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TAME MODULES OVER DISTRIBUTIVELY GENERATED NEARRINGS

BY

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Abstract. The class of tame d. g. modules is initiated to study the general theory of endomorphism nearrings acting on not necessarily abelian groups. Noetherian quotients and in particular, the annihilators are used to develop the structure theory throughout this paper. Examples are provided to demonstrate and delimit this theory.

1. Introduction. A set N with two binary operations $+$ and $\cdot$ is a (left) nearring if $(N,+)$ is a group, $(N,\cdot)$ is a semigroup and $a(b+c)=ab+ac$ for all $a,b,c\in N$. N is said to be *0-symmetric* if $0a=0$ for all $a\in N$. Let G be an additively written group (not necessarily abelian). Both $M(G)=\{f\colon G\to G\}$ the set of all mappings of G and $M_0(G)=\{f\colon G\to G\mid 0f=0\}$ the set of all zero preserving mappings of G are nearrings with pointwise addition and composition of functions. The nearring $M_0(G)$ is clearly a 0-symmetric nearring. An element $d\in N$ is called *distributive* if $(a+b)d=ad+bd$ for all $a,b\in N$. A nearring N is called *distributively generated* (abbre. d. g.) if $(N,+)$ is generated as a group by a semigroup $(S,\cdot)$ of distributive elements in N. A d. g. nearring N generated by S is denoted $N[S]$ when a knowledge of the exact set S is necessary.

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The investigation of d. g. nearrings can be traced back to Fitting in 1932 [1], who study the sum of two endomorphisms of a nonabelian group. Followed by Neumann [8], the d. g. nearrings associated with free groups in a variety are introduced. Fröhlich [2] investigated the nearring $I(G)$ generated by all inner automorphisms $Inn(G)$ of a finite simple nonabelian group G.

The study of nearring module can be interpreted as studying the representation of a nearring on the nearring of group mappings $M(G)$ for some group G. Explicitly, we call G a nearring N-module (or simply N-module) if there exists a nearring homomorphism $\theta\colon N \to M(G)$. This homomorphism θ is called a *representation* of N on G. It is called *faithful* if $\ker \theta = 0$. A representation of a d. g. nearring $N[S]$, $\theta\colon N[S] \to M_0(G)$, is called a *d. g. representation* if $S\theta \subseteq End(G)$, and G is called a *d. g. module* [7, p.226] or $N[S]$*-module.*

Let G be an N-module. A subgroup H of G such that $hn \in H$ for all $h \in H$, $n \in N$ is called an N*-submodule* of G. An N*-ideal* of G is a normal subgroup K of G such that $(a+k)n - an \in K$ for all $a \in G$, $k \in K$, $n \in N$. If N has a unity 1, then G is a *unitary* N-module provided that $a1 = a$ for all $a \in G$. A unitary N-module G is called *tame* [10] if every its N-submodule is an N-ideal or equivalently, for any $a, b \in G$, $n \in N$, there exists $m \in N$ such that $(a+b)n - an = bm$ [10, Proposition 2.1]. A necessary condition for an N-module to be tame is that every N-submodule must be a normal subgroup. These motivate the following definition.

An $N[S]$-module G is called a *tame* $N[S]$*-module* or *tame d. g. module* if all N-submodules are normal subgroups of G. Tame d. g. modules are abundant. For instance, let S be a semigroup of endomorphisms of an arbitrary group G containing all inner automorphisms $Inn(G)$ and N the nearring generated by S. Then G is a tame $N[S]$-module. Also, every ring R-module is a tame $R[R]$-module. $N[S]$ is called a *tame d. g. nearring* if $N[S]$ has a faithful tame d. g. module. Let G be an arbitrary group then

G is a faithful tame $E(G)[End(G)]$-module (or faithful tame $A(G)[Aut(G)]$-module, $I(G)[Inn(G)]$-module). Therefore $E(G)[End(G)]$, $A(G)[Aut(G)]$, $I(G)[Inn(G)]$ are all tame d. g. nearrings for any group G. If G is Hamiltonian, then $E(G)[End(G)]$ and all its subrings $R[R]$ (not necessarily with the same unity in $E(G)$) are tame d. g. nearrings. (A group is called *Hamiltonian* if every subgroup is a normal subgroup [11]). A tame $N[T]$-module then gives us a *tame d. g. representation* of $N[T]$ on G. Consider the ring of integers $\mathbb{Z}$ as a d. g. nearring $\mathbb{Z}[1]$, then $\theta\colon \mathbb{Z}[1] \to M_0(G)[End(G)]$ defined by $1\theta = id_G$, the identity map of G, is a d. g. representation for every group G but not a tame d. g. representation in general. However, this representation is tame d. g. if and only if G is a Hamiltonian group. Furthermore, consider the ring of integers $\mathbb{Z}$ as a d. g. nearring $\mathbb{Z}[\mathbb{Z}]$. Define $\theta\colon \mathbb{Z}[\mathbb{Z}] \to M_0(G)[EndG]$ via $1\theta = id_G$, the identity map of G. This representation is d. g. if and only if G is abelian [7, p.226]. Since when G is abelian, all subgroups are normal. It follows that θ is tame d. g. if and only if G is abelian. From these examples, it is essential to consider the generating semigroup for a d. g. nearring, especially in its representation on a group G.

The following result shows that a tame $N[S]$-module is indeed "tame" in the sense of Scott [10].

Theorem 1.1. *Let G be a tame $N[T]$-module and H a subgroup of G. Then the following are equivalent.*

(1) *H is an N-ideal of G.*
(2) *H is an N-submodule of G.*
(3) *H is a normal subgroup of G and $HT \subseteq H$.*

Proof. Since the d. g. nearring $N[T]$ is 0-symmetric, it follows that (1) implies (2). Assume H is an N-submodule of G. Then H is a normal subgroup of G by definition and $HT \subseteq HN \subseteq H$. Therefore (2) implies (3). It remains to show (3) implies (1). Let $h \in H$, $a \in G$ and $n \in N$. Since N

is additively generated by T, write $n=\sum_{i=1}^{k}\varepsilon_i\alpha_i\in N$ where $\varepsilon_i\in\{1,-1\}$, $\alpha_i\in T$ for $i=1,2,\cdots,k$. Let $n_j=\sum_{i=1}^{j}\varepsilon_i\alpha_i\in N$ for $j=1,2,\cdots,k$. Observe that $n_k=n$ and $-an_j+an_{j+1}=a\varepsilon_{j+1}\alpha_{j+1}$. Then

$$\begin{aligned}(a+h)n-an &= (a+h)\left(\sum_{i=1}^{k}\varepsilon_i\alpha_i\right)-an_k\\ &= \left(\sum_{i=1}^{k}\varepsilon_i(a+h)\alpha_i\right)-an_k\\ &= a\varepsilon_1\alpha_1+\left(\sum_{i=1}^{k-1}(h\varepsilon_i\alpha_i+a\varepsilon_{i+1}\alpha_{i+1})\right)+h\varepsilon_k\alpha_k-an_k\\ &= an_1+\left(\sum_{i=1}^{k-1}(h\varepsilon_i\alpha_i+(-an_j+an_{j+1}))\right)+h\varepsilon_k\alpha_k-an_k\\ &= \sum_{i=1}^{k}(an_i+h\varepsilon_i\alpha_i-an_i)\in H.\end{aligned}$$

Therefore H is an N-ideal of G. This completes the proof.

In section two, the properties of noetherian quotient are investigated. These results are used to show that in a tame $N[T]$-module, the quotient $(K:H)$ is an ideal of $N[T]$ provided that both K and H are N-submodules of G in Proposition 2.9. Conditions such that $(K:H)$ is both a ring and an ideal of N are given in Theorem 2.11. In section three, characterizations of different types of primitivities are presented in Theorem 3.4. A subdirect product decomposition of a tame d. g. nearring is given in Theorem 3.7. Examples are provided to illustrated and delimit our results. For terminologies not defined in this paper, please refer [7, 9] but note that [9] use right nearrings instead of left nearrings.

2. Noetherian quotients.

Let G be an N-module and X, Y be nonempty subsets of G. Define the noetherian quotient of X by Y as

$$(X:Y)=\{f\in N\mid yf\in X \text{ for all } y\in Y\}.$$

If $X = \{0\}$, we write $(0 : Y)$ for $(\{0\} : Y)$. The set $(0 : Y)$ is called the annihilator of Y in N and is denoted by $Ann_N(Y)$ or $Ann(Y)$. When $Y = G$, we may use $N(G, X)$ to denote $(X : G)$ [3]. If X is a normal subgroup of G, the set $(X : G)$ or $N(G, X)$ is equal to $Ann(G/X)$. We first summarize some known results concerning $(X : Y)$ when X, Y are subsets of an N-module G [7, 2.31].

Proposition 2.1. *Let G be an N-module and let X, Y be subsets of G.*

(1) *If X is a subgroup of G, then $(X : Y)$ is a subgroup of N.*
(2) *If X is a normal subgroup of G, then $(X : Y)$ is a normal subgroup of N.*
(3) *If X is an N-submodule of G, then $(X : Y)$ is a right N-subgroup of N.*
(4) *If X is a subgroup of G and $YN \leq Y$, then $(X : Y)$ is a left N-subgroup of N.*
(5) *If X is an N-ideal of G, then $(X : Y)$ is a right ideal of N.*

When K is a subgroup of G and Y is a subset of G, define a subgroup K_Y of G as $K_Y = gp(\cup\{Yf \mid f \in (K : Y)\})$, the subgroup of G generated by $\cup\{Yf \mid f \in (K : Y)\}$. It is easy to see that K_Y is a subgroup of K.

Proposition 2.2. *Let G be an N-module and K a subgroup of G. Let X, Y be subsets of G. Then :*

(1) $(K_Y : Y) = (K : Y)$,
(2) *If K_G is abelian, then $(K : G)$ is abelian.*
(3) *If $YN \leq Y$, then $(X : Y)$ is left invariant in N, that is $N(X : Y) \subseteq (X : Y)$.*

Proof. (1) Since $K_Y \subseteq K$, therefore $Yf \subseteq K_Y \subseteq K$ for all $f \in (K_Y : Y)$. Hence $f \in (K : Y)$ and $(K_Y : Y) \subseteq (K : Y)$. On the other hand, if

$f \in (K : Y)$ then $Yf \subseteq K_Y$ by definition of K_Y and so $f \in (K_Y : Y)$ or $(K : Y) \subseteq (K_Y : Y)$.

(2) is trivial.

(3) Let $f \in N$ and $g \in (X : Y)$, $a \in Y$. Then $a(fg) = (af)g \in Yg \subseteq X$. Therefore $fg \in (X : Y)$ and $(X : Y)$ is left invariant in N.

From Proposition 2.1(2), if K is a normal subgroup of G, then $(K : Y)$ is a normal subgroup of N. Thus $(K_Y : Y)$ is a normal subgroup of N by Proposition 2.2(1). However this connection can be strengthen when we consider K_Y and $(K_Y : Y)$. Let G be an N-module. We say that N acts *transitively* on G if for all $0 \neq a,\ b \in G$, there exists $f \in N$ such that $af = b$.

Proposition 2.3. *Let G be an N-module, K a subgroup of G and Y a subset of G.*

(1) *If K_Y is a normal subgroup of G, then $(K_Y : Y)$ is a normal subgroup of N.*
(2) *If $(K_Y : Y)$ is a normal subgroup of N, then K_Y is a normal subgroup of G provided that N acts transitively on G.*

Proof. (1) Suppose K_Y is a normal subgroup of G. Let $f \in (K_Y : Y)$ and $s \in N$, $a \in Y$. Then $s+f-s \in N$ and $a(s+f-s) = as+af-as \in K_Y$. Therefore $s+f-s \in (K_Y : Y)$ and $(K_Y : Y)$ is a normal subgroup of N.

(2) Let $0 \neq a \in Yf$ for some $f \in (K : Y)$. Therefore $a = xf$ for some nonzero $x \in Y$. Let $b \in G$. The hypothesis implies that there exists $s \in N$ such that $xs = b$. Now we have $b+a-b = xs+xf-xs = x(s+f-s) \in K_Y$. Further let $c \in K_Y$. Then $c = \sum_{i=1}^{m} \varepsilon_i a_i f_i$ where $\varepsilon_i \in \{1,\ -1\}$, $a_i \in Y$ and $f_i \in (K : Y)$ for $i \in \{1,\ 2, \cdots, m\}$. So

$$b+c-b \;=\; b+\sum_{i=1}^{m} \varepsilon_i a_i f_i - b$$

$$= \sum_{i=1}^{m}(b+\varepsilon_i a_i f_i - b) \in K_Y.$$

Hence K_Y is a normal subgroup of G.

Proposition 2.4. *Let G be an N-module. If K_Y or K is a normal subgroup of G and Y is an N-submodule of G, then $(K:Y)$ is a left ideal of N.*

Proof. This follows from Proposition 2.2(3) and Proposition 2.3(1).

Theorem 2.5. *Let G be an N-module and provided that N acts transitively on G. Suppose that K is a subgroup of G and Y is an N-submodule of G. Then the following are equivalent.*

(1) *K_Y is a normal subgroup of G.*

(2) *$(K_Y:Y)$ is a normal subgroup of N.*

(3) *$(K:Y)$ is a normal subgroup of N.*

(4) *$(K:Y)$ is a left ideal of N.*

(5) *$(K_Y:Y)$ is a left ideal of N.*

Proof. (1) $\Leftrightarrow$ (2) : By Proposition 2.3.

(2) $\Leftrightarrow$ (3) : By Proposition 2.2(1).

(1) $\Rightarrow$ (4) : By Proposition 2.4.

(4) $\Rightarrow$ (1) : Suppose $(K:Y)$ is a left ideal of N. Then $(K:Y)$ is a normal subgroup of N. As N acts transitively on G, we know K_Y is a normal subgroup of G by Proposition 2.3(2).

(4) $\Leftrightarrow$ (5) : By Proposition 2.2(1).

Corollary 2.6. *Let G be an N-module and provided that N acts tran-*

sitively on G. Suppose that K is a subgroup of G and Y is a submodule of G. If K is a normal subgroup of G, then K_Y is a normal subgroup of G.

Proof. By Proposition 2.4 and Theorem 2.5.

We now apply the previous results to tame modules over a d. g. nearring $N[T]$.

Theorem 2.7. *Soppose G is a tame $N[T]$-module. Let K be a subgroup of G and H an N-submodule of G. Then the following are equivalent.*

(1) *K_H is an N-submodule of G.*
(2) *$(K : H)$ is right invariant in $N[T]$.*
(3) *$(K_H : H)$ is right invariant in $N[T]$.*
(4) *$(K : H)$ is an ideal of $N[T]$.*
(5) *$(K_H : H)$ is an ideal of $N[T]$.*

Proof. The equivalence of (2), (3) and (4), (5) follows from Proposition 2.2(1).

(1) $\Rightarrow$ (2): Let $a \in H$, $f \in (K : H)$, $g \in N[T]$. Then $af \in K_H$ and so $a(fg) = (af)g \in K_H$. Hence $(K : H)$ is right invariant in $N[T]$.

(2) $\Rightarrow$ (1): Assume $(K : H)$ is right invariant in $N[T]$. It suffices to show that for all $a \in Hf$, where $f \in (K : H)$, we have $a\alpha \in K_H$ for all $\alpha \in T$. As $a = xf$ for some $x \in H$, and our hypothesis gives $f\alpha \in (K : H)$. Therefore $a\alpha = (xf)\alpha = x(f\alpha) \in K_H$ and so K_H is a submodule of G.

(4) $\Rightarrow$ (2): This is clear from the fact that $N[T]$ is a d. g. nearring.

(1) $\Rightarrow$ (4): Since G is a tame $N[T]$-module, all submodules are normal subgroups of G. Proposition 2.4 implies that $(K : H)$ is a left ideal of N. Since (1) implies (2), $(K : H)$ is right invariant in N. Since $N[T]$ is a d. g. nearring, using a similar argument in Theorem 1.1, $(K : H)$ is an ideal of $N[T]$.

Before going further, we first observe some examples of tame $N[S]$-modules.

Example 2.8. The symmetric group S_3 is a tame $E(S_3)[End(S_3)]$-module. From [6], we have $|E(S_3)| = 54$ and $|E(S_3, A_3)| = 27$ which is a ring with right identity and is an ideal of $E(S_3)$. Note that all the nilpotent elements in $E(S_3)$ are contained in $E(S_3, A_3)$. Further, consider the generalized quaternion group $Q_n = \langle\, a, b \mid 2^{n-1}a = 0,\ a + b = b - a,\ a^{n-2}a = 2b\,\rangle$. The subgroup generated by a is a fully invariant subgroup of Q_n and $E(Q_n, \langle a\rangle)$ is a subring and is an ideal of $E(Q_n)$ [5].

The following results show that the rings $E(S_3, A_3)$ and $E(Q_n, \langle a\rangle)$ being ideals of $E(S_3)$, $E(Q_n)$ respectively are not particular cases.

Proposition 2.9. *Let G be a tame $N[T]$-module, K a subgroup of G and H a submodule of G. Then :*

(1) *If K is a submodule of G, then $(K : H)$ is an ideal of $N[T]$. In particular, $(0 : K) = Ann(K)$ and $(K : G) = N(G, K) = Ann(G/K)$ are ideals of $N[T]$.*

(2) *If K is a submodule of G, then K_H is a submodule of G.*

Proof. (1) Since $(K : H)$ is left invariant in $N[T]$ by Proposition 2.2(3), and $N[T]$ is a d. g. nearring. It suffices to show that $(K : H)$ is right invariant and a normal subgroup of $N[T]$. The normality follows from Proposition 2.4. To show $(K : H)$ is right invariant in $N[T]$. Let $a \in H$, $f \in (K : H)$ and $g \in N[T]$. Then $a(fg) = (af)g \in Kg \subseteq K$. Therefore $(K : H)$ is right invariant in $N[T]$.

(2) This follows immediately from (1) above and Theorem 2.7.

It is well known that an abelian d. g. nearring is a ring. The following lemma shows that it is also true for the right ideal of a d. g. nearring even the right ideal is not necessarily distributively generated.

Lemma 2.10. *Let $N[S]$ be a d. g. nearring and A a right ideal of $N[S]$. Then the following are equivalent.*

(1) *A is a ring.*

(2) *A is abelian.*

(3) *A is a normal right S-subgroup of N.*

Proof. The equivalence of (1) and (3) follows from [7, Corollary 9.22]. It remains to show the equivalence of (1) and (2). If A is a ring then it is clear that A is abelian. Suppose that A is an abelian right ideal of $N[S]$. To show that A is a ring, we need only to verify the right distributive law. Let f, g, $h \in A$. Since $N[S]$ is a d. g. nearring, we may find $\varepsilon_i \in \{1, -1\}$ and $\alpha_i \in S$ such that such that $h = \sum_{i=1}^{m} \varepsilon_i \alpha_i$. Now

$$\begin{aligned}(f+g)h &= (f+g)\sum_{i=1}^{m}\varepsilon_i\alpha_i = \sum_{i=1}^{m}\varepsilon_i(f+g)\alpha_i \\ &= \sum_{i=1}^{m}\varepsilon_i(f\alpha_i + g\alpha_i) = \sum_{i=1}^{m}\varepsilon_i f\alpha_i + \sum_{i=1}^{m}\varepsilon_i g\alpha_i \\ &= fh + gh.\end{aligned}$$

Thus the right distributive law holds and A is a ring.

Theorem 2.11. *Let G be a tame $N[T]$-module and K a submodule of G. If K or K_G is abelian then $(K : G)$ is both a ring and an ideal of $N[T]$.*

Proof. By Proposition 2.2(2), Theorem 2.7, $(K : H)$ is an abelian ideal of $N[T]$. Therefore $(K : H)$ is a ring by Lemma 2.10.

3. Primitivity and tame d. g. nearrings. Let N be a nearring and G, H two N-modules. Let S be a nonempty subset of N. A subgroup K of G such that $KS \subseteq K$ is called an *S-subgroup* of G and G is called *S-simple* if it contains no proper nontrivial S-subgroups. A group homomorphism

$\alpha: G \to H$ is called an *S-homomorphism* if $(a\alpha)s = (as)\alpha$ for all $a \in G$, $s \in S$. The following proposition is well known for $N[S]$-modules [7, p.158].

Proposition 3.1. *Let G, H be $N[S]$-modules and K a subgroup of G. Let $\alpha: G \to H$ be a group homomorphism. Then*

(1) *K is an N-submodule of G if and only if K is an S-subgroup of G.*

(2) *α is an N-homomorphism if and only if α is an S-homomorphism.*

Once $N[S]$-module is defined, a natural question to ask is: *Is there a difference between N-submodules and $N[S]$-submodules*? We first address this question. Let G be an $N[S]$-module and K a subgroup of G. Let θ be the d. g. representation of $N[S]$ on $M_0(G)$. It is natural to say K is called an $N[S]$-*submodule* of G if $KN \subseteq K$ and $S\theta \subseteq End(K)$. Let G be an arbitrary group and consider G as $E(G)[End(G)]$-module. Then all the fully invariant subgroups of G are $E(G)[End(G)]$-submodules, and vice versa. The following proposition shows that N-submodules is the same as $N[S]$-submodules.

Proposition 3.2. *Let G be an $N[S]$-module and K a subgroup of G. Then the following are equivalent.*

(1) *K is an N-submodule of G.*

(2) *K is an S-subgroup of G.*

(3) *K is an $N[S]$-submodule of G.*

Proof. (1) $\Leftrightarrow$ (2) : This follows from Proposition 3.1(1).

(3) $\Rightarrow$ (1) : This is clear from the definition.

(1) $\Rightarrow$ (3) : Since G is an $N[S]$-module, we have $(a + b)s = as + bs$ for all $a, b \in G$ and $s \in S$. This property will be inherited by its subgroup K. Since $KS \subseteq K$ by our assumption, the acting $s|_K$ of s restricted on K is an endomorphism of K. Therefore K is an $N[S]$-submodule.

Proposition 3.3. *Let G be a tame $N[T]$-module and K a subgroup of G. Then the following are equivalent.*

(1) *K is an N-ideal of G.*

(2) *K is a T-subgroup of G.*

(3) *K is an N-submodule of G.*

(4) *K is an $N[T]$-submodule of G.*

(5) *K is a tame $N[T]$-submodule of G.*

Proof. The equivalence of (2), (3), (4) follows from Proposition 3.2 and the equivalence of (1) and (3) follows from Theorem 1.1. Note that (5) implies (4) is clear. It remains to show that (4) implies (5). If H is an N-submodule of K then H is an N-submodule of G. It follows that H is a normal subgroup of G because G is tame. Therefore H is a normal subgroup of K and K is tame.

Theorem 3.4. *Let $N[T]$ be a tame d. g. nearring on G. Then the following are equivalent.*

(1) *$N[T]$ is 0-primitive on G.*

(2) *$N[T]$ is 1-primitive on G.*

(3) *$N[T]$ is 2-primitive on G.*

(4) *G is T-simple.*

(5) *$N[T]$ acts transitively on G.*

Proof. The equivalence of (1), (2), (3) and (4) follows immediately from Proposition 3.3. It remains to show (5) is equivalent to other conditions.

(2) $\Rightarrow$ (5) : Suppose $N[T]$ is 1-primitive on G. Let $0 \neq a \in G$. Then $aN = 0$ or $aN = G$. Recall that the quotient $[0 : T] = \{a \in G \mid at = 0 \text{ for all } t \in T\}$ is an N-submodule of G by [4, Proposition 2.9(4)] and so $[0 : T]$ is an N-ideal of G by Proposition 3.3. If $aN = 0$ then $aT = 0$ and

so $a \in [0 : T]$ is nonempty. Therefore $[0 : T] = G$ or $GT = 0$. Which then implies $GN = 0$ for $N[T]$ is a d. g. nearring generated by T. Since G is a faithful $N[T]$-module, this implies that $G = 0$ and $N[T] = 0$. If $aN = G$ then clearly $N[T]$ acts transitively on G.

(5) $\Rightarrow$ (3) : Suppose $N[T]$ acts transitively on G. Then clearly $aN = G$ for any nonzero $a \in G$. Therefore G is a faithful type-2 $N[T]$-module and $N[T]$ is 2-primitive on G.

The results above have shown the close connections between the different aspects which can be defined on an N-module G. Most results on tame endomorphism nearrings in the sense of Meldrum can be generalized to tame d. g. nearrings. However, the further development will pursuade us that tame d. g. module provide us a suitable platform to study the interplay properties of groups and d. g. nearrings.

Lemma 3.5. *Let G be a tame $N[T]$-module, and let $I = Ann(G)$ be the annihilator of G. Then $N[T]/I$ is a tame d. g. nearring on G.*

Proof. Observe that I is an ideal of N by Proposition 2.9. It is clear that $\tilde{N}[\tilde{T}] = N[T]/I$ is additively generated by $\tilde{T} = \{\ t+I \mid t \in T\ \}$. Denote $\tilde{f} = f + I \in \tilde{N}[\tilde{T}]$. Define $a\tilde{f} = af$ for all $a \in G$ and $\tilde{f} \in \tilde{N}[\tilde{T}]$. It suffices to show that G is a faithful tame $\tilde{N}[\tilde{T}]$-module.

Let $a \in G$ and $\tilde{f},\ \tilde{g} \in \tilde{N}[\tilde{T}]$. If $\tilde{f} = \tilde{g}$ then $f = g + h$ for some $h \in I$, and so

$$a\tilde{f} = af = a(g + h) = ag + ah = ag = a\tilde{g}.$$

Hence the action of $\tilde{N}[\tilde{T}]$ on G is well defined. It is now easy to verify that G is an $\tilde{N}$-module. Let $\tilde{t} \in \tilde{T}$, and $a, b \in G$. Then

$$(a + b)\tilde{t} = (a + b)t = at + bt = a\tilde{t} + b\tilde{t}.$$

Therefore G is an $\tilde{N}[\tilde{T}]$-module. Now let K be an $\tilde{N}$-submodule of G. Let $f \in N$. Then $kf = k\tilde{f} \in K$ for all $k \in K$. K is thus an N-submodule of G and so is normal in G because G is a tame $N[T]$-module. Hence G is a tame $\tilde{N}[\tilde{T}]$-module. If $a\tilde{f} = 0$ then $af = 0$ for all $a \in G$ and so $f \in I$. Therefore G is faithful and $N[T]/I$ is a tame d. g. nearring on G.

Proposition 3.6. *Let G be a tame $N[T]$-module and K a T-subgroup of G. Then:*

(1) *K is a tame $N[T]$-module and $N[T]/(0:K)$ is a tame d. g. nearring on K.*

(2) *G/K is a tame $N[T]$-module and $N[T]/(K:G)$ is a tame d. g. nearring on G/K.*

Proof. (1) From Proposition 3.2, K is an $N[T]$-module. If H is an $N[T]$-submodule of K then it is an $N[T]$-submodule of G and so is normal in G. It follows that H is a normal subgroup of K and so K is tame. We now have a tame d. g. representation $\theta\colon N[T] \to M_0(K)$ via $f\theta = f|_K$, the restriction of f on K. Now

$$\ker \theta = \{ f \in N \mid f|_K = 0 \} = \{ f \in N \mid Kf = 0 \} = (0:K).$$

Therefore K is a faithful tame $N[T]/(0:K)$-module, and $N[T]/(0:K)$ is a tame d. g. nearring on K by Lemma 3.5.

(2) Let G be a tame $N[T]$-module and K a T-subgroup of G. From Proposition 3.3, K is an N-ideal of G and so G/K is an N-module given by $(a+K)f = af + K$, for all $a + K \in G/K$ and $f \in N$. It is easy to see that G/K is an $N[T]$-module. If H/K is an $N[T]$-submodule of G/K then $(h+K)f = hf + K \in H/K$ for all $h \in H$, $f \in N$. Therefore $hf \in H$ for all $h \in H$ and $f \in N$. So H is an N-submodule of G. Since G is tame, H must be a normal subgroup of G and so H/K is a normal subgroup of

G/K. This proves that G/K is a tame $N[T]$-module. Now, consider the d. g. representation $\theta: N[T] \to M_0(G/K)$ via $(a+K)(f\theta) = af + K$ for all $a + K \in G/K$ and $f \in N[T]$. We have

$$\begin{aligned} \ker\theta &= \{\, f \in N[T] \mid af \in K \text{ for all } a \in G \,\} \\ &= (K:G) = Ann(G/K) \end{aligned}$$

Therefore G/K is a faithful tame $N[T]/(K:G)$-module and $N[T]/(K:G)$ is a tame d. g. nearring on G/K by Lemma 3.5.

The following result provides a link between the tame d. g. nearring $N[T]$ and the two tame d. g. nearrings in Proposition 3.6. In an N-module G, denote $N(G,K,0) = \{n \in N \mid an \in K,\ kn = 0 \text{ for } a \in G,\ k \in K\}$ where $K \subseteq G$.

Theorem 3.7. *Let $N[T]$ be a tame d. g. nearring on a group G, K a T-subgroup of G. Then the nearring $N/N(G,K,0)$ is a subdirect product of the tame d. g. nearrings $N/(0:K)$ and $N/(K:G)$.*

Proof. Observe that $N(G,K,0) = N(G,K) \cap N(K,0) = (K:G) \cap (0:K)$ is an ideal of N by Proposition 2.9 and Proposition 3.3. Define functions $\alpha: N \to N/(0:K)$ via $n\alpha = n + (0:K)$ and $\beta: N \to N/(K:G)$ via $n\beta = n + (K:G)$. Then α, β are the natural homomorphisms with $\ker\alpha \cap \ker\beta = N(G,K,0)$. Thus $N/N(G,K,0)$ is a subdirect product of $N/(0:K)$ and $N/(K:G)$ by [7, Lemma 7.2].

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