BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 33, Number 4, December 2005

LIMIT THEOREMS FOR ARRAYS OF RATIOS OF ORDER STATISTICS

BY

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Abstract. Let $\{X_{nk}, 1 \leq k \leq m_n, n \geq 1\}$ be independent random variables from the Pareto distribution. Let $X_{n(k)}$ be the k^{th} largest order statistic from the n^{th} row of our array. Then set $R_{nij} = X_{n(j)}/X_{n(i)}$ where $j < i$. This paper establishes limit theorems involving weighted sums from the sequence $\{R_{nij}, n \geq 1\}$.

Consider independent random variables $\{X, X_{nk}, 1 \leq k \leq m_n, n \geq 1\}$ with density $f_X(x) = px^{-p-1}I(x \geq 1)$, where $p > 0$. Let $X_{n(k)}$ be the k^{th} largest order statistic from each row of our array. Hence $X_{n(m_n)} \leq X_{n(m_n-1)} \leq \cdots \leq X_{n(2)} \leq X_{n(1)}$. Next define $R_n = R_{nij} = X_{n(j)}/X_{n(i)}$ where $j < i$ for all $n \geq 1$. Thus the density of R_n is

$$f_{R_n}(r) = \frac{p(i-1)!}{(i-j-1)!(j-1)!}(1-r^{-p})^{i-j-1}r^{-pj-1}I(r \geq 1).$$

It is important to note that the density of R_n is free of m_n. In this paper we will study limit theorems involving weighted sums of $\{R_n, n \geq 1\}$ where $j < i$. If pj exceeds one, then ER_n is finite and the associated theorems are straight forward and unremarkable, see Theorems 6, 7 and 8. If $pj < 1$, then these limit theorems fail to exist, see Theorem 5. The most interesting case

Received by the editors June 26, 2004.

2000 Subject Classification: 60F05, 60F15.

Key words and phrases: Almost sure convergence, strong law of large numbers, weak law of large numbers, generalized law of the iterated logarithm.

of all occurs when $pj = 1$. Strange and unusual limit theorems occur when examining random variables that barely do or do not have a first moment.

As usual, we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. Also we use the constant C to denote a generic real number that is not necessarily the same in each appearance. Our first theorem establishes an unusual strong law where all the variables p, j and i are fixed. Thus in Theorem 1 we are considering i.i.d. Pareto random variables. In all the other theorems we are allowing $i_n \to \infty$.

Theorem 1. *If $pj = 1$ and $\alpha > -2$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{(\lg n)^\alpha}{n} R_n}{(\lg N)^{\alpha+2}} = \frac{\binom{i-1}{j}}{(\alpha+2)} \qquad \textit{almost surely.}$$

Proof. Let $a_n = (\lg n)^\alpha/n$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = b_n/a_n = n(\lg n)^2$. We use the usual Khintchine-Kolmogorov Convergence Theorem argument, see Chow and Teicher(1). We partition our sum into the following three terms:

$$\begin{aligned}\frac{\sum_{n=1}^{N} a_n R_n}{b_N} &= \frac{\sum_{n=1}^{N} a_n [R_n I(1 \le R_n \le c_n) - ER_n I(1 \le R_n \le c_n)]}{b_N} \\ &\quad + \frac{\sum_{n=1}^{N} a_n R_n I(R_n > c_n)}{b_N} + \frac{\sum_{n=1}^{N} a_n ER_n I(1 \le R_n \le c_n)}{b_N}.\end{aligned}$$

The first term vanishes almost surely since

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-2} ER_n^2 I(1 \le R_n \le c_n) &< C\sum_{n=1}^{\infty} c_n^{-2} \int_1^{c_n} dr < C\sum_{n=1}^{\infty} c_n^{-1} \\ &= C\sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.\end{aligned}$$

The second term vanishes almost surely since

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-2} dr = C\sum_{n=1}^{\infty} c_n^{-1} = C\sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

As for the third term

$$\begin{aligned}
& ER_nI(1 \le R_n \le c_n) \\
= \;& \frac{p(i-1)!}{(i-j-1)!(j-1)!}\int_1^{c_n}(1-r^{-p})^{i-j-1}r^{-pj}dr \\
= \;& \binom{i-1}{j}\int_1^{c_n}(1-r^{-p})^{i-j-1}r^{-1}dr \\
= \;& \binom{i-1}{j}\left[\int_1^{c_n}\frac{dr}{r}+\sum_{k=1}^{i-j-1}\binom{i-j-1}{k}(-1)^k\int_1^{c_n}r^{-pk-1}dr\right] \\
= \;& \binom{i-1}{j}\left[\lg c_n + j\sum_{k=1}^{i-j-1}\frac{\binom{i-j-1}{k}(-1)^k}{k}+j\sum_{k=1}^{i-j-1}\frac{\binom{i-j-1}{k}(-1)^{k+1}}{kc_n^{pk}}\right] \\
\sim \;& \binom{i-1}{j}\lg n
\end{aligned}$$

since i and j are fixed and c_n goes to infinity as $n \to \infty$. Thus

$$\frac{\sum_{n=1}^N a_nER_nI(1 \le R_n \le c_n)}{b_N} \sim \frac{\binom{i-1}{j}\sum_{n=1}^N \frac{(\lg n)^{\alpha+1}}{n}}{(\lg N)^{\alpha+2}} \to \frac{\binom{i-1}{j}}{(\alpha+2)}$$

concluding the proof.

Next we look at the case of p and j fixed, while i_n now grows to infinity.

Theorem 2. *If $pj = 1$ and $\alpha > -2$, then*

$$\lim_{N\to\infty}\frac{\sum_{n=1}^N \frac{(\lg n)^\alpha}{ni_n^j}R_n}{(\lg N)^{\alpha+2}} = \frac{1}{(\alpha+2)j!} \qquad \textit{almost surely.}$$

Proof. Let $a_n = (\lg n)^\alpha/(ni_n^j)$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = b_n/a_n = ni_n^j(\lg n)^2$.

As in the first proof we partition our sum into the following three terms:

$$\frac{\sum_{n=1}^N a_nR_n}{b_N} = \frac{\sum_{n=1}^N a_n[R_nI(1 \le R_n \le c_n) - ER_nI(1 \le R_n \le c_n)]}{b_N}$$

$$+\frac{\sum_{n=1}^{N} a_n R_n I(R_n > c_n)}{b_N} + \frac{\sum_{n=1}^{N} a_n E R_n I(1 \le R_n \le c_n)}{b_N}.$$

The first two terms vanish since

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \le R_n \le c_n) &< C \sum_{n=1}^{\infty} \frac{(i_n-1)!}{(i_n-j-1)! c_n^2} \int_1^{c_n} dr \\ &< C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty\end{aligned}$$

and

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C \sum_{n=1}^{\infty} \frac{(i_n-1)!}{(i_n-j-1)!} \int_{c_n}^{\infty} r^{-2} dr < C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n} < \infty.$$

Next, we turn our attention to the third term. From page 4 of Gradshteyn and Ryzhik[3] we have

$$\sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k} = -\sum_{k=1}^{i_n-j-1} \frac{1}{k}.$$

Thus

$$\begin{aligned}&E R_n I(1 \le R_n \le c_n) \\ = &\binom{i_n-1}{j}\left[\lg c_n + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k} + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{k c_n^{pk}}\right] \\ = &\binom{i_n-1}{j}\left[\lg n + j \lg i_n + 2\lg_2 n - j \sum_{k=1}^{i_n-j-1} \frac{1}{k} + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{k c_n^{pk}}\right] \\ = &\binom{i_n-1}{j}\left[\lg n + j\left[\lg i_n - \sum_{k=1}^{i_n-j-1} \frac{1}{k}\right] + 2\lg_2 n + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{k c_n^{pk}}\right] \\ \sim &\frac{i_n^j}{j!} \lg n\end{aligned}$$

since

$$\lg i_n - \sum_{k=1}^{i_n-j-1} \frac{1}{k} = O(1)$$

and

$$\Big|\sum_{k=1}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{kc_n^{pk}}\Big| < \sum_{k=1}^{i_n}\frac{i_n^k}{[ni_n^j(\lg n)^2]^{pk}} = \sum_{k=1}^{i_n}\frac{i_n^k}{i_n^{pjk}[n(\lg n)^2]^{pk}}$$
$$= \sum_{k=1}^{i_n}\frac{1}{[n(\lg n)^2]^{pk}} = \sum_{k=1}^{i_n}\Big[\frac{1}{[n(\lg n)^2]^p}\Big]^k < \sum_{k=1}^{\infty}\Big(\frac{1}{2}\Big)^k = 1$$

by selecting any n such that $n(\lg n)^2 > 2^{1/p}$. Thus

$$\frac{\sum_{n=1}^N a_n ER_n I(1\le R_n\le c_n)}{b_N} \sim \frac{\sum_{n=1}^N(\frac{(\lg n)^\alpha}{ni_n^j})\cdot(\frac{i_n^j\lg n}{j!})}{(\lg N)^{\alpha+2}}$$
$$= \frac{\sum_{n=1}^N\frac{(\lg n)^{\alpha+1}}{n}}{(\lg N)^{\alpha+2}j!} \to \frac{1}{(\alpha+2)j!}$$

concluding this proof.

The next theorem answers the question as to the behavior of our partial sums when we slightly increase our coefficients. In this case we cannot obtain an Exact Strong Law, we can only obtain an Exact Weak Law (see Theorem 4).

Theorem 3. *If* $pj = 1$ *and* $\alpha > -1$, *then*

$$\frac{\sum_{n=1}^N\frac{n^\alpha}{i_n^j}R_n}{N^{\alpha+1}\lg N}\stackrel{P}{\to}\frac{1}{(\alpha+1)j!}\quad\text{as}\quad N\to\infty.$$

Proof. Set $a_n = n^\alpha/i_n^j$ and $b_N = N^{\alpha+1}\lg N$. From the Degenerate Convergence Theorem, which can be found on page 356 of Chow and Teicher[(1)], we have for all $\epsilon > 0$

$$\sum_{n=1}^N P\{R_n > \epsilon b_N/a_n\} < C\sum_{n=1}^N\frac{(i_n-1)!}{(i_n-j-1)!}\int_{\epsilon b_N/a_n}^\infty r^{-2}dr$$
$$< C\sum_{n=1}^N i_n^j\int_{\epsilon b_N/a_n}^\infty r^{-2}dr < \frac{C}{b_N}\sum_{n=1}^N a_n i_n^j$$

$$= \frac{C\sum_{n=1}^N n^\alpha}{N^{\alpha+1}\lg N} < \frac{CN^{\alpha+1}}{N^{\alpha+1}\lg N} = \frac{C}{\lg N} \to 0.$$

Similarly, the variance term in the Degenerate Convergence Theorem is bounded above by

$$\begin{aligned}
&\sum_{n=1}^N \frac{a_n^2}{b_N^2} ER_n^2 I(1 \le R_n \le b_N/a_n) < \frac{C}{b_N^2}\sum_{n=1}^N a_n^2 i_n^j \int_1^{b_N/a_n} dr \\
< \ & \frac{C\sum_{n=1}^N a_n i_n^j}{b_N} = \frac{C\sum_{n=1}^N n^\alpha}{N^{\alpha+1}\lg N} \\
< \ & \frac{CN^{\alpha+1}}{N^{\alpha+1}\lg N} = \frac{C}{\lg N} \to 0.
\end{aligned}$$

Hence the limit of our normalized partial sums will be the limit of

$$\begin{aligned}
&\sum_{n=1}^N E\Big(\frac{a_n R_n}{b_N} I(1 \le R_n \le b_N/a_n)\Big) \\
= \ & \frac{1}{b_N}\sum_{n=1}^N a_n \binom{i_n - 1}{j} \int_1^{b_N/a_n} (1 - r^{-p})^{i_n - j - 1} r^{-1} dr \\
= \ & \frac{1}{b_N j!}\sum_{n=1}^N \frac{a_n (i_n - 1)!}{(i_n - j - 1)!}\Bigg[\int_1^{b_N/a_n} \frac{dr}{r} \\
& + \sum_{k=1}^{i_n - j - 1} \binom{i_n - j - 1}{k} (-1)^k \int_1^{b_N/a_n} r^{-pk-1} dr\Bigg] \\
= \ & \frac{1}{b_N j!}\sum_{n=1}^N \frac{a_n (i_n - 1)!}{(i_n - j - 1)!}\Bigg[\lg(b_N/a_n) + \sum_{k=1}^{i_n - j - 1} \frac{\binom{i_n - j - 1}{k}(-1)^k}{pk} \\
& + \sum_{k=1}^{i_n - j - 1} \frac{\binom{i_n - j - 1}{k}(-1)^{k+1} a_n^{pk}}{pk b_N^{pk}}\Bigg] \\
= \ & \frac{1}{b_N j!}\sum_{n=1}^N \frac{a_n (i_n - 1)!}{(i_n - j - 1)!}\Bigg[\lg b_N - \lg a_n + j\sum_{k=1}^{i_n - j - 1} \frac{\binom{i_n - j - 1}{k}(-1)^k}{k} \\
& + j\sum_{k=1}^{i_n - j - 1} \frac{\binom{i_n - j - 1}{k}(-1)^{k+1} a_n^{pk}}{k b_N^{pk}}\Bigg] \\
= \ & \frac{1}{j! N^{\alpha+1}\lg N}\sum_{n=1}^N \frac{n^\alpha (i_n - 1)!}{i_n^j (i_n - j - 1)!}\Big[(\alpha+1)\lg N + \lg_2 N - \alpha \lg n + j \lg i_n
\end{aligned}$$

$$
\begin{aligned}
& -j\sum_{k=1}^{i_n-j-1}\frac{1}{k}+j\sum_{k=1}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}(n^\alpha/i_n^j)^{pk}}{k(N^{\alpha+1}\lg N)^{pk}}\Bigg]\\
= \;& \frac{1}{j!N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha(i_n-1)!}{i_n^j(i_n-j-1)!}\Bigg[(\alpha+1)\lg N+\lg_2 N-\alpha\lg n\\
& +j\Bigg[\lg i_n-\sum_{k=1}^{i_n-j-1}\frac{1}{k}\Bigg]+j\sum_{k=1}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}(n^\alpha/i_n^j)^{pk}}{k(N^{\alpha+1}\lg N)^{pk}}\Bigg]\\
\sim \;& \frac{1}{j!N^{\alpha+1}\lg N}\Bigg[(\alpha+1)\lg N\sum_{n=1}^{N}n^\alpha-\alpha\sum_{n=1}^{N}n^\alpha\lg n\Bigg]\\
= \;& \frac{1}{j!}\Bigg[\frac{(\alpha+1)\sum_{n=1}^N n^\alpha}{N^{\alpha+1}}-\frac{\alpha\sum_{n=1}^N n^\alpha\lg n}{N^{\alpha+1}\lg N}\Bigg]\\
\to \;& \frac{1}{j!}\Bigg[1-\frac{\alpha}{\alpha+1}\Bigg]=\frac{1}{(\alpha+1)j!}.
\end{aligned}
$$

For, if $\alpha \geq 0$, then

$$
\begin{aligned}
& \Bigg|\sum_{k=1}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}(n^\alpha/i_n^j)^{pk}}{k(N^{\alpha+1}\lg N)^{pk}}\Bigg| < \sum_{k=1}^{i_n}\frac{i_n^k n^{\alpha pk}}{i_n^{pjk}(N^{\alpha+1}\lg N)^{pk}}\\
< \;& \sum_{k=1}^{\infty}\frac{n^{\alpha pk}}{(N^{\alpha+1}\lg N)^{pk}}=\sum_{k=1}^{\infty}\frac{n^{\alpha pk}}{N^{\alpha pk}(N\lg N)^{pk}}\\
< \;& \sum_{k=1}^{\infty}\frac{1}{(N\lg N)^{pk}}=\sum_{k=1}^{\infty}\Bigg(\frac{1}{(N\lg N)^p}\Bigg)^k=O(1)
\end{aligned}
$$

while, if $-1 < \alpha < 0$, then

$$
\begin{aligned}
& \Bigg|\sum_{k=1}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}(n^\alpha/i_n^j)^{pk}}{k(N^{\alpha+1}\lg N)^{pk}}\Bigg| < \sum_{k=1}^{i_n}\frac{i_n^k n^{\alpha pk}}{i_n^{pjk}(N^{\alpha+1}\lg N)^{pk}}\\
< \;& \sum_{k=1}^{\infty}\frac{n^{\alpha pk}}{(N^{\alpha+1}\lg N)^{pk}}<\sum_{k=1}^{\infty}\frac{1}{(N^{\alpha+1}\lg N)^{pk}}=\sum_{k=1}^{\infty}\Bigg(\frac{1}{(N^{\alpha+1}\lg N)^p}\Bigg)^k=O(1)
\end{aligned}
$$

which completes this proof.

It is important to note that under the hypotheses of our Weak Law (Theorem 3) a Strong Law fails to hold. The ensuing result (Theorem 4) shows the almost sure behavior of the normalized partial sums observed in that Weak Law. Notice that the weak limit obtained in Theorem 3 is the almost sure lower bound of these sums. This is precisely what happens in these situations. A famous example of this is the St. Petersburg Game, see Feller(2), page 251. The classical St. Petersburg distribution is

$$P\{X_n = 2^n\} = \frac{1}{2^n}.$$

On page 253 of Feller(2) it is shown that

$$\frac{\sum_{n=1}^N X_n}{N \lg N} \xrightarrow{P} c \quad \text{as} \quad N \to \infty$$

where c is a nonzero constant that depends on the base of the logarithm used. Hence if one wishes to use $cN \lg N$ as the cumulative entrance fee for the St. Petersburg game, then it would seem to be fair to both the house and the player.

However this is not the case, since it turns out that

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^N X_n}{N \lg N} = c \quad \text{almost surely}$$

while

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N X_n}{N \lg N} = \infty \quad \text{almost surely.}$$

Thus it would not make sense for the house to partake in this game since the upper limit for the player is infinite, while the lower limit is finite. The following theorem shows this type of behavior. It goes by the name of a Generalized Law of the Iterated Logarithm.

Theorem 4. *If $pj = 1$ and $\alpha > -1$, then*

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^N \frac{n^\alpha}{i_n^j} R_n}{N^{\alpha+1} \lg N} = \frac{1}{(\alpha+1)j!} \quad \textit{almost surely}$$

while

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{n^\alpha}{i_n^j} R_n}{N^{\alpha+1}\lg N} = \infty \quad \textit{almost surely.}$$

Proof. Let $a_n = n^\alpha/i_n^j$, $b_n = n^{\alpha+1}\lg n$ and $c_n = b_n/a_n = ni_n^j \lg n$. From Theorem 3 we can conclude that

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{n^\alpha}{i_n^j} R_n}{N^{\alpha+1}\lg N} \le \frac{1}{(\alpha+1)j!} \quad \text{almost surely.} \tag{1}$$

In order to establish the opposite inequality we note that

$$\begin{aligned} \frac{\sum_{n=1}^{N} a_n R_n}{b_N} &\ge \frac{\sum_{n=1}^{N} a_n R_n I(1 \le R_n \le ni_n^j)}{b_N} \\ &= \frac{\sum_{n=1}^{N} a_n [R_n I(1 \le R_n \le ni_n^j) - ER_n I(1 \le R_n \le ni_n^j)]}{b_N} \\ &\quad + \frac{\sum_{n=1}^{N} a_n ER_n I(1 \le R_n \le ni_n^j)}{b_N}. \end{aligned}$$

The first term vanishes almost surely by the Khintchine-Kolmogorov Convergence Theorem and Kronecker's lemma since

$$\begin{aligned} &\sum_{n=1}^{\infty} c_n^{-2} ER_n^2 I(1 \le R_n \le ni_n^j) < C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^2} \int_1^{ni_n^j} dr \\ < \;& C \sum_{n=1}^{\infty} \frac{ni_n^{2j}}{(ni_n^j \lg n)^2} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty. \end{aligned}$$

As for the second term

$$\begin{aligned} &ER_n I(1 \le R_n \le ni_n^j) \sim \frac{i_n^j}{j!} \int_1^{ni_n^j} (1 - r^{-p})^{i_n - j - 1} r^{-1} dr \\ = \;& \frac{i_n^j}{j!} \left[\int_1^{ni_n^j} \frac{dr}{r} + \sum_{k=1}^{i_n-j-1} \binom{i_n - j - 1}{k} (-1)^k \int_1^{ni_n^j} r^{-pk-1} dr \right] \\ = \;& \frac{i_n^j}{j!} \left[\lg(ni_n^j) + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k} + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{k(ni_n^j)^{pk}} \right] \\ = \;& \frac{i_n^j}{j!} \left[\lg n + j \left[\lg i_n - \sum_{k=1}^{i_n-j-1} \frac{1}{k} \right] + j \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{k(ni_n^j)^{pk}} \right] \end{aligned}$$

$$\sim \quad \frac{i_n^j \lg n}{j!}$$

since

$$\lg i_n - \sum_{k=1}^{i_n-j-1} \frac{1}{k} = O(1)$$

and

$$\left| \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}}{k(ni_n^j)^{pk}} \right| < \sum_{k=1}^{i_n} \frac{i_n^k}{(ni_n^j)^{pk}} < \sum_{k=1}^{\infty} \left(\frac{1}{n^p}\right)^k = O(1).$$

Thus

$$\frac{\sum_{n=1}^N a_n E R_n I(1 \le R_n \le n i_n^j)}{b_N} \sim \frac{\sum_{n=1}^N (\frac{n^\alpha}{i_n^j}) \cdot (\frac{i_n^j \lg n}{j!})}{N^{\alpha+1} \lg N}$$

$$= \quad \frac{\sum_{n=1}^N n^\alpha \lg n}{j! N^{\alpha+1} \lg N} \to \frac{1}{(\alpha+1) j!}$$

whence

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^N \frac{n^\alpha}{i_n^j} R_n}{N^{\alpha+1} \lg N} \ge \frac{1}{(\alpha+1)j!} \quad \text{almost surely}$$

which, when combined with (1) proves that

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^N \frac{n^\alpha}{i_n^j} R_n}{N^{\alpha+1} \lg N} = \frac{1}{(\alpha+1)j!} \quad \text{almost surely.}$$

Next we obtain the upper limit for our normalized sum. Note that for all $M > 0$

$$\sum_{n=1}^{\infty} P\{R_n > Mc_n\} = \sum_{n=1}^{\infty} \binom{i_n - 1}{j} \int_{Mc_n}^{\infty} (1 - r^{-p})^{i_n-j-1} r^{-2} dr$$

$$= \quad \frac{1}{j!} \sum_{n=1}^{\infty} \frac{(i_n-1)!}{(i_n-j-1)!} \sum_{k=0}^{i_n-j-1} \binom{i_n-j-1}{k} (-1)^k \int_{Mc_n}^{\infty} r^{-pk-2} dr$$

$$= \quad \frac{1}{j!} \sum_{n=1}^{\infty} \frac{(i_n-1)!}{(i_n-j-1)!} \sum_{k=0}^{i_n-j-1} \binom{i_n-j-1}{k} \frac{(-1)^k}{(pk+1)(Mc_n)^{pk+1}}$$

$$= \frac{1}{M(j-1)!}\sum_{n=1}^{\infty}\frac{(i_n-1)!}{c_n(i_n-j-1)!}\left[\frac{1}{j}+\sum_{k=1}^{i_n-j-1}\binom{i_n-j-1}{k}\frac{(-1)^k}{(k+j)(Mc_n)^{pk}}\right]$$
$$> C\sum_{n=1}^{\infty}\frac{(i_n-1)!}{c_n(i_n-j-1)!}$$

since

$$\left|\sum_{k=1}^{i_n-j-1}\binom{i_n-j-1}{k}\frac{(-1)^k}{(k+j)(Mc_n)^{pk}}\right| < \sum_{k=1}^{i_n}\frac{i_n^k}{(Mc_n)^{pk}} = \sum_{k=1}^{i_n}\frac{i_n^k}{M^{pk}[ni_n^j\lg n]^{pk}}$$
$$< \sum_{k=1}^{\infty}\frac{i_n^k}{M^{pk}i_n^k[n\lg n]^{pk}} = \sum_{k=1}^{\infty}\left(\frac{1}{[Mn\lg n]^p}\right)^k = \frac{1}{[Mn\lg n]^p-1}\to 0.$$

Hence

$$\sum_{n=1}^{\infty}P\{R_n > Mc_n\} > C\sum_{n=1}^{\infty}\frac{(i_n-1)!}{(i_n-j-1)!ni_n^j\lg n} > C\sum_{n=1}^{\infty}\frac{1}{n\lg n}=\infty$$

which implies that

$$\limsup_{n\to\infty}\frac{a_nR_n}{b_n}=\infty \quad \text{almost surely}$$

which in turn allows us to conclude that

$$\limsup_{N\to\infty}\frac{\sum_{n=1}^{N}a_nR_n}{b_N}=\infty \quad \text{almost surely}$$

completing this proof.

The next question is what happens when pj is less than one. In this case we cannot even obtain an Exact Weak.

Theorem 5. *If $pj < 1$ and a_n and b_N are positive constants where*

(2) $$\max_{1\le n\le N} a_n^p i_n = o(b_N^p)$$

then the only finite limit of our normalized sums is zero, i.e.,

$$\frac{\sum_{n=1}^{N} a_n R_n}{b_N} \xrightarrow{P} 0 \quad \text{as} \quad N \to \infty.$$

Proof. Let w_n be the median of R_n. Thus

$$\begin{aligned} \frac{1}{2} &= \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \int_{w_n}^{\infty} (1-r^{-p})^{i_n-j-1} r^{-pj-1} dr \\ &< \frac{p i_n^j}{(j-1)!} \int_{w_n}^{\infty} r^{-pj-1} dr = \frac{i_n^j}{j! w_n^{pj}}. \end{aligned}$$

Hence, we can conclude that $w_n < C i_n^{1/p}$. Thus from (2) we have

$$\frac{\max_{1\le n\le N} a_n w_n}{b_N} < \frac{C \max_{1\le n\le N} a_n i_n^{1/p}}{b_N} \to 0.$$

Assuming that a Weak Law holds we have, from the Degenerate Convergence Theorem

$$\begin{aligned} 0 &\leftarrow \sum_{n=1}^{N} P\{R_n > b_N/a_n\} \\ &= \frac{p}{(j-1)!} \sum_{n=1}^{N} \frac{(i_n-1)!}{(i_n-j-1)!} \int_{b_N/a_n}^{\infty} (1-r^{-p})^{i_n-j-1} r^{-pj-1} dr \\ &= \frac{p}{(j-1)!} \sum_{n=1}^{N} \frac{(i_n-1)!}{(i_n-j-1)!} \sum_{k=0}^{i_n-j-1} \binom{i_n-j-1}{k} (-1)^k \int_{b_N/a_n}^{\infty} r^{-p(k+j)-1} dr \\ &= \frac{p}{(j-1)!} \sum_{n=1}^{N} \frac{(i_n-1)!}{(i_n-j-1)!} \sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k (a_n/b_N)^{p(k+j)}}{p(k+j)} \\ &= \frac{1}{(j-1)!} \sum_{n=1}^{N} \frac{(i_n-1)!}{(i_n-j-1)!} \left(\frac{a_n}{b_N}\right)^{pj} \left[\frac{1}{j} + \sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k+j} \left(\frac{a_n}{b_N}\right)^{pk}\right] \\ &> C \sum_{n=1}^{N} \frac{(i_n-1)!}{(i_n-j-1)!} \left(\frac{a_n}{b_N}\right)^{pj} \end{aligned}$$

since, for if we select N large enough so that $a_n^p i_n < \epsilon b_N^p$ for all $1 \le n \le N$

and $0 < \epsilon < 1/2$, it follows that

$$\left|\sum_{k=1}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k+j}\left(\frac{a_n}{b_N}\right)^{pk}\right| < \sum_{k=1}^{i_n-j-1}\binom{i_n-j-1}{k}\left(\frac{a_n}{b_N}\right)^{pk}$$
$$< \sum_{k=1}^{\infty} i_n^k\left(\frac{a_n}{b_N}\right)^{pk} < \sum_{k=1}^{\infty}\left(\frac{a_n^p i_n}{b_N^p}\right)^k < \sum_{k=1}^{\infty}\epsilon^k < 2\epsilon.$$

Thus we have

$$\sum_{n=1}^{N}\frac{(i_n-1)!}{(i_n-j-1)!}\left(\frac{a_n}{b_N}\right)^{pj} \to 0.$$

Hence, by once again utilizing the Degenerate Convergence Theorem, the limit of our normailzed partial sum is zero since

$$\sum_{n=1}^{N}\frac{a_n}{b_N}ER_nI(1 \le R_n \le b_N/a_n) < \frac{C}{b_N}\sum_{n=1}^{N}\frac{a_n(i_n-1)!}{(i_n-j-1)!}\int_1^{b_N/a_n} r^{-pj}dr$$
$$< \frac{C}{b_N}\sum_{n=1}^{N}\frac{a_n(i_n-1)!}{(i_n-j-1)!}\left(\frac{b_N}{a_n}\right)^{-pj+1} = C\sum_{n=1}^{N}\frac{(i_n-1)!}{(i_n-j-1)!}\left(\frac{a_n}{b_N}\right)^{pj} \to 0$$

which completes the proof.

Finally, for completeness sake we establish strong laws when $pj > 1$. In these final three theorems we use the generalized partition:

$$\frac{\sum_{n=1}^N a_nR_n}{b_N} = \frac{\sum_{n=1}^N a_n[R_nI(1\le R_n\le c_n) - ER_nI(1\le R_n\le c_n)]}{b_N}$$
$$+\frac{\sum_{n=1}^N a_nR_nI(R_n > c_n)}{b_N} + \frac{\sum_{n=1}^N a_nER_nI(1\le R_n\le c_n)}{b_N}$$

where a_n, b_n and hence $c_n = b_n/a_n$ are not predetermined constants. The hypotheses of these theorems depend on the growth of c_n.

Theorem 6. *If* $1 < pj < 2$ *and* $\sum_{n=1}^{\infty} i_n^j/c_n^{pj} < \infty$, *then*

$$\lim_{N\to\infty}\frac{\sum_{n=1}^N a_nR_n}{b_N} = \frac{\Gamma(j-1/p)}{(j-1)!}\lim_{N\to\infty}\frac{\sum_{n=1}^N \frac{a_n(i_n-1)!}{\Gamma(i_n-1/p)}}{b_N} \quad \text{almost surely.}$$

Proof. Using our partition we show that the first term vanishes almost surely since

$$\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \le R_n \le c_n) < C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^2} \int_1^{c_n} r^{-pj+1} dr$$
$$< \quad C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^2} c_n{}^{-pj+2} = C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^{pj}} < \infty.$$

As for the second term

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C \sum_{n=1}^{\infty} i_n^j \int_{c_n}^{\infty} r^{-pj-1} dr = C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^{pj}} < \infty.$$

Finally

$$\begin{aligned}
& E R_n I(1 \le R_n \le c_n) \\
= & \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \int_1^{c_n} (1-r^{-p})^{i_n-j-1} r^{-pj} dr \\
= & \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \sum_{k=0}^{i_n-j-1} \binom{i_n-j-1}{k} (-1)^k \int_1^{c_n} r^{-p(k+j)} dr \\
= & \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \Big[\sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{p(k+j)-1} \\
& + \sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1} c_n^{-p(k+j)+1}}{p(k+j)-1} \Big] \\
= & \frac{(i_n-1)!}{(i_n-j-1)!(j-1)!} \Big[\sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k+j-1/p} \\
& + p \sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1} c_n^{-p(k+j)+1}}{p(k+j)-1} \Big] \\
\sim & \frac{(i_n-1)!}{(i_n-j-1)!(j-1)!} \left[\frac{(i_n-j-1)!}{(j-1/p)\cdots(i_n-1/p-1)} \right] \\
= & \frac{(i_n-1)!\Gamma(j-1/p)}{(j-1)!\Gamma(i_n-1/p)}.
\end{aligned}$$

Where we used problem 5 from page 29 of Riordan[(4)] to obtain the first sum. As for the second term, first note that since $\sum_{n=1}^{\infty} i_n^j / c_n^{pj} < \infty$ we have

$(i_n/c_n^p) < 1/2$ for all large n. Thus

$$\left|\sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1}c_n^{-p(k+j)+1}}{p(k+j)-1}\right| < C\sum_{k=0}^{i_n} i_n^k c_n^{-p(k+j)+1}$$
$$< \quad Cc_n^{-pj+1}\sum_{k=0}^{\infty} i_n^k/c_n^{pk} = Cc_n^{-pj+1}\sum_{k=0}^{\infty}(i_n/c_n^p)^k < Cc_n^{-pj+1} \to 0$$

as $n \to \infty$ since $pj > 1$ which concludes this proof.

Note that if $a_n = 1$, $b_n = n$ and $i_n = i$, then our limit is

$$\frac{(i-1)!\Gamma(j-1/p)}{(j-1)!\Gamma(i-1/p)}$$

which is just the expectation of R_n.

Theorem 7. *If* $pj = 2$ *and* $\sum_{n=1}^{\infty} i_n^j \lg c_n/c_n^2 < \infty$, *then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} a_n R_n}{b_N} = \frac{\Gamma(j/2)}{(j-1)!}\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\frac{a_n(i_n-1)!}{\Gamma(i_n-j/2)}}{b_N} \quad \textit{almost surely.}$$

Proof. Our first two terms vanish almost surely since

$$\sum_{n=1}^{\infty} c_n^{-2}ER_n^2 I(1 \le R_n \le c_n) < C\sum_{n=1}^{\infty}\frac{i_n^j}{c_n^2}\int_1^{c_n}\frac{dr}{r} = C\sum_{n=1}^{\infty}\frac{i_n^j \lg c_n}{c_n^2} < \infty$$

and

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C\sum_{n=1}^{\infty} i_n^j \int_{c_n}^{\infty} r^{-3}dr < C\sum_{n=1}^{\infty}\frac{i_n^j}{c_n^2} < C\sum_{n=1}^{\infty}\frac{i_n^j\lg c_n}{c_n^2} < \infty.$$

As for our truncated expectation, we have

$$\begin{aligned} &ER_nI(1 \le R_n \le c_n) \\ = \;& \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!}\int_1^{c_n}(1-r^{-p})^{i_n-j-1}r^{-2}dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!}\sum_{k=0}^{i_n-j-1}\binom{i_n-j-1}{k}(-1)^k\int_1^{c_n} r^{-pk-2}dr \\
&= \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!}\Bigg[\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^k}{pk+1} \\
&\quad+\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}c_n^{-pk-1}}{pk+1}\Bigg] \\
&= \frac{(i_n-1)!}{(i_n-j-1)!(j-1)!}\Bigg[\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^k}{k+1/p} \\
&\quad+p\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}c_n^{-pk-1}}{pk+1}\Bigg] \\
&\sim \frac{(i_n-1)!}{(i_n-j-1)!(j-1)!}\left[\frac{p(i_n-j-1)!}{(1/p+1)\cdots(1/p+i_n-j-1)}\right] \\
&= \frac{p(i_n-1)!\Gamma(1/p+1)}{(j-1)!\Gamma(i_n-j+1/p)} \\
&= \frac{(i_n-1)!\Gamma(j/2)}{(j-1)!\Gamma(i_n-j/2)}.
\end{aligned}
$$

Where we used problem 5 from page 29 of Riordan(4) to once again obtain the first sum. As for the second term, again note that $\sum_{n=1}^{\infty} i_n^j \lg c_n/c_n^2 < \infty$ implies that $i_n/c_n^p \to 0$. Thus

$$
\left|\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}c_n^{-pk-1}}{pk+1}\right| < C\sum_{k=0}^{i_n}\frac{i_n^k}{c_n^{pk+1}} < \frac{C}{c_n}\sum_{k=0}^{\infty}(i_n/c_n^p)^k < \frac{C}{c_n}\to 0
$$

which concludes this proof.

At this point lets observe a rather simple example. This example shows the ease of these theorems. Let $p=1$, which means that the underlying density is $f_X(x) = x^{-2}I(x\geq 1)$. Next set $j=2$, which means that the numerator in the ratio of our order statistics is the second largest one. Then by setting $a_n = n^\alpha$ and $i_n = n$ we are forced, in order to obtain a finite nonzero constant as our limit, to set $b_n = n^{\alpha+2}$, where $\alpha > -2$. Note that $c_n = n^2$, hence the hypothesis of Theorem 7 holds. Then, via a very simple calculation we have

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} n^{\alpha} R_n}{N^{\alpha+2}} = \frac{1}{\alpha+2} \quad \text{almost surely.}$$

Our final theorem explores the situation of $pj > 2$.

Theorem 8. *If* $pj > 2$ *and* $\sum_{n=1}^{\infty} i_n^j / c_n^2 < \infty$, *then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} a_n R_n}{b_N} = \frac{\Gamma(j-1/p)}{(j-1)!} \lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{a_n (i_n-1)!}{\Gamma(i_n-1/p)}}{b_N} \quad \textit{almost surely.}$$

Proof. The first term converges to zero with probability one since

$$\sum_{n=1}^{\infty} c_n^{-2} E R_n^2 I(1 \le R_n \le c_n) < C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^2} \int_1^{c_n} r^{-pj+1} dr < C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^2} < \infty.$$

While

$$\sum_{n=1}^{\infty} P\{R_n > c_n\} < C \sum_{n=1}^{\infty} i_n^j \int_{c_n}^{\infty} r^{-pj-1} dr = C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^{pj}} < C \sum_{n=1}^{\infty} \frac{i_n^j}{c_n^2} < \infty.$$

Therefore, as in the proof of Theorem 6

$$\begin{aligned}
& E R_n I(1 \le R_n \le c_n) \\
&= \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \int_1^{c_n} (1-r^{-p})^{i_n-j-1} r^{-pj} dr \\
&= \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \sum_{k=0}^{i_n-j-1} \binom{i_n-j-1}{k} (-1)^k \int_1^{c_n} r^{-p(k+j)} dr \\
&= \frac{p(i_n-1)!}{(i_n-j-1)!(j-1)!} \left[\sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{p(k+j)-1} \right. \\
&\quad \left. + \sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^{k+1} c_n^{-p(k+j)+1}}{p(k+j)-1} \right] \\
&= \frac{(i_n-1)!}{(i_n-j-1)!(j-1)!} \left[\sum_{k=0}^{i_n-j-1} \frac{\binom{i_n-j-1}{k}(-1)^k}{k+j-1/p} \right.
\end{aligned}$$

$$+p\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}c_n^{-p(k+j)+1}}{p(k+j)-1}\Bigg]$$

$$\sim \frac{(i_n-1)!}{(i_n-j-1)!(j-1)!}\left[\frac{(i_n-j-1)!}{(j-1/p)\cdots(i_n-1/p-1)}\right]$$

$$=\frac{(i_n-1)!\Gamma(j-1/p)}{(j-1)!\Gamma(i_n-1/p)}.$$

In this case we have $i_n^j/c_n^2 \to 0$. This in turn implies that $i_n/c_n^p \to 0$. So, if we select n large enough so that $(i_n/c_n^p) < 1/2$ it follows that

$$\left|\sum_{k=0}^{i_n-j-1}\frac{\binom{i_n-j-1}{k}(-1)^{k+1}c_n^{-p(k+j)+1}}{p(k+j)-1}\right| < C\sum_{k=0}^{i_n} i_n^k c_n^{-p(k+j)+1}$$

$$< \quad Cc_n^{-pj+1}\sum_{k=0}^{\infty} i_n^k c_n^{-pk} = Cc_n^{-pj+1}\sum_{k=0}^{\infty}(i_n/c_n^p)^k < Cc_n^{-pj+1} \to 0$$

as $n \to \infty$ which concludes this proof.

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