1 = p_2 = p_3 = 2, p_4 = 3, p_5 = \cdots = p_r = 1,$

(p) $g' = 0,\ N = 16,\ p_1 = 2, p_2 = 4, p_3 = 8, p_4 = \cdots = p_r = 1,$

(r) $g' = 0,\ N = 24,\ p_1 = 2, p_2 = 4, p_3 = 6, p_4 = \cdots = p_r = 1,$

(s) $g' = 0,\ N = 24,\ p_1 = p_2 = 3, p_3 = 4, p_4 = \cdots = p_r = 1,$

(t) $g' = 0,\ N = 48,\ p_1 = 2, p_2 = 3, p_3 = 8, p_4 = \cdots = p_r = 1.$

Thus $m_i = 2$ for $p_i = 1$ and $m_i = \varepsilon_i p_i$, where $\varepsilon_i = 1$ or 2, for the remaining ones. The number of $p_i = 1$ can be calculated by Proposition 2.1 and it shall be denoted by t. So we obtain all candidates for the signatures of the group Λ up to permutation of periods. The images of x_i in $\tilde{G}$ have orders p_i and in all cases but (a) they generate $\tilde{G}$. In the exceptional case we have to add $\tilde{\theta}(a_1)$ and $\tilde{\theta}(b_1)$ to the set of generators. Thus according to the classification of finite group actions on a genus 2 surface given up to topological equivalence by Broughton in [3], $\tilde{G}$ is isomorphic to one of the groups listed in Table 1. According to these cases we shall study conformal actions on 2-hyperelliptic Riemann surfaces and we shall refer to an action of the group G as to $(\tilde{G})$-action.

3. Classification of the actions on 2-hyperelliptic Riemann surfaces. In this section we determine all finite groups acting on 2-hyperelliptic Riemann surface and decide which of their actions are equivalent. Let $G = \Lambda/\Gamma$ be an automorphism group of 2-hyperelliptic Riemann surface and let $\sigma(\Lambda) = (g'; m_1, \ldots, m_r)$. The generating vector of $\tilde{G} = G/\rho$ given in Table 1 determines the epimorphism $\tilde{\theta}$ up to topological equivalence. Since $\tilde{\theta} = \pi\theta$, any generating vector of G can be written as $(g_1\rho^{k_1}, \ldots, g_{r-t}\rho^{k_{r-t}}, \rho, \overset{t}{\ldots}, \rho)$, where $\pi(g_i) = \tilde{\theta}(x_i)$ and $k_i \in \{0,1\}$. We shall denote it by v_k. Let $g_i^{p_i} = \rho^{\alpha_i}$ for $\alpha_i = 0$ or 1 and $i = 1, \ldots, r$. By Proposition 2.1, the order of $\theta(x_i)$ is equal to m_i, since Γ is a surface Fuchsian group. More precisely $\theta(x_i)^{p_i} = \rho^{r_i}$, where $r_i = \varepsilon_i - 1$. Thus for $i = 1, \ldots, r$,

$$\alpha_i = r_i \text{ if } p_i \equiv 0\ (2) \text{ or } \alpha_i \equiv k_i + r_i\ (2) \text{ if } p_i \equiv 1\ (2). \tag{3}$$

Table 1

Case	Presentation of $\tilde{G}$	Branching data	Generating vector
a	$Z_2 = \langle \tilde{x} : \tilde{x}^2 \rangle$	$(1 : 2^2)$	$(1, 1, \tilde{x}, \tilde{x})$
b	$Z_2 = \langle \tilde{x} : \tilde{x}^2 \rangle$	(2^6)	$\tilde{x}, \tilde{x}, \tilde{x}, \tilde{x}, \tilde{x}, \tilde{x})$
c	$Z_3 = \langle \tilde{x} : \tilde{x}^3 \rangle$	(3^4)	$(\tilde{x}, \tilde{x}, \tilde{x}^{-1}, \tilde{x}^{-1})$
d	$Z_4 = \langle \tilde{x} : \tilde{x}^4 \rangle$	$(2^2, 4^2)$	$\tilde{x}^2, \tilde{x}^2, \tilde{x}, \tilde{x}^{-1})$
e	$Z_2 \oplus Z_2 = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^2, [\tilde{x}, \tilde{y}] \rangle$	(2^5)	$(\tilde{x}, \tilde{x}, \tilde{x}, \tilde{y}, \tilde{x}\tilde{y})$
f	$Z_5 = \langle \tilde{x} : \tilde{x}^5 \rangle$	$(5, 5, 5)$	$(\tilde{x}, \tilde{x}, \tilde{x}^3)$
g	$Z_6 = \langle \tilde{x} : \tilde{x}^6 \rangle$	$(3, 6, 6)$	$(\tilde{x}^4, \tilde{x}, \tilde{x})$
h.1	$Z_6 = \langle \tilde{x} : \tilde{x}^6 \rangle$	$(2^2, 3^2)$	$(\tilde{x}^3, \tilde{x}^3, \tilde{x}^2, \tilde{x}^4)$
h.2	$D_3 = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^3, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y} \rangle$	$(2^2, 3^2)$	$(\tilde{x}, \tilde{x}, \tilde{y}, \tilde{y}^{-1})$
i	$Z_8 = \langle \tilde{x} : \tilde{x}^8 \rangle$	$(2, 8, 8)$	$(\tilde{x}^4, \tilde{x}^3, \tilde{x})$
j	$\tilde{D}_2 = \langle \tilde{x}, \tilde{y} : \tilde{x}^4, \tilde{y}^4, \tilde{x}^2\tilde{y}^2, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y} \rangle$	$(4, 4, 4)$	$(\tilde{x}, \tilde{y}, \tilde{y}\tilde{x})$
k	$D_4 = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^4, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y} \rangle$	$(2^3, 4)$	$(\tilde{x}, \tilde{x}\tilde{y}, \tilde{y}^2, \tilde{y})$
l	$Z_{10} = \langle \tilde{x} : \tilde{x}^{10} \rangle$	$(2, 5, 10)$	$(\tilde{x}^5, \tilde{x}^4, \tilde{x})$
m	$Z_2 \oplus Z_6 = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^6, [\tilde{x}, \tilde{y}] \rangle$	$(2, 6, 6)$	$(\tilde{x}, \tilde{x}\tilde{y}, \tilde{y}^{-1})$
n	$D_{4,3,-1} = \langle \tilde{x}, \tilde{y} : \tilde{x}^4, \tilde{y}^3, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y} \rangle$	$(3, 4, 4)$	$(\tilde{y}, (\tilde{x}\tilde{y})^{-1}, \tilde{x})$
o	$D_6 = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^6, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y} \rangle$	$(2^3, 3)$	$(\tilde{x}, \tilde{x}\tilde{y}, \tilde{y}^3, \tilde{y}^2)$
p	$D_{2,8,3} = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^8, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^5$	$(2, 4, 8)$	$(\tilde{x}, (\tilde{y}\tilde{x})^{-1}, \tilde{y})$
r	$(Z_2 \oplus Z_2 \oplus Z_3) \ltimes Z_2 =$ $\langle \tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} : \tilde{x}^2, \tilde{y}^2, \tilde{z}^2, \tilde{w}^3, [\tilde{y}, \tilde{z}], [\tilde{y}, \tilde{w}],$ $[\tilde{z}, \tilde{w}], [\tilde{x}, \tilde{y}], \tilde{x}\tilde{w}\tilde{x}^{-1}\tilde{w}, \tilde{x}\tilde{z}\tilde{x}^{-1}\tilde{y}^{-1}\tilde{z}^{-1} \rangle$	$(2, 4, 6)$	$(\tilde{x}, (\tilde{z}\tilde{w}\tilde{x})^{-1}, \tilde{z}\tilde{w})$
s	$SL_2(3) = \langle \tilde{x}, \tilde{y} : \tilde{x}^3, \tilde{y}^4, (\tilde{x}\tilde{y})^3, \tilde{x}\tilde{y}^2\tilde{x}^{-1}\tilde{y}^2 \rangle$	$(3, 3, 4)$	$(\tilde{x}, (\tilde{y}\tilde{x})^{-1}, \tilde{y})$
t	$GL_2(3) = \langle \tilde{x}, \tilde{y} : \tilde{x}^2, \tilde{y}^3, (\tilde{x}\tilde{y})^8, ((\tilde{x}\tilde{y})^4 x)^2 \rangle$	$(2, 3, 8)$	$(\tilde{x}, \tilde{y}, (\tilde{x}\tilde{y})^{-1}$

We say that epimorphisms $\theta : \Lambda \to G$ and $\theta' : \Lambda' \to G'$ are equivalent or that a pair (ψ, φ) of isomorphisms $\psi : \Lambda \to \Lambda'$ and $\varphi : G \to G'$ induces their equivalence if

$$\varphi\theta = \theta'\psi. \tag{4}$$

By Remark 0.2.6 [10], the actions of G and G' may be equivalent only if Λ and Λ' have the same signatures up to permutation of periods. Conversely any permutation of periods give rise to isomorphic groups with the same topological type of the action. However when the relations in the presentation of some group admit parameters not depending on the signature of Λ then actions corresponding to different values of such parameters may not be equivalent.

For $i = 1, \dots, r$, let x_i and x'_i be the canonical elliptic generators of Λ and Λ' corresponding to periods m_i and m'_i respectively. Then by Theorem 3 [17], for any isomorphism $\psi : \Lambda \to \Lambda'$, there exists a homeomorphism $t : \mathcal{H} \to \mathcal{H}$ such that $\psi(\lambda) = t\lambda t^{-1}$ for all $\lambda \in \Lambda$. Thus $\psi(x_i)$ is conjugate to some x'_j or x'^{-1}_j such that $m_i = m'_j$ and consequently ψ induces the permutation $\overline{\psi}$ of the set $\{1, \dots, r\}$ preserving branches orders i.e., $m_i = m_{\overline{\psi}(i)}$. In particular if $G = \Lambda/\Gamma$ is abelian and a pair (ψ, φ) induces the equivalence of epimorphisms θ_k and θ_l from Λ onto G then by (4), for $i = 1, \dots, r$

$$\varphi\theta_k(x_i) = \theta_l(x^j_{\overline{\psi}(i)}) \tag{5}$$

where $j = 1$ or $j = -1$.

We define some pairs of isomorphisms of Fuchsian groups and abstract groups which we shall use along the paper to prove the equivalence of generating vectors of G. By Ψ_i we shall denote a pair (ψ_i, id_G), where ψ_i is defined by the assignment $\psi_i(x_i) = x_{i+1}, \psi_i(x_{i+1}) = x^{-1}_{i+1}x_ix_{i+1}, \psi_i(x_j) = x_j$ for $j \neq i$. In the cases when G is generated by x, y and ρ we shall need pairs $\Phi_{p,q} = (\mathrm{id}_\Lambda, \varphi_{p,q})$, where $\varphi_{p,q}$ is defined by $\varphi_{p,q}(x) = x\rho^p, \varphi_{p,q}(y) = y\rho^q, \varphi_{p,q}(\rho) = \rho$. Finally when G is generated by x and ρ, we shall use pairs $\Omega_{p,q} = (\mathrm{id}_\Lambda, \omega_{p,q})$, where $\omega_{p,q}(x) = x^p\rho^q, \omega_{p,q}(\rho) = \rho$. We shall apply these pairs only in the situation when $\psi_i, \varphi_{p,q}, \omega_{p,q}$ induce automorphisms of considered groups.

Theorem 3.1. *The topological type of the* (Z_n)*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and generating vectors listed in Table* 2*.*

Proof. Let G be a group of automorphisms of 2-hyperelliptic Riemann surface such that $\tilde{G}$ is isomorphic to $Z_n = \langle \tilde{x} \rangle$ and let $x \in \pi^{-1}(\tilde{x})$. Then G is generated by x and ρ which satisfy the relations $\rho^2 = x^n\rho^\alpha = [x, \rho] = 1$ for some $\alpha \in \{0, 1\}$. So $G = Z_n \oplus Z_2$ if $\alpha = 0$ or $G = Z_{2n}$ if $\alpha = 1$.

Cases a, b. Here $\tilde{G} = \langle \tilde{\theta}(x_1) \rangle = Z_2$. Thus $\rho^{r_1} = \theta(x_1)^2 = x^2$ which implies $\alpha = r_1$. If $r_1 = 0$ then all periods in the signature of Λ are equal to 2. In a contrary suppose that $m_i = 4$ for some $2 \le i \le r$. Then $G = \langle \theta(x_i) \rangle$ and $\theta(x_1) = \theta(x_i)^2$. Thus $\pi\theta(x_1) = \pi(\rho) = 1$ which means that $p_1 = 1$, a contradiction. Consequently Λ has the signature $(1; 2, 2, 2, \overset{g-3}{\ldots}, 2)$, in the case (a) or $(0; 2, 2, 2, 2, 2, 2, 2 \overset{g-3}{\ldots}, 2)$ in (b) and $G = \langle x \rangle \oplus \langle \rho \rangle = Z_2 \oplus Z_2$. Similarly we can argue that if $r_1 = 1$ then $m_i = 4$ for all $p_i \neq 1$. Thus Λ has the signature $(1; 4, 4, 2, \overset{g-4}{\ldots}, 2)$ in the case (a) or $(0; 4, 4, 4, 4, 4, 4, 2, \overset{g-3}{\ldots}, 2)$ in (b) and $G = Z_4$.

The generating vector of G has a form $v_k^{(a)} = (\rho^\beta, \rho^\gamma, x\rho^{k_1}, x\rho^{k_2}, \rho, \overset{t}{\ldots}, \rho)$ or $v_k^{(b)} = (x\rho^{k_1}, x\rho^{k_2}, x\rho^{k_3}, x\rho^{k_4}, x\rho^{k_5}, x\rho^{k_6}, \rho \overset{t}{\ldots}, \rho)$ according to the case (a) or (b) respectively. Suppose that a pair (ψ, φ) induces the equivalence of two vectors with exponents k_i and l_i respectively. Then $\varphi = \omega_{1,q}$ for $q = 1$ or 0 and by (5), we obtain $k_i = l_{\tilde{\psi}(i)} + q$ for $i = 1, \ldots, r - t$. Thus either $\sum_{i=1}^{r-t} k_i = \sum_{i=1}^{r-t} l_i$ if $q = 0$ or $\sum_{i=1}^{r-t} k_i = (r - t) - \sum_{i=1}^{r-t} l_i$ if $q = 1$. Consequently vectors indexed by (a) or (b) in Table 2 are not equivalent. Using pairs $(\psi_\beta, \mathrm{id}_G)$ and $(\psi_\gamma, \mathrm{id}_G)$, where ψ_β and ψ_γ are defined by assignments $\psi_\beta(a) = x_r a, \psi_\beta(b) = b, \psi_\beta(x_i) = x_i$ for $i = 1, 2$, $\psi_\beta(x_3) = x_2^{-1}x_1^{-1}bx_rb^{-1}x_1x_2, \psi_\beta(x_4) = x_3, \ldots, \psi_\beta(x_r) = x_{r-1}$ and $\psi_\gamma(a) =$

$a, \psi_\gamma(b) = x_r b, \psi_\gamma(x_i) = x_r x_i x_r^{-1}$ for $i = 1, \ldots, r-1, \psi_\gamma(x_r) = a x_r^{-1} a^{-1}$, it is easy to check that vectors $v_k^{(a)}$ and $(1, 1, x\rho^{k_1}, x\rho^{k_2}, \rho, \overset{t}{\ldots}, \rho)$ are equivalent. So composing pairs Ψ_i and $\Omega_{1,1}$ in an appropriate way we can show that $v_k^{(a)}$ and $v_k^{(b)}$ are equivalent to one of vectors indexed by (a) or (b) respectively in Table 2.

Case c. Here $n = 3$ and $v_k = (x\rho^{k_1}, x\rho^{k_2}, x^{-1}\rho^{k_3}, x^{-1}\rho^{k_4}, \rho, \overset{t}{\ldots}, \rho)$. Thus by (3), $\alpha \equiv k_i + r_i$ (2) and so using pair $\Omega_{1,1}$ we can prove that v_k is equivalent to $(x\rho^{r_1}, x\rho^{r_2}, x^{-1}\rho^{r_3}, x^{-1}\rho^{r_4}\rho, \overset{t}{\ldots}, \rho)$ and G is isomorphic to $Z_3 \oplus Z_2$.

Case d. Now $n = 4$ and $v_k = (x^2\rho^{k_1}, x^2\rho^{k_2}, x\rho^{k_3}, x^{-1}\rho^{k_4}, \rho, \overset{t}{\ldots}, \rho)$. Thus $\rho^\alpha = x^4 = \rho^{r_1} = \rho^{r_2} = \rho^{r_3} = \rho^{r_4}$. Consequently $\sigma(\Lambda) = (4, 4, 8, 8, 2, \overset{(g-6)/2}{\ldots}, 2)$ and $G = Z_8$ or $\sigma(\Lambda) = (2, 2, 4, 4, 2, \overset{(g-3)/2}{\ldots}, 2)$ and $G = Z_4 \oplus Z_2$.

If a pair (ψ, φ) induces the equivalence of two vectors corresponding to parameters k_i and l_i respectively then $\tilde{\psi}$ permutes the sets $\{1, 2\}$ and $\{3, 4\}$. Moreover $\varphi = \omega_{p,q}$, for some p co-prime with the order of x and $q = 0$ or 1. Thus putting all possible values of p into (5), we obtain that for $G = Z_8$, two vectors are equivalent only if $k_1 + k_2 = l_1 + l_2$ or $k_1 + k_2 = 2 - (l_1 + l_2)$ while for $G = Z_4 \oplus Z_2$, only if the first condition is satisfied. It proves that none two of vectors indexed by (d) in Table 2 are equivalent. Moreover using pairs $\Omega_{1,1}$ and $(\psi_3, \omega_{3,q})$ in the first case or Ψ_1 and $\Omega_{1,1}$ in the second one we can show that v_k is equivalent to one of them.

The cases f, g, h, i are similar and so we omit them.

Theorem 3.2. *The topological type of the* $(SL_2(3))$*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and generating vectors listed in Table* 3.

Proof. Let G be a group of automorphisms of 2-hyperelliptic Riemann surface such that $\tilde{G} = \langle \tilde{x}, \tilde{y} : \tilde{x}^3, \tilde{y}^4, (\tilde{y}\tilde{x})^3, \tilde{x}\tilde{y}^2\tilde{x}^{-1}\tilde{y}^2 \rangle$ and let x, y belong to $\pi^{-1}(\tilde{x})$ and $\pi^{-1}(\tilde{y})$ respectively. Then x, y, ρ generate G and satisfy the relations $\rho^2 = [x,\rho] = [y,\rho] = x^3\rho^\alpha = y^4\rho^\beta = (yx)^3\rho^\gamma = xy^2x^{-1}y^2\rho^\delta = 1$ for some $\alpha,\beta,\gamma,\delta \in \{0,1\}$.

A generating vector of G has a form $v_k = (x\rho^{k_1}, (yx)^{-1}\rho^{k_2}, y\rho^{k_3}\rho, \overset{t}{\ldots}, \rho)$. Thus by (3), $\beta = r_3$, $\alpha \equiv k_1 + r_1$ (2) and $\gamma \equiv k_2 + r_2$ (2). We can assume that $\alpha = \gamma = 0$ since the pairs $\Phi_{1,1}$ and $\Phi_{0,1}$ induce the equivalence of two actions corresponding to different values of α or γ respectively. Consequently v_k is equivalent to $(x\rho^{r_1}, (yx)^{-1}\rho^{r_2}, y\rho^{r_1+r_2+t}, \rho, \overset{t}{\ldots}, \rho)$. Finally by the relation $xy^2x^{-1}y^2\rho^\delta = 1$ we obtain $y^2x^{-1}y^{-2} = x^{-1}\rho^{\delta+r_3}$ which implies $\rho^\alpha = y^2x^{-3}y^{-2} = x^{-3}\rho^{\delta+r_3} = \rho^{\alpha+\delta+r_3}$ and consequently $\delta = r_3$. So the presentation of G depends only on r_3 and using Programme GAP we check that it has required order 48 only for $r_3 = 0$.

Theorem 3.3. *The topological type of the* $(Z_2 \oplus Z_n)$*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and a generating vector listed in Table* 4.

Proof. Let G be a group of automorphisms of 2-hyperelliptic Riemann surface such that $\tilde{G}$ is isomorphic to $Z_2 \oplus Z_n = \langle\tilde{x}\rangle \oplus \langle\tilde{y}\rangle$ and let x and y belong to $\pi^{-1}(\tilde{x})$ and $\pi^{-1}(\tilde{y})$ respectively. Then G is generated by x, y and ρ which satisfy the relations $x^2\rho^\alpha = y^n\rho^\beta = [x,y]\rho^\gamma = \rho^2 = [\rho,x] = [\rho,y] = 1$ for some $\alpha,\beta,\gamma \in \{0,1\}$. Clearly for any values of these parameters the group G has order $4n$.

Case e. Here $n = 2$ and $v_k = (x\rho^{k_1}, x\rho^{k_2}, x\rho^{k_3}, y\rho^{k_4}, yx\rho^{k_5}, \rho, \overset{t}{\ldots}, \rho)$. Since $x^2 = \theta(x_i)^2 = \rho^{r_i}$ for $i = 1,2,3$, it follows that $r_1 = r_2 = r_3 = \alpha$. Furthermore $\rho^{r_4} = \theta(x_4)^2 = y^2$ and $\rho^{r_5} = (yx)^2 = x^2y^2\rho^\gamma = \rho^{r_1+r_4+\gamma}$ imply that $\beta = r_4$ and $\gamma \equiv r_1 + r_4 + r_5$ (2).

Using pairs $\Phi_{p,q}$ and Ψ_i for $i = 1,2$, we can show that v_k is equivalent to $v_0 = (x,x,x,y,yx\rho^{\beta+t}, \rho, \overset{t}{\ldots}, \rho)$ or $v_1 = (x\rho, x, x, y, yx\rho^{\beta+t+1}, \rho, \overset{t}{\ldots}, \rho)$. Moreover for $\gamma = 1$, a pair (ψ,φ) induces the equivalence of v_0 and v_1, where

φ and ψ are defined by the assignments $\varphi(x) = x^{-1}\rho$, $\varphi(y) = y^{-1}, \varphi(\rho) = \rho$ and $\psi(x_1) = x_1^{-1}$, $\psi(x_2) = x_4^{-1}x_3^{-1}x_4$, $\psi(x_3) = x_4^{-1}x_2^{-1}x_4$, $\psi(x_4) = x_4^{-1}$, $\psi(x_5) = x_1^{-1}x_5^{-1}x_1$, $\psi(x_6) = x_1^{-1}x_5x_r^{-1}x_5^{-1}x_1, \ldots, \psi(x_r) = x_1^{-1}x_5x_6^{-1}x_5^{-1}x_1$. We shall show that they are not equivalent for $\gamma = 0$. For, assume that a pair (ψ', φ') induces the equivalence of epimorphisms θ_0 and θ_1 corresponding to v_0 and v_1 respectively. Then by (5), $\theta_1(x_{\tilde{\psi}'(i)}) = \varphi'(x)$ for $i \in \{1, 2, 3\}$, which is impossible and so v_1 and v_0 are nonequivalent indeed.

Case m. Now $n = 6$ and $v_k = (x\rho^{k_1}, x^{-1}y\rho^{k_2}, y^{-1}\rho^{k_3}, \rho, \overset{t}{\ldots}, \rho)$. Thus $\alpha = r_1, \beta = r_3$ and since $\rho^{r_2} = (xy)^6 = (x^2y^2\rho^\gamma)^3 = \rho^{r_1+r_3+\gamma}$, it follows that $\gamma \equiv r_1 + r_2 + r_3$ (2). Using pairs $\Phi_{1,0}$ and $\Phi_{0,1}$ we can show v_k is equivalent to $(x, x^{-1}y, y^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$.

Theorem 3.4. *The topological type of the* $(\tilde{D}_2)$*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and generating vectors listed in Table* 5.

Proof. Let G be an automorphism group of 2-hyperelliptic Riemann surface such that $\tilde{G} = \langle \tilde{x}, \tilde{y} : \tilde{x}^4, \tilde{y}^4, \tilde{x}^2\tilde{y}^2, \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}\rangle$ and let x, y belong to $\pi^{-1}(\tilde{x}), \pi^{-1}(\tilde{y})$ respectively. Then x, y and ρ generate G and satisfy the relations $x^4 = \rho^\alpha, y^4 = \rho^\beta, x^2 = y^2\rho^\delta, xyx^{-1} = y^{-1}\rho^\gamma$ for some $\alpha, \beta, \gamma, \delta \in \{0, 1\}$. The pairs $\Phi_{1,0}$ and $\Phi_{0,1}$ induce the equivalence of any generating vector of G and $(x, y, y^{-1}x^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$. Thus by (3), $\alpha = r_1, \beta = r_2$ and the relations $x^2 = y^2\rho^\delta$ and $\rho^{r_3} = \theta(x_3)^4 = (yx)^4 = x^4 = \rho^{r_1}$ imply that $r_1 = r_2 = r_3$. Furthermore since $y^2 = x^2\rho^\delta = x(x^2\rho^\delta)x^{-1} = xy^2x^{-1} = (y^{-1}\rho^\gamma)^2 = y^{-2}$, it follows that $r_2 = \beta = 0$. Consequently Λ has the signature $(0; 4, 4, 4, 2, \overset{(g-3)/4}{\ldots}, 2)$ and G is isomorphic to $G_{\gamma,\delta} = \langle x, y, \rho : x^4, y^4, \rho^2, [x, \rho], [y, \rho], x^2y^2\rho^\delta, xyx^{-1}y\rho^\gamma\rangle$. Using Programme GAP we check that $G_{\gamma,\delta}$ has order 16 for any values of parameters γ, δ. Let us define φ_1 by the assignment $\varphi_1(x) = y, \varphi_1(y) = y^{-1}xy, \varphi_1(\rho) = \rho$. Then $\varphi_1(y)^{-1} = y^{-1}x^{-1}y = x^{-1}(xy^{-1}x^{-1})y = x^{-1}y^2\rho^\gamma = x\rho^{\gamma+\delta} = \varphi_1(x)\varphi_1(y)\varphi_1(x)^{-1}\rho^{\gamma+\delta}$ and $\varphi_1(y)^2 = y^{-1}x^2y = y^2\rho^\delta = \varphi_1(x)^2\rho^\delta$. So it is easy to check that the pair (ψ_1, φ_1) induces the equivalence of actions of $G_{0,1}$ and $G_{1,1}$. Next let us define φ_2 by the assignment $\varphi_2(x) = x, \varphi_2(y) = y^{-1}x^{-1}\rho^t, \varphi_2(\rho) = \rho$. Then $\varphi_2(x)\varphi_2(y)\varphi_2(x)^{-1} = xy^{-1}x^{-2}\rho^t = xy^{-3}\rho^{\delta+t} = xy\rho^{t+\delta} = \varphi_2(y)^{-1}\rho^\delta$ and $\varphi_2(y)^2 = (y^{-1}x^{-1})^2 = x^{-1}(xy^{-1}x^{-1})y^{-1}x^{-1} = x^{-2}\rho^\gamma = \varphi_2(x)^2\rho^\gamma$. Thus a pair (ψ_2, φ_2) induces the equivalence of actions of $G_{0,1}$ and $G_{1,0}$. Finally let us notice that there does not exist any isomorphism $\varphi : G_{1,0} \to G_{0,0}$ since otherwise $\varphi(x) = y^a x^b \rho^c, \varphi(y) = y^d x^e \rho^f$ for some integers in range $0 \le a, c, d, f \le 1$ and $1 \le b, e \le 3$ and so the relation $\rho = \varphi(\rho) = \varphi(x^2y^2)$ implies that $\rho \in \langle x, y\rangle$, a contradiction. Consequently there are two nonequivalent actions represented by $G_{0,0}$ and $G_{1,0}$.

Theorem 3.5. *The topological type of the* $(D_{n,m,k})$*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and generating vectors listed in Table* 6.

Proof. Let G be an automorphism group of 2-hyperelliptic Riemann surface such that $\tilde{G} = \langle \tilde{x}, \tilde{y} : \tilde{x}^n = \tilde{y}^m = 1, \tilde{x}\tilde{y}\tilde{x}^{-1} = \tilde{y}^k\rangle$ and let x, y belong to $\pi^{-1}(x)$ and $\pi^{-1}(\tilde{y})$ respectively. Then x, y and ρ generate G and satisfy the relations $x^n\rho^\alpha = y^m\rho^\beta = \rho^2 = [x, \rho] = [y, \rho] = 1$ and $xyx^{-1} = y^k\rho^\gamma$ for some $\alpha, \beta, \gamma \in \{0, 1\}$.

Case n. Here $n = 4, m = 3, k = -1$ and the generating vector of G has a form $v_k = (y\rho^{k_1}, (xy)^{-1}\rho^{k_2}, x\rho^{k_3}, \rho, \overset{t}{\ldots}, \rho)$. By (3), $\beta \equiv k_1 + r_1$ (2). Since $\Phi_{0,1}$ and $\Phi_{1,0}$ induce the equivalence of two actions with different values of β and k_3 respectively, we can assume that $\beta = 0$ and v_k is equivalent to $(y\rho^{r_1}, (xy)^{-1}\rho^{t+r_1}, x, \rho, \overset{t}{\ldots}, \rho)$. Furthermore the relations $\rho^{r_3} = \theta(x_3)^4 = x^4 = \theta(x_2)^4 = \rho^{r_2}$ and $1 = xy^3x^{-1} = (y^{-1}\rho^\gamma)^3 = \rho^\gamma$ imply that $r_2 = r_3 = \alpha$ and $\gamma = 0$. It is easy to see that for such parameters α, β, γ, the group G has order 24 and so it actually define $(D_{n,m,k})$-action.

Case p. Now $n = 2, m = 8, k = 3$ and the generating vector v_k of G is equal to $(x\rho^{k_1}, (yx)^{-1}\rho^{k_2}, y\rho^{k_3}, \rho \overset{t}{\ldots}, \rho)$. Using pairs $\Phi_{0,1}$ and $\Phi_{1,0}$ we can show that v_k is equivalent to $(x,(yx)^{-1}\rho^t,y,\rho,\overset{t}{\ldots},\rho)$. By (3), $\alpha=r_1,\beta=r_3$ and $r_2=r_3$ since $\rho^{r_2}=\theta(x_2)^2=(yx)^4=y^8=\rho^{r_3}$. Furthermore the relation $y=x^2yx^{-2}=x(y^3\rho^\gamma)x^{-1}=(xyx^{-1})^3\rho^\gamma=y^9$ implies that $y^8=1$ and so $\beta=r_2=0$. Thus Λ has one of signatures $(0;2,4,8,2^{(g-3)/8})$ or $(0;4,4,8,2^{(g-7)/8})$ and G is isomorphic to $G^2_\gamma=\langle x,y,\rho:\rho^2,x^2,y^8,[x,\rho],[y,\rho],xyx^{-1}y^5\rho^\gamma\rangle$ or $G^4_\gamma=\langle x,y:x^4,y^8,xyx^{-1}y^5x^{2\gamma},[x^2,y]\rangle$ respectively. Using Programme GAP we check that G^2_γ and G^4_γ have order 32 for arbitrary γ. Suppose that there exists an isomorphism $\varphi:G^i_1\to G^i_0$ for $i=2$ or 4. Then $\varphi(y)=y^ax^b\rho^c$ for some integers in range $1\le a\le 8$ and $0\le b,c\le 1$. If $b=1$ then $\varphi(y)^4=1$ or ρ, which is impossible while for $b=1$, the relation $\varphi(x)\varphi(y)\varphi(x)^{-1}=\varphi(y)^3\rho$ is not satisfied for any $\varphi(x)$. Thus the actions of G^i_1 and G^i_0 are not equivalent.

Theorem 3.6. *The topological type of the $(GL_2(3))$-action on 2-hyperelliptic Riemann surface of genus $g>9$ is determined by a group of automorphisms, branching data and generating vectors listed in Table 7.*

Proof. Let G be an automorphism group of 2-hyperelliptic Riemann surface such that $\tilde{G}=\langle\tilde{x},\tilde{y}:\tilde{x}^2,\tilde{y}^3,(\tilde{x}\tilde{y})^8,((\tilde{x}\tilde{y})^4x)^2\rangle$ and let x,y belong to $\pi^{-1}(\tilde{x})$ and $\pi^{-1}(\tilde{y})$ respectively. Then G is generated by x,y,ρ which satisfy the relations $x^2\rho^\alpha=y^3\rho^\beta=(xy)^8\rho^\gamma=\rho^2=[x,\rho]=[y,\rho]=(xy)^4x(xy)^4x\rho^\delta=1$ for some $\alpha,\beta,\gamma,\delta\in\{0,1\}$. The generating vector of G has a form $v_k=(x\rho^{k_1},y\rho^{k_2},(xy)^{-1}\rho^{k_3},\rho,\overset{t}{\ldots},\rho)$. Thus by (3), $\alpha=r_1,\gamma=r_3$ and $\beta\equiv k_2+r_2$ (2). The pair $\Phi_{0,1}$ induces the equivalence of two actions corresponding to different values of β and therefore we can assume that $\beta=0$ and $k_2=r_2$. Using in addition the pair $\Phi_{1,0}$ it easy to prove that v_k is equivalent to $(x,y\rho^{r_2},(xy)^{-1}\rho^{t+r_2},\rho,\overset{t}{\ldots},\rho)$. Moreover the relation $(xy)^4x(xy)^4x\rho^\delta$ implies $(xy)^4y(xy)^{-4}=y\rho^{\delta+r_3+r_1}$ and so $\delta\equiv r_1+r_3$ (2). We check that the group G has the required order 96 only for $\gamma=0$.

Theorem 3.7. *The topological type of the* (D_n)*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and generating vectors listed in Table* 8.

Proof. Let G be a group of automorphisms of 2-hyperelliptic Riemann surface such that $\tilde{G} = \langle \tilde{x}, \tilde{y} : \tilde{x}^2 = \tilde{y}^n = 1, \tilde{x}\tilde{y}\tilde{x}^{-1} = \tilde{y}^{-1} \rangle$ and let x and y belong to $\pi^{-1}(\tilde{x})$ and $\pi^{-1}(\tilde{y})$ respectively. Then G is generated by x, y and ρ which satisfy the relations $x^2\rho^\alpha = y^n\rho^\beta = \rho^2 = [\rho, x] = [\rho, y] = [x, y]\rho^\gamma = 1$ for some $\alpha, \beta, \gamma \in \{0, 1\}$.

Case h.2. Here $n = 3$ and $v_k = (x\rho^{k_1}, x\rho^{k_2}, y\rho^{k_3}, y^{-1}\rho^{k_4}, \rho, .\overset{t}{.}., \rho)$. Since $\rho^\beta = xy^3x^{-1} = y^{-3}\rho^\gamma = \rho^{\beta+\gamma}$, it follows that $\gamma = 0$. Furthermore by (3), $r_1 = r_2 = \alpha$ and $k_i \equiv \beta + r_i$ (2) for $i = 3, 4$. We can assume that $\beta = 0$ and $k_i = r_i$ for $i = 3, 4$ since the pair $\Phi_{0,1}$ induces the equivalence of actions corresponding to different values of β. Using in addition the pair $\Phi_{1,0}$ we can show that v_k is equivalent to $(x\rho^{t+\alpha+r_3+r_4}, x, y\rho^{r_3}, y^{-1}\rho^{k_4}, \rho, .\overset{t}{.}., \rho)$.

Case k. Now $n = 4$ and $v_k = (x\rho^{k_1}, xy\rho^{k_2}, y^2\rho^{k_3}, y\rho^{k_4}, \rho, .\overset{t}{.}., \rho)$. Thus $\alpha = r_1, \beta = r_4 = r_3$ and since $\rho^{r_2} = (xy)^2 = (xyx^{-1})x^2y = \rho^{r_1+\gamma}$, it follows that $\gamma \equiv r_1 + r_2$ (2). Using pairs $\Phi_{p,q}$, we can assume that $k_1 = k_2 = 0$. Moreover for $\beta = 1$, a pair (ψ, φ) induces the equivalence of two vectors with different values of k_3, where ψ and φ are defined by the assignments $\psi(x_1) = x_4x_1^{-1}x_4^{-1}, \psi(x_2) = x_2^{-1}, \psi(x_3) = x_2x_3^{-1}x_2^{-1}, \psi(x_4) = x_4^{-1}, \psi(x_5) = x_4x_1^{-1}x_r^{-1}x_1x_4^{-1}, \ldots, \psi(x_r) = x_4x_1^{-1}x_5^{-1}x_1x_4^{-1}$ and $\varphi(y) = y^{-1}\rho, \varphi(x) = yx^{-1}y^{-1}$ respectively. We shall show that for $\beta = 0$ two vectors with different values of k_3 cannot be equivalent. For, suppose that a pair (ψ', φ') induces the equivalence of v_k and v_l corresponding to exponents k_i and l_i respectively. Then $\tilde{\psi}'(4) = 4$ since otherwise by (4), we obtain that $\varphi'(y)^2$ is equal to 1 or ρ which is impossible. Thus $\varphi'(y) = y^p\rho^q$ for $p = 1$ or -1 and $q = 0$ or 1 and so applying (4) for $i = 3$, we obtain that $k_3 = l_3$. Consequently there is one action represented by generating vector $(x, xy, y^2, y^{5+4(r_1+t)}, y^4, .\overset{t}{.}., y^4)$ or two nonequivalent actions represented by vectors $(x, xy, y^2, y\rho^{r_1+t}, \rho, \overset{t}{\ldots}, \rho)$ and $(x, xy, y\rho, y\rho^{r_1+t+1}, \rho, \overset{t}{\ldots}, \rho)$ according to $r_3 = 1$ or $r_3 = 0$.

Case o. Here $n = 6$ and $v_k = (x\rho^{k_1}, xy\rho^{k_2}, y^3\rho^{k_3}, y^2\rho^{k_4}, \rho, \overset{t}{\ldots}, \rho)$. Thus $\alpha = r_1, \beta = r_3$ and since $\rho^{r_2} = \theta(x_2)^2 = \rho^{r_1+\gamma}$, it follows that $\gamma \equiv r_1 + r_2$ (2). Moreover $\rho^{r_4} = \theta(x_4)^3 = \rho^{r_3+k_4}$ implies $k_4 \equiv r_3 + r_4$ (2). Using pairs $\Phi_{i,j}$ we can show the equivalence of v_k and $(x\rho^{t+r_1+r_4}, xy, y^3, y^2\rho^{r_3+r_4}, \rho, \overset{t}{\ldots}, \rho)$.

We check that all groups determined in theorem have order $4n$ for $n = 3, 4, 6$ respectively.

Theorem 3.8. *The topological type of the* $(Z_2 \ltimes (Z_2 \oplus Z_2 \oplus Z_3))$*-action on* 2*-hyperelliptic Riemann surface of genus* $g > 9$ *is determined by a group of automorphisms, branching data and generating vectors listed in Table* 9.

Proof. Let G be an automorphism group of 2-hyperelliptic Riemann suthace such that $\tilde{G} = \langle \tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} : \tilde{x}^2 = \tilde{y}^2 = \tilde{z}^2 = \tilde{w}^3 = [\tilde{y}, \tilde{z}], [\tilde{z}, \tilde{w}] = [\tilde{y}, \tilde{w}] = 1, \tilde{x}\tilde{y}\tilde{x}^{-1} = \tilde{y}, \tilde{x}\tilde{z}\tilde{x}^{-1} = \tilde{z}\tilde{y}, \tilde{x}\tilde{w}\tilde{x}^{-1} = \tilde{w}^{-1}\rangle$ and let x, y, z, w belong to $\pi^{-1}(\tilde{x}), \pi^{-1}(\tilde{y}), \pi^{-1}(\tilde{z})$ and $\pi^{-1}(\tilde{w})$ respectively. Then x, y, z, w and ρ generate G and satisfy the relations $\rho^2 = [x, \rho] = [y, \rho] = [z, \rho] = [w, \rho] = x^2\rho^\alpha = y^2\rho^\beta = z^2\rho^\gamma = w^3\rho^\delta = [y, z]\rho^{\alpha_1} = [z, w]\rho^{\alpha_2} = [y, w]\rho^{\alpha_3} = 1$, $xyx^{-1} = y\rho^{\beta_1}, xzx^{-1} = zy\rho^{\beta_2}, xwx^{-1} = w^{-1}\rho^{\beta_3}$ for some $\alpha, \beta, \gamma, \delta, \alpha_i, \beta_i \in \{0, 1\}$. A generating vector of G has a form $v_k = (x\rho^{k_1}, (zwx)^{-1}\rho^{k_2}, zw\rho^{k_3}, \rho, \overset{t}{\ldots}, \rho)$. Let an automorphism $\varphi_{i,j,k,l}$ of G be defined by $\varphi_{i,j,k,l}(x) = x\rho^i, \varphi_{i,j,k,l}(y) = y\rho^j, \varphi_{i,j,k,l}(z) = z\rho^k, \varphi_{i,j,k,l}(w) = w\rho^l, \varphi_{i,j,k,l}(\rho) = \rho$ and $\Phi_{i,j,k,l}$ denotes a pair $(\mathrm{id}_\Lambda, \varphi_{i,j,k,l})$. Then the pairs $\Phi_{1,0,0,0}$ and $\Phi_{0,0,1,0}$ induce the equivalence of v_k and $(x, (zwx)^{-1}, zw\rho^t, \rho, \overset{t}{\ldots}, \rho)$. Since the pairs $\Phi_{0,0,0,1}$ and $\Phi_{0,1,0,0}$ induce the equivalence of actions corresponding to different values of δ and β_2 respectively, we shall assume that $\delta = \beta_2 = 0$. The remaining parameters except β_2 can be calculated from given branching data. Indeed by (3), $\alpha = r_1, \beta = r_2$ and $\gamma = r_3$. Furthermore $1 = w^3 = zw^3z^{-1} = (w^{-1}\rho^{\alpha_2})^3 = \rho^{\alpha_2}$ implies $\alpha_2 = 0$ and similarly we can argue that $\alpha_3 = \beta_3 = 0$ either.

Finally raising the equation $xyzx^{-1} = z\rho^{\beta_1+r_2}$ to second power we obtain $\rho^{r_2+r_3+\alpha_1} = x(y^2z^2\rho^{\alpha_1})x^{-1} = z^2 = \rho^{r_3}$ which implies $\alpha_1 = r_2$ and similarly from the equality $x(zwx)x^{-1} = xzw$ we obtain $\beta_1 = r_2$. Thus we found all parameters corresponding to given brunching data. Furthermore for any such parameters G has the required order 48.

4. Full actions on 2-hyperelliptic Riemann surfaces.

Theorem 4.1. *A group G is the full group of automorphisms of some 2-hyperelliptic Riemann surface of genus $g > 9$ if and only if G is one of the groups listed in Tables* 2, 3, 4, 5, 6, 7, 8, 9 *and the pair* (G, g) *is different from* $(s.3, 11)$ *and* $(s.6, 14)$.

Proof. Let $G = \Lambda/\Gamma$ be a group of automorphisms of 2-hyperelliptic Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g > 9$. If the signature τ of Λ does not appear in the first column of the tables 1.5.1 or 1.5.2 in [18] then Λ can be chosen to be a maximal [18] and so G can be assumed to be full group of automorphisms of X. In the other case Λ is always contained in an NEC group Λ' and signatures τ' of such groups are given in the second column of the corresponding row. For $g > 9$, we have only four such cases given in the following table.

τ	Position τ	g	τ'	Position τ'	g'
$(2,2,12,12)$	$m.3, t=1$	11	$(2,2,2,12)$	none	
$(4,4,12)$	$r.7, t=0$	11	$(2,4,24)$	none	
$(6,6,4)$	$s.3, t=0$	11	$(2,6,8)$	$t.2, t=0$	11
$(6,6,8)$	$s.6, t=0$	14	$(2,6,16)$	$t.4, t=0$	14

The two first rows provide full automorphism groups since τ' does not correspond to any group acting on 2-hyperelliptic surface. So assume that Λ has one of signatures $(6,6,4)$ or $(6,6,8)$. We shall prove that for every epimorphism $\theta : \Lambda \to G$ whose kernel Γ has signature $(g; -)$ and $\mathcal{H}/\Gamma$ is 2-hyperelliptic, there exists a Fuchsian group Λ', a group G', group embeddings $i : \Lambda \hookrightarrow \Lambda'$, $j : G \hookrightarrow G'$ and an epimorphism $\theta' : \Lambda' \to G'$ such that $1 \neq [\Lambda' : \Lambda] = [G : G']$ and $\theta' \cdot i = j \cdot \theta$. Then $G = \Lambda/\Gamma \subsetneq \Lambda'/\Gamma = G' \subseteq \mathrm{Aut}(X)$, for all elliptic-hyperelliptic surfaces X of genus g on which G acts as a group of automorphisms.

First assume that $\tau = [6, 6, 4]$ and $\tau' = [2, 6, 8]$. Let Λ' be a Fuchsian group with the signature τ' containing Λ and let y_1, y_2, y_3 be its canonical generators. Clearly $x_1 = y_2, x_2 = y_3y_2y_3^{-1}$ and $x_3 = y_3^2$ belong to Λ and have orders $6, 6$ and 4 respectively. Moreover it is easy to see that $x_1x_2x_3 = 1$ and x_1, x_2, x_3 generate the normal subgroup of Λ' of index 2. So x_1, x_2, x_3 form a system of canonical generators for Λ and consequently, the embedding $i : \Lambda \hookrightarrow \Lambda'$ is induced by the assignment:

$$i(x_1) = y_2, i(x_2) = y_3y_2y_3^{-1}, i(x_3) = y_3^2.$$

By Theorems 3.2 and 3.6, G and G' have the presentations $s.3$ and $t.2$ respectively. It is easy to check that the assignment

$$j(x) = y, j(y) = (xy)^6$$

gives a group monomorphism $j : G \to G'$ consistent with an epimorphisms $\theta : \Lambda \to G$ and $\theta' : \Lambda' \to G'$ given in above theorems.

The case of a Fuchsian group Λ with the signature $\tau = (6, 6, 8)$ is similar. Here $\tau' = (2, 6, 16)$ appears in Theorem 3.6 and so G' has presentation $t.4$. Now the embedding $i : \Lambda \hookrightarrow \Lambda'$ may be induced by the assignment:

$$i(x_1) = y_1y_2y_1, i(x_2) = y_2, i(x_3) = y_3^2$$

and a group monomorphism $j : G \to G'$ for which $\theta' i = j\theta$ is defined by

$$j(x) = xyx, j(y) = (xy)^{-2}.$$

Case	Presentation of G	Branching data	t	Generating vectors
a.1	$\langle x : x^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(1 : 2^2, 2^t)$	$g-3$	$(1, 1, x, x\rho^t, \rho, \overset{t}{\ldots}, \rho)$
a.2	$\langle x : x^4\rangle$	$(1 : 4^2, 2^t)$	$g-4$	$(1, 1, x, x^{3+2t}, x^2, \overset{t}{\ldots}, x^2)$
b.1	$\langle x : x^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^6, 2^t)$	$g-3$	$(x, x, x, x, x, x\rho^t, \rho, \overset{t}{\ldots}, \rho)$,$(x, x, x, x\rho, x\rho, x\rho^t, \rho, \overset{t}{\ldots}, \rho)$
b.2	$\langle x : x^4\rangle$	$(4^6, 2^t)$	$g-6$	$(x, x, x, x, x, x^{3+2t}, x^2, \overset{t}{\ldots}, x^2)$,$(x, x, x, x^3, x^3, x^{3+2t}, x^2, \overset{t}{\ldots}, x^2)$
c.1	$\langle x : x^3\rangle \oplus \langle \rho : \rho^2\rangle$	$(3^4, 2^t)$	$(2g-6)/3$	$(x, x, x^{-1}, x^{-1}, \rho, \overset{t}{\ldots}, \rho)$
c.2	$\langle x : x^3\rangle \oplus \langle \rho : \rho^2\rangle$	$(6, 3^3, 2^t)$	$(2g-7)/3$	$(x\rho, x, x^{-1}, x^{-1}, \rho, \overset{t}{\ldots}, \rho)$
c.3	$\langle x : x^3\rangle \oplus \langle \rho : \rho^2\rangle$	$(6^2, 3^2, 2^t)$	$(2g-8)/3$	$(x\rho, x\rho, x^{-1}, x^{-1}, \rho, \overset{t}{\ldots}, \rho)$
c.4	$\langle x : x^3\rangle \oplus \langle \rho : \rho^2\rangle$	$(6^3, 3, 2^t)$	$(2g-9)/3$	$(x\rho, x\rho, x^{-1}\rho, x^{-1}, \rho, \overset{t}{\ldots}, \rho)$
c.5	$\langle x : x^3\rangle \oplus \langle \rho : \rho^2\rangle$	$(6^4, 2^t)$	$(2g-10)/3$	$(x\rho, x\rho, x^{-1}\rho, x^{-1}\rho, \rho, \overset{t}{\ldots}, \rho)$
d.1	$\langle x : x^4\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 4^2, 2^t)$	$(g-3)/2$	$(x^2, x^2, x, x^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$, $(x^2\rho, x^2, x, x^{-1}\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$ $(x^2\rho, x^2\rho, x, x^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$
d.2	$\langle x : x^8\rangle$	$(4^2, 8^2, 2^t)$	$(g-6)/2$	$(x^2, x^2, x, x^{3+4t}, x^4, \overset{t}{\ldots}, x^4)$, $(x^6, x^2, x, x^{7+4t}, x^4, \overset{t}{\ldots}, x^4)$
f.1	$\langle x : x^5\rangle \oplus \langle \rho : \rho^2\rangle$	$(5^3, 2^t)$	$(2g-6)/5$	$(x, x, x^3, \rho, \overset{t}{\ldots}, \rho)$
f.2	$\langle x : x^5\rangle \oplus \langle \rho : \rho^2\rangle$	$(10, 5^2, 2^t)$	$(2g-7)/5$	$(x\rho, x, x^3, \rho, \overset{t}{\ldots}, \rho)$
f.3	$\langle x : x^5\rangle \oplus \langle \rho : \rho^2\rangle$	$(10^2, 5, 2^t)$	$(2g-8)/5$	$(x\rho, x\rho, x^3, \rho, \overset{t}{\ldots}, \rho)$
f.4	$\langle x : x^5\rangle \oplus \langle \rho : \rho^2\rangle$	$(10^3, 2^t)$	$(2g-9)/5$	$(x\rho, x\rho, x^3\rho, \rho, \overset{t}{\ldots}, \rho)$
g.1	$\langle x : x^6\rangle \oplus \langle \rho : \rho^2\rangle$	$(3, 6^2, 2^t)$	$(g-3)/3$	$(x^4, x, x\rho^t, \rho, \overset{t}{\ldots}, \rho)$
g.2	$\langle x : x^6\rangle \oplus \langle \rho : \rho^2\rangle$	$(6^3, 2^t)$	$(g-4)/2$	$(x^4\rho, x, x\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
g.3	$\langle x : x^{12}\rangle$	$(3, 12^2, 2^t)$	$(g-4)/3$	$(x^4, x, x^{7+6t}, x^6, \overset{t}{\ldots}, x^6)$
g.4	$\langle x : x^{12}\rangle$	$(6, 12^2, 2^t)$	$(g-5)/3$	$(x^{10}, x, x^{1+6t}, x^6, \overset{t}{\ldots}, x^6)$

Table 2

Case	Presentation of G	Branching data	t	Generating vectors
h.a.1	$\langle x : x^6\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 3^2, 2^t)$	$(g-3)/3$	$(x^3\rho^t, x^3, x^2, x^4, \rho, \overset{t}{\ldots}, \rho)$
h.a.2	$\langle x : x^6\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 6, 3, 2^t)$	$(g-4)/3$	$(x^3\rho^{t+1}, x^3, x^2\rho, x^4, \rho, \overset{t}{\ldots}, \rho)$
h.a.3	$\langle x : x^6\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 6^2, 2^t)$	$(g-5)/3$	$(x^3\rho^t, x^3, x^2\rho, x^4\rho, \rho, \overset{t}{\ldots}, \rho)$
h.a.4	$\langle x : x^{12}\rangle$	$(4^2, 6^2, 2^t)$	$(g-8)/3$	$(x^{3+6t}, x^9, x^2, x^{10}, x^6, \overset{t}{\ldots}, x^6)$
h.a.5	$\langle x : x^{12}\rangle$	$(4^2, 6, 3, 2^t)$	$(g-7)/3$	$(x^{3+6t}, x^3, x^2, x^4, x^6, \overset{t}{\ldots}, x^6)$
h.a.6	$\langle x : x^{12}\rangle$	$(4^2, 3^2, 2^t)$	$(g-6)/3$	$(x^{3+6t}, x^9, x^8, x^4, \rho, \overset{t}{\ldots}, \rho)$
i.1	$\langle x : x^8\rangle \oplus \langle \rho : \rho^2\rangle$	$(2, 8^2, 2^t)$	$(g-3)/4$	$(x^4, x^3, x\rho^t, \rho, \overset{t}{\ldots}, \rho)$,$(x^4\rho, x^3, x\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
i.2	$\langle x : x^{16}\rangle$	$(4, 16^2, 2^t)$	$(g-6)/4$	$(x^4, x^3, x^{9+8t}, x^8, \overset{t}{\ldots}, x^8)$
l.1	$\langle x : x^{10}\rangle \oplus \langle \rho : \rho^2\rangle$	$(2, 5, 10, 2^t)$	$(g-3)/5$	$(x^5, x^4, x\rho^t, \rho, \overset{t}{\ldots}, \rho)$
l.2	$\langle x : x^{10}\rangle \oplus \langle \rho : \rho^2\rangle$	$(2, 10, 10, 2^t)$	$(g-4)/5$	$(x^5, x^4\rho, x\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
l.3	$\langle x : x^{20}\rangle$	$(4, 5, 20, 2^t)$	$(g-6)/5$	$(x^5, x^4, x^{11+10t}, x^{10}, \overset{t}{\ldots}, x^{10})$
l.4	$\langle x : x^{20}\rangle$	$(4, 10, 20, 2^t)$	$(g-7)/5$	$(x^5, x^{14}, x^{1+10t}, x^{10}, \overset{t}{\ldots}, x^{10})$

Table 2 (continued)

Case	Presentation of G	Branching data	t	Generating vectors
s.1	$\langle x, y : x^3, y^4, (yx)^3, xy^2x^{-1}y^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(3^2, 4, 2^t)$	$(g-3)/12$	$(x, (yx)^{-1}, y\rho^t, \rho, \overset{t}{\ldots}, \rho)$
s.2	$\langle x, y : x^3, y^4, (yx)^3, xy^2x^{-1}y^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(3, 6, 4, 2^t)$	$(g-7)/12$	$(x, (yx)^{-1}\rho, y\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
s.3	$\langle x, y : x^3, y^4, (yx)^3, xy^2x^{-1}y^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(6^2, 4, 2^t)$	$(g-11)/12$	$(x\rho, (yx)^{-1}\rho, y\rho^t, \rho, \overset{t}{\ldots}, \rho)$

Table 3

Case	Presentation of G	Branching data	t	Generating vectors
e.1	$\langle x : x^2\rangle \oplus \langle y : y^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^5, 2^t)$	$(g-3)/2$	$(x, x, x, y, yx\rho^t, \rho, \overset{t}{\ldots}, \rho)$,$(x\rho, x, x, y, yx\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
e.2	$\langle x, y, \rho : x^2, y^2, \rho^2, [x, \rho], [y, \rho], [x, y]\rho\rangle$	$(2^4, 4, 2^t)$	$(g-4)/2$	$(x, x, x, y, yx\rho^t, \rho, \overset{t}{\ldots}, \rho)$,$(x\rho, x, x, y, yx\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
e.3	$\langle x : x^2\rangle \oplus \langle y : y^4\rangle$	$(2^3, 4^2, 2^t)$	$(g-5)/2$	$(x, x, x, y, y^{3+2t}x, y^2, \overset{t}{\ldots}, y^2)$,$(xy^2, x, x, y, y^{1+2t}x, y^2, \overset{t}{\ldots}, y^2)$
e.4	$\langle x, y : x^4, y^2, yxy^{-1}x\rangle$	$(4^3, 2^{2+t})$	$(g-6)/2$	$(x, x, x, y, yx^{1+2t}, x^2, \overset{t}{\ldots}, x^2)$,$(x^3, x, x, y, yx^{3+2t}, x^2, \overset{t}{\ldots}, x^2)$
e.5	$\langle x, y : x^4, x^2y^2, [x, y]\rangle$	$(4^4, 2^{t+1})$	$(g-7)/2$	$(x, x, x, y, y^3x^{1+2t}, x^2, \overset{t}{\ldots}, x^2)$,$(x^3, x, x, y, yx^{1+2t}, x^2, \overset{t}{\ldots}, x^2)$
e.6	$\langle x, y : x^4, x^2y^2, xyx^{-1}y\rangle$	$(4^5, 2^t)$	$(g-8)/2$	$(x, x, x, y, y^3x^{1+2t}, x^2, \overset{t}{\ldots}, x^2)$,$(x^3, x, x, y, yx^{1+2t}, x^2, \overset{t}{\ldots}, x^2)$
m.1	$\langle x, y, \rho : x^2, y^6, \rho^2, [x, y], [x, \rho], [y, \rho]\}$	$(2, 6^2, 2^t)$	$(g-3)/6$	$(x, xy, y^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$
m.2	$\langle x, y, \rho : x^2, y^6, \rho^2, [x, \rho], [y, \rho], [x, y]\rho\rangle$	$(2, 12, 6, 2^t)$	$(g-4)/6$	$(x, xy, y^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$
m.3	$\langle x, y : x^2, y^{12}, [x, y]\rangle$	$(2, 12^2, 2^t)$	$(g-5)/6$	$(x, xy, y^{-1+6t}, y^6, \overset{t}{\ldots}, y^6)$
m.4	$\langle x, y : x^4, y^6, yxy^{-1}x\rangle$	$(4, 6^2, 2^t)$	$(g-6)/6$	$(x^1, x^3y, y^{-1}x^{2t}, x^2, \overset{t}{\ldots}, x^2)$
m.5	$\langle x^4, y^6, [x, y]\rangle$	$(4, 12, 6, 2^t)$	$(g-7)/6$	$(x^1, x^3y, y^{-1}x^{2t}, x^2, \overset{t}{\ldots}, x^2)$
m.6	$\langle x, y : x^4, x^2y^6, xyx^{-1}y^5\rangle$	$(4, 12^2, 2^t)$	$(g-8)/6$	$(x^1, x^3y, y^{-1}x^{2t}, x^2, \overset{t}{\ldots}, x^2)$

Table 4

Case	Presentation of G	Branching data	t	Generating vectors
j.1	$\langle x, y, \rho : x^4, y^4, \rho^2, [x, \rho], [y, \rho], x^2y^2, xyx^{-1}y\rangle$	$(4^3, 2^t)$	$(g-3)/4$	$(x, y, y^{-1}x^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$
j.2	$\langle x, y, \rho : x^4, y^4, \rho^2, [x, \rho], [y, \rho], x^2y^2, xyx^{-1}y\rho\rangle$	$(4^3, 2^t)$	$(g-3)/4$	$(x, y, y^{-1}x^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$

Table 5

Case	Presentation of G	Branching data	t	Generating vectors
n.1	$\langle x, y : x^4, y^3, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(3, 4^2, 2^t)$	$(g-3)/6$	$(y, (xy)^{-1}\rho^t, x, \rho, \overset{t}{\ldots}, \rho)$
n.2	$\langle x, y : x^4, y^3, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(6, 4^2, 2^t)$	$(g-5)/6$	$(y\rho, (xy)^{-1}\rho^{t+1}, x, \rho, \overset{t}{\ldots}, \rho)$
n.3	$\langle x, y : x^8, y^3, xyx^{-1}y\rangle$	$(3, 8^2, 2^t)$	$(g-6)/6$	$(y, y^{-1}x^{4t-1}, x, x^4, \overset{t}{\ldots}, x^4)$
n.4	$\langle x, y : x^8, y^3, xyx^{-1}y\rangle$	$(6, 8^2, 2^t)$	$(g-8)/6$	$(x^4y, y^{-1}x^{4t+3}, x, x^4, \overset{t}{\ldots}, x^4)$
p.1	$\langle x, y : x^2, y^8, xyx^{-1}y^5\rangle \oplus \langle \rho : \rho^2\rangle$	$(2, 4, 8, 2^t)$	$(g-3)/8$	$(x, xy^7\rho^t, y, \rho, \overset{t}{\ldots}, \rho)$
p.2	$\langle x, y : x^2, y^8, \rho^2, [x,\rho], [y,\rho], xyx^{-1}y^5\rho\rangle$	$(2, 4, 8, 2^t)$	$(g-3)/8$	$(x, xy^7\rho^t, y, \rho, \overset{t}{\ldots}, \rho)$
p.3	$\langle x, y : x^4, y^8, xyx^{-1}y^5\rangle$	$(4^2, 8, 2^t)$	$(g-7)/8$	$(x, x^3y^7, y, x^2, \overset{t}{\ldots}, x^2)$
p.4	$\langle x, y : x^4, y^8, xyx^{-1}y^5, [x^2, y]\rangle$	$(4^2, 8, 2^t)$	$(g-7)/8$	$(x, x^3y^7, y, x^2, \overset{t}{\ldots}, x^2)$

Table 6

Case	Presentation of G	Branching data	t	Generating vectors
t.1	$\langle x, y : x^2, y^3, (xy)^8, ((xy)^4x)^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(2, 3, 8, 2^t)$	$(g-3)/24$	$(x, y, (xy)^{-1}\rho^t, \rho, \overset{t}{\ldots}, \rho)$
t.2	$\langle x, y : x^2, y^3, (xy)^8, ((xy)^4x)^2\rangle \oplus \langle \rho : \rho^2\rangle$	$(2, 6, 8, 2^t)$	$(g-11)/24$	$(x, y\rho, (xy)^{-1}\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
t.3	$\langle x, y : x^4, y^3, (xy)^8, (xy)^4x(xy)^4x^{-1}, x^2yx^2y^{-1}\rangle$	$(4, 3, 8, 2^t)$	$(g-15)/24$	$(x, y, (xy)^{-1}x^{2t}, x^2, \overset{t}{\ldots}, x^2)$
t.4	$\langle x, y : x^4, y^3, (xy)^8, (xy)^4x(xy)^4x^{-1}, x^2yx^2y^{-1}\rangle$	$(4, 6, 8, 2^t)$	$(g-23)/24$	$(x, x^2y, (xy)^{-1}x^{2+2t}, x^2, \overset{t}{\ldots}, x^2)$

Table 7

	Presentation of G	Data	t	Generating vectors
h.2.1	$\langle x, y : x^2, y^3, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 3^2, 2^t)$	$(g-3)/3$	$(x\rho^t, x, y, y^{-1}, \rho, \overset{t}{\ldots}, \rho)$
h.2.2	$\langle x, y : x^2, y^3, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 6, 3, 2^t)$	$(g-4)/3$	$(x\rho^{t+1}, x, y\rho, y^{-1}, \rho, \overset{t}{\ldots}, \rho)$
h.2.3	$\langle x, y : x^2, y^3, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^2, 6^2, 2^t)$	$(g-5)/3$	$(x\rho^t, x, y\rho, y^{-1}\rho, \rho, \overset{t}{\ldots}, \rho)$
h.2.4	$\langle x, y : x^4, y^3, xyx^{-1}y\rangle$	$(4^2, 3^2, 2^t)$	$(g-6)/3$	$(x^{3+2t}, x, y, y^{-1}, x^2, \overset{t}{\ldots}, x^2)$
h.2.5	$\langle x, y : x^4, y^3, xyx^{-1}y\rangle$	$(4^2, 6, 3, 2^t)$	$(g-7)/3$	$(x^{1+2t}, x, x^2y, y^{-1}, x^2, \overset{t}{\ldots}, x^2)$
h.2.6	$\langle x, y : x^4, y^3, xyx^{-1}y\rangle$	$(4^2, 6^2, 2^t)$	$(g-8)/3$	$(x^{3+2t}, x, yx^2, x^2y^{-1}, x^2, \overset{t}{\ldots}, x^2)$
k.1	$\langle x, y : x^2, y^4, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^3, 4, 2^t)$	$(g-3)/4$	$(x, xy, y^2, y\rho^t, \rho, \overset{t}{\ldots}, \rho)$, $(x, xy, y^2\rho, y\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
k.2	$\langle x, y, \rho : x^2, y^4, \rho^2, [x,\rho], [y,\rho], xyx^{-1}y\rho\rangle$	$(2, 4, 2, 4, 2^t)$	$(g-5)/4$	$(x, xy, y^2, y\rho^t, \rho, \overset{t}{\ldots}, \rho)$, $(x, xy, y^2\rho, y\rho^{t+1}, \rho, \overset{t}{\ldots}, \rho)$
k.3	$\langle x, y : x^2, y^8, xyx^{-1}y\rangle$	$(2^2, 4, 8, 2^t)$	$(g-6)/4$	$(x, xy, y^2, y^{5+4t}, y^4, \overset{t}{\ldots}, y^4)$
k.4	$\langle x, y : x^4, y^4, xyx^{-1}y\rangle$	$(4^2, 2, 4, 2^t)$	$(g-7)/4$	$(x, xy, y^2, yx^{2+2t}, x^2, \overset{t}{\ldots}, x^2)$, $(x, xy, y^2x^2, yx^{2t}, x^2, \overset{t}{\ldots}, x^2)$
k.5	$\langle x, y : x^2, y^8, xyx^{-1}y^5\rangle$	$(2, 4^2, 8, 2^t)$	$(g-8)/4$	$(x, xy, y^2, y^{5+4t}, y^4, \overset{t}{\ldots}, y^4)$
k.6	$\langle x, y : x^4, x^2y^4, xyx^{-1}y\rangle$	$(4^3, 8, 2^t)$	$(g-10)/4$	$(x, xy, y^2, y^{1+4t}, y^4, \overset{t}{\ldots}, y^4)$
o.1	$\langle x, y : x^2, y^6, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^3, 3, 2^t)$	$(g-3)/6$	$(x\rho^t, xy, y^3, y^2, \rho, \overset{t}{\ldots}, \rho)$
o.2	$\langle x, y : x^2, y^6, xyx^{-1}y\rangle \oplus \langle \rho : \rho^2\rangle$	$(2^3, 6, 2^t)$	$(g-5)/6$	$(x\rho^{t+1}, xy, y^3, y^2\rho, \rho, \overset{t}{\ldots}, \rho)$
o.3	$\langle x, y : x^2, y^{12}, xyx^{-1}y\rangle$	$(2^2, 4, 3, 2^t)$	$(g-6)/6$	$(y^{6t}x, xy, y^3, y^8, y^6, \overset{t}{\ldots}, y^6)$
o.4	$\langle x, y : x^2, y^{12}, xyx^{-1}y\rangle$	$(2^2, 4, 6, 2^t)$	$(g-8)/6$	$(y^{6(t+1)}x, xy, y^3, y^2, y^6, \overset{t}{\ldots}, y^6)$
o.5	$\langle x, y : x^2, y^{12}, xyx^{-1}y^7\rangle$	$(2, 4^2, 3, 2^t)$	$(g-9)/6$	$(y^{6t}x, xy, y^3, y^8, y^6, \overset{t}{\ldots}, y^6)$
o.6	$\langle x, y : x^2, y^{12}, xyx^{-1}y^7\rangle$	$(2, 4^2, 6, 2^t)$	$(g-11)/6$	$(y^{6(t+1)}x, xy, y^3, y^2, y^6, \overset{t}{\ldots}, y^6)$
o.7	$\langle x, y : x^4, x^2y^6, xyx^{-1}y\rangle$	$(4^3, 3, 2^t)$	$(g-12)/6$	$(x^{3+2t}, xy, y^3, y^2x^2, x^2, \overset{t}{\ldots}, x^2)$
o.8	$\langle x, y : x^4, x^2y^6, xyx^{-1}y\rangle$	$(4^3, 6, 2^t)$	$(g-14)/6$	$(x^{1+2t}, xy, y^3, y^2, x^2, \overset{t}{\ldots}, x^2)$

Table 8

Case	Presentation of G	Branching data	t	Generating vectors
r.1	$\langle x,y,z,w : x^2, y^2, z^2, w^3, [y,w], [z,w], [y,z], [x,y], xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle \oplus \langle \rho : \rho^2\rangle$	$(2,4,6,2^t)$	$(g-3)/12$	$(x\rho^t, (zwx)^{-1}, zw, \rho, \overset{t}{\ldots}, \rho)$
r.2	$\langle x,y,z,w : x^2, y^2, z^4, w^3, [y,w], [z,w], [y,z], [x,y], xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(2,4,12,2^t)$	$(g-5)/12$	$(z^{2t}x, (zwx)^{-1}, zw, z^2, \overset{t}{\ldots}, z^2)$
r.3	$\langle x,y,z,w : x^2, y^4, z^2, w^3, [y,w], [z,w], zyz^{-1}y, xyx^{-1}y, xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(2,8,6,2^t)$	$(g-6)/12$	$(y^{2t}x, (zwx)^{-1}, zw, y^2, \overset{t}{\ldots}, y^2)$
r.4	$\langle x,y,z,w : x^2, y^4, z^2y^2, w^3, [y,w], [z,w], zyz^{-1}y, xyx^{-1}y, xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(2,8,12,2^t)$	$(g-8)/12$	$(y^{2t}x, (zwx)^{-1}, zw, y^2, \overset{t}{\ldots}, y^2)$
r.5	$\langle x,y,z,w : x^4, y^2, z^2, w^3, [y,w], [z,w], [y,z], [x,y], xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(4,4,6,2^t)$	$(g-9)/12$	$(x^{1+2t}, (zwx)^{-1}, zw, x^2, \overset{t}{\ldots}, x^2)$
r.6	$\langle x,y,z,w : x^4, y^2x^2, z^2, w^3, [y,w], [z,w], zyz^{-1}y, xyx^{-1}y, xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(4,8,6,2^t)$	$(g-12)/12$	$(x^{1+2t}, (zwx)^{-1}, zw, x^2, \overset{t}{\ldots}, x^2)$
r.7	$\langle x,y,z,w : x^4, y^2, z^2x^2, w^3, [y,w], [z,w], [z,y], [x,y], xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(4,4,12,2^t)$	$(g-11)/12$	$(x^{1+2t}, (zwx)^{-1}, zw, x^2, \overset{t}{\ldots}, x^2)$
r.8	$\langle x,y,z,w : x^4, y^2x^2, z^2x^2, w^3, [y,w], [z,w], zyz^{-1}y, xyx^{-1}y, xzx^{-1}y^{-1}z^{-1}, xwx^{-1}w\rangle$	$(4,8,12,2^t)$	$(g-14)/12$	$(x^{1+2t}, (zwx)^{-1}, zw, x^2, \overset{t}{\ldots}, x^2)$

Table 9

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