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SOME APPLICATIONS OF δ-PREOPEN SETS IN TOPOLOGICAL SPACES

BY

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Abstract. In 1993, Raychaudhuri and Mukherjee [10] introduced the notions of δ-preopen sets and δ-preclosure. In this paper, we introduce some weak separation axioms by utilizing the notions of δ-preopen sets and the δ-preclosure operator.

1. Introduction. Mashhour et al. [7] introduced preopen sets and precontinuous functions in topological spaces. On the other hand, Veličko [13] introduced the notion of δ-open sets which are stronger than open sets. Since then, δ-open sets have been widely used in order to introduce new spaces and functions. Recently, Raychaudhuri and Mukherjee [10] have introduced the notions of δ-preopen sets and δ-almost continuity in topological spaces. The class of δ-preopen sets is larger than that of preopen sets. By using δ-preopen sets, in [11], they introduced and investigated δp-closed spaces.

In this paper, we introduce some weak separation axioms by utilizing the notions of δ-preopen sets and the δ-preclosure operator. We show that (δ, p)-T_1 spaces, (δ, p)-R_0 spaces and (δ, p)-symmetric spaces are all equivalent. The following property is fundamental and important: a subset A of a topological space (X, τ) is δ-preopen in (X, τ) if and only if it is preopen in (X, τ_s), where τ_s is the semi-regularization of τ. This fact plays an important role in the sequel.

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2. Preliminaries. Let (X, τ) (or X) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be regular open if $Int(Cl(A)) = A$. A point x of X is called a δ-cluster point of A if $A \cap V \neq \emptyset$ for every regular open set V containing x. The set of all δ-cluster point of A is called the δ-closure of A and denoted by $Cl_\delta(A)$. If $Cl_\delta(A) = A$, then A is said to be δ-closed [13]. The complement of a δ-closed set is said to be δ-open. The union of all δ-open sets contained in A is called the δ-interior of A and is denoted by $Int_\delta(A)$. It is well-known that the family of regular open sets in a topological space (X, τ) is a base for a topology which is called the semi-regularization of τ and is denoted by τ_s. The family of all δ-open sets in (X, τ) forms a topology which is weaker than τ. Actually this topology coincides with τ_s.

Definition 1. A subset A of a topological space (X, τ) is said to be δ-preopen [10] (resp. preopen [7]) if $A \subset Int(Cl_\delta(A))$ (resp. $A \subset Int(Cl(A))$).

The complement of a δ-preopen (resp. preopen) set is said to be δ-preclosed (resp. preclosed). The family of all δ-preopen (resp. δ-preclosed, preopen, preclosed) sets in a topological space (X, τ) is denoted by δPO(X, τ) (resp. δPC(X, τ), PO(X, τ), PC(X, τ)).

Definition 2. Let A be a subset of a topological space (X, τ). The intersection of all δ-preclosed (resp. preclosed) sets containing A is called the δ-preclosure [10] (resp. preclosure [3]) of A and is denoted by $pCl_\delta(A)$ (resp. $pCl(A)$).

Definition 3. Let (X, τ) be a topological space. A subset U of X is called a (δ, p)-neighbourhood of a point $x \in X$ if there exists a δ-preopen set V such that $x \in V \subseteq U$.

In [3], basic properties of the preclosure are obtained. Moreover, for the δ-preclosure the following properties are shown in [10]:

Lemma 2.1. (Raychaudhuri and Mukherjee [10]). *For the δ-preclosure of subsets A, B in a topological space (X,τ), the following properties hold:*

(1) *A is δ-preclosed in (X,τ) if and only if $A = pCl_\delta(A)$,*
(2) *If $A \subset B$, then $pCl_\delta(A) \subset pCl_\delta(B)$,*
(3) *$pCl_\delta(A)$ is δ-preclosed , that is, $pCl_\delta(A) = pCl_\delta(pCl_\delta(A)))$,*
(4) *$x \in pCl_\delta(A)$ if and only if $A \cap V \neq \emptyset$ for every $V \in \delta PO(X,\tau)$ containing x.*

Lemma 2.2. *For a family $\{A_\alpha \mid \alpha \in \triangle\}$ of subsets a topological space (X,τ), the following properties hold:*

(1) *$pCl_\delta(\cap\{A_\alpha : \alpha \in \triangle\}) \subset \cap\{pCl_\delta(A_\alpha) : \alpha \in \triangle\}$.*
(2) *$pCl_\delta(\cup\{A_\alpha : \alpha \in \triangle\}) \supset \cup\{pCl_\delta(A_\alpha) : \alpha \in \triangle\}$.*

Proof. (1) Since $\cap_{\alpha\in\triangle} A_\alpha \subset A_\alpha$ for each $\alpha \in \triangle$, by Lemma 2.1 we have $pCl_\delta(\cap_{\alpha\in\triangle} A_\alpha) \subset pCl_\delta(A_\alpha)$ for each $\alpha \in \triangle$ and hence $pCl_\delta(\cap_{\alpha\in\triangle} A_\alpha) \subset \cap_{\alpha\in\triangle} pCl_\delta(A_\alpha)$.

(2) Since $A_\alpha \subset \bigcup_{\alpha\in\triangle} A_\alpha$ for each $\alpha \in \triangle$, by Lemma 2.1 we have $pCl_\delta(A_\alpha) \subset pCl_\delta(\bigcup_{\alpha\in\triangle} A_\alpha)$ and hence $\bigcup_{\alpha\in\triangle} pCl_\delta(A_\alpha) \subset pCl_\delta(\bigcup_{\alpha\in\triangle} A_\alpha)$.

Lemma 2.3. (Veličko [13]). *For a subset A of a topological space (X,τ), the following properties hold:*

(1) *If A is open in (X,τ), then $Cl_\delta(A) = Cl(A)$,*
(2) *If A is closed in (X,τ), then $Int_\delta(A) = Int(A)$.*

Lemma 2.4. *Let A be a subset of a topological space (X,τ). Then the following properties hold:*

(1) *If A is preopen in (X,τ), then it is δ-preopen in (X,τ),*

(2) *A is δ-preopen in (X,τ) if and only if it is preopen in (X,τ_s),*

(3) *A is δ-preclosed in (X,τ) if and only if it is preclosed in (X,τ_s).*

Proof. (1) This is obvious since $Cl(A) \subset Cl_\delta(A)$ for any subset A of X.

(2) Since $Cl_\delta(A)$ is closed in (X,τ), this property follows from Lemma 2.3 and the fact that $Int(Cl_\delta(A)) = Int_\delta(Cl_\delta(A)) = \tau_s\text{-}Int(\tau_s\text{-}Cl(A))$, where τ_s-$Int(A)$ (resp. τ_s-$Cl(A)$) denotes the interior (resp. the closure) of A in the space (X,τ_s).

(3) This is obvious by (2).

Lemma 2.5. (Maki et al. [6]). *Let (X,τ) be a topological space. For each point $x \in X$, $\{x\}$ is preopen or preclosed.*

3. $D_{(\delta,p)}$-sets and associated separation axioms.

Definition 4. A subset A of a topological space (X,τ) is called a $D_{(\delta,p)}$-set (resp. D_p-set [1, 4]) if there are two $U, V \in \delta PO(X,\tau)$ (resp. $PO(X,\tau)$) such that $U \neq X$ and A=$U \backslash V$.

It is true that every δ-preopen (resp. preopen) set U different from X is a $D_{(\delta,p)}$-set (resp. D_p-set) if A=U and V=$\emptyset$.

Definition 5. A topological space (X,τ) is said to be

(1) (δ,p)-D_0 (resp. pre-D_0 [1, 4]) if for any distinct pair of points x and y of X there exist a $D_{(\delta,p)}$-set (resp. D_p-set) of X containing x but not y or a $D_{(\delta,p)}$-set (resp. D_p-set) of X containing y but not x,

(2) (δ,p)-D_1 (resp. pre-D_1 [1, 4]) if for any distinct pair of points x and y of X there exist a $D_{(\delta,p)}$-set (resp. D_p-sets) of X containing x but not y and a $D_{(\delta,p)}$-set (resp. D_p-set) of X containing y but not x,

(3) (δ,p)-D_2 (resp. pre-D_2 [1, 4]) if for any distinct pair of points x and y of X there exists disjoint $D_{(\delta,p)}$-sets (resp. D_p-sets) G and E of X containing x and y, respectively.

Definition 6. A topological space (X,τ) is said to be

(1) (δ,p)-T_0 (resp. pre-T_0 [5, 8]) if for any distinct pair of points x and y in X, there exist a δ-preopen (resp. preopen) set U in X containing x but not y or a δ-preopen (resp. preopen) set V in X containing y but not x,

(2) (δ,p)-T_1 (resp. pre-T_1 [5, 8]) if for any distinct pair of points x and y in X, there exist a δ-preopen (resp. preopen) set U in X containing x but not y and a δ-preopen (resp. preopen) set V in X containing y but not x,

(3) (δ,p)-T_2 (resp. pre-T_2 [5, 8]) if for any distinct pair of points x and y in X, there exist δ-preopen (resp. preopen) sets U and V in X containing x and y, respectively, such that $U\cap V=\emptyset$.

Remark 3.1.

(i) If (X,τ) is (δ,p)-T_i , then it is (δ,p)- T_{i-1}, $i=1,2$.
(ii) If (X,τ) is (δ,p)-T_i , then (X,τ) is (δ,p)-D_i for $i=0,1,2$.
(iii) If (X,τ) is (δ,p)-D_i , then it is (δ,p)-D_{i-1}, $i=1,2$.

By Remark 3.1, we have the following diagram:

$$\begin{array}{ccccc} \text{pre-}T_2 & \rightarrow & (\delta,p)\text{-}T_2 & \rightarrow & (\delta,p)\text{-}D_2 \\ \downarrow & & \downarrow & & \downarrow \\ \text{pre-}T_1 & \rightarrow & (\delta,p)\text{-}T_1 & \rightarrow & (\delta,p)\text{-}D_1 \\ \downarrow & & \downarrow & & \downarrow \\ \text{pre-}T_0 & \rightarrow & (\delta,p)\text{-}T_0 & \rightarrow & (\delta,p)\text{-}D_0 \end{array}$$

Theorem 3.2. *For a topological space (X,τ), the following properties hold:*

(1) (X,τ) is (δ,p)-D_2 (resp. (δ,p)-D_1) if and only if (X,τ_s) is pre-D_2 (resp. pre-D_1),

(2) (X,τ) is (δ,p)-T_2 (resp. (δ,p)-T_1) if and only if (X,τ_s) is pre-T_2 (resp. pre-T_1).

Proof. This follows easily from Lemma 2.4(2).

Theorem 3.3. *For a topological space (X,τ), the following statements are true: (X,τ) is (δ,p)-D_1 (resp. pre-D_1) if and only if it is (δ,p)-D_2 (resp. pre-D_2).*

Proof. We prove only the first statement, the second being given in [1, 4].

Sufficiency. This follows from Remark 3.1(3).

Necessity. Suppose X is a (δ,p)-D_1. Then for each distinct pair $x,y\in X$, we have $D_{(\delta,p)}$-sets G_1,G_2 such that $x\in G_1$, $y\notin G_1$; $y\in G_2$, $x\notin G_2$. Let $G_1= U_1\backslash U_2$, $G_2=U_3\backslash U_4$. From $x\notin G_2$ we have either $x\notin U_3$ or $x\in U_3$ and $x\in U_4$. We discuss the two cases separately.

(1) $x\notin U_3$. From $y\notin G_1$ we have two subcases:
   (a) $y\notin U_1$. From $x\in U_1\backslash U_2$ we have $x\in U_1\backslash\ (U_2\cup\ U_3)$ and from $y\in U_3\backslash U_4$ we have $y\in U_3\backslash(U_1\cup U_4)$. It is easy to see that $(U_1\backslash(U_2\cup U_3))\cap(U_3\backslash(U_1\cup U_4))=\emptyset$.
   (b) $y\in U_1$ and $y\in U_2$. We have $x\in U_1\backslash U_2$, $y\in U_2$ and $(U_1\backslash U_2)\cap U_2=\emptyset$.

(2) $x\in U_3$ and $x\in U_4$. We have $y\in U_3\backslash U_4$, $x\in U_4$ and $(U_3\backslash U_4)\cap U_4=\emptyset$.

From the discussion above we know that the space X is (δ,p)- D_2.

Definition 7. A point $x\in X$ which has only X as the (δ,p)-neighborhood is called a (δ,p)-neat point.

Theorem 3.4. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) *is* (δ,p)-D_1*;*

(2) (X,τ) *has no* (δ,p)*-neat point.*

Proof. (1) $\rightarrow$ (2). Since (X,τ) is (δ,p)-D_1, so each point x of X is contained in a $D_{(\delta,p)}$-set $O{=}U\backslash V$ and thus in U. By definition $U \neq X$. This implies that x is not a (δ,p)-neat point.

(2) $\rightarrow$ (1). By Lemma 2.5 for each distinct pair of points $x, y \in X$, at least one of them, x(say) has a (δ,p)-neighborhood U containing x and not y. Thus U which is different from X is a $D_{(\delta,p)}$-set. If X has no (δ,p)-neat point, then y is not a (δ,p)-neat point. This means that there exists a (δ,p)-neighborhood V of y such that $V \neq X$. Thus $y \in (V\backslash U)$ but not x and $V\backslash U$ is a $D_{(\delta,p)}$-set. Hence X is (δ,p)-D_1.

Definition 8. A topological space (X,τ) is (δ,p)-symmetric if for x and y in X, $x \in pCl_\delta(\{y\})$ implies $y \in pCl_\delta(\{x\})$.

Theorem 3.5. *For a topological space (X,τ), the following properties are equivalent:*

(1) (X,τ) *is* (δ,p)*-symmetric;*

(2) *For each* $x \in X$, $\{x\}$ *is* δ*-preclosed;*

(3) (X,τ) *is* (δ,p)-T_1.

Proof. (1) $\rightarrow$ (2) : Let x be any point of X. Let y be any distinct point from x. By Lemma 2.5, $\{y\}$ is preopen or preclosed in (X,τ).
(i) In case when $\{y\}$ is preopen, put $V_y = \{y\}$ then $V_y \in \delta PO(X,\tau)$. (ii) In case when $\{y\}$ is preclosed $x \notin \{y\} = pCl(\{y\})$ and $x \notin pCl_\delta(\{y\})$. By (1), $y \notin pCl_\delta(\{x\})$. Now put $V_y = X\backslash pCl_\delta(\{x\})$. Then $x \notin V_y$, $y \in V_y$ and $V_y \in \delta PO(X,\tau)$. Therefore, we obtain that for each $y \in X\backslash\{x\}$ there exists $V_y \in \delta PO(X,\tau)$ such that $x \notin V_y$, $y \in V_y \in \delta PO(X,\tau)$. Hence

$X\backslash\{x\} = \bigcup_{y\in X\backslash\{x\}} V_y \in \delta PO(X,\tau)$. This shows that $\{x\}$ is δ-preclosed in (X,τ).

(2) $\to$ (3) : Suppose $\{p\}$ is δ-preclosed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X\backslash\{x\}$. Hence $X\backslash\{x\}$ is a δ-preopen set containing y but not containing x. Similarly $X\backslash\{y\}$ is a δ-preopen set containing x but not containing y. Accordingly X is a (δ,p)-T_1 space.

(3) $\to$ (1) : Suppose that $y \notin pCl_\delta(\{x\})$. Then, since $x \neq y$, by (3) there exists a δ-preopen set U containing x such that $y \notin U$ and hence $x \notin pCl_\delta(\{y\})$. This shows that $x \in pCl_\delta(\{y\})$ implies $y \in pCl_\delta(\{x\})$. Therefore, (X,τ) is (δ,p)-symmetric.

Definition 9. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be

(i) δ-precontinuous if for each $x \in X$ and each δ-preopen set V containing $f(x)$, there is a δ-preopen set U in X containing x such that $f(U) \subset V$,

(ii) δ^*-almost-continuous [9] if $f^{-1}(V) \in \delta PO(X,\tau)$ for each $V \in \delta PO(Y,\tau)$,

(iii) preirresolute [12] if $f^{-1}(V) \in PO(X,\tau)$ for each $V \in PO(Y,\tau)$.

Lemma 3.6. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(i) *f is δ-precontinuous;*

(ii) *f is δ^*-almost-continuous;*

(iii) *$f : (X,\tau_s) \to (Y,\sigma_s)$ is preirresolute.*

Theorem 3.7. *If $f : (X,\tau) \to (Y,\sigma)$ is a δ-precontinuous surjective function and E is a $D_{(\delta,p)}$-set in Y, then the inverse image $f^{-1}(E)$ is a $D_{(\delta,p)}$-set in X.*

Proof. Let E be a $D_{(\delta,p)}$-set in Y. Then there are δ-preopen sets U_1 and U_2 in Y such that $E = U_1\backslash U_2$ and $U_1 \neq Y$. By the δ-precontinuity of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are δ-preopen in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1)\backslash f^{-1}(U_2)$ is a $D_{(\delta,p)}$-set.

Theorem 3.8. *If (Y, σ) is (δ, p)-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is a δ-precontinuous bijection, then (X, τ) is (δ, p)-D_1.*

Proof. Suppose that Y is a (δ, p)-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is (δ, p)-D_1, there exist $D_{(\delta,p)}$-sets G_x and G_y of Y containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 3.7 , $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $D_{(\delta,p)}$-sets in X containing x and y, respectively, such that $y \notin f^{-1}(G_x)$ and $x \notin f^{-1}(G_y)$. This implies that X is a (δ, p)-D_1 space.

Theorem 3.9. *A topological space (X, τ) is (δ, p)-D_1 if and only if for each pair of distinct points $x, y \in X$, there exists a δ-precontinuous surjective function $f : (X, \tau) \to (Y, \sigma)$ such that $f(x)$ and $f(y)$ are distinct, where (Y, σ) is a (δ, p)-D_1 space.*

Proof. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a δ-precontinuous, surjective function f of a space X onto a (δ, p)-D_1 space Y such that $f(x) \neq f(y)$. By Theorem 3.3, there exist disjoint $D_{(\delta,p)}$-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is δ-precontinuous and surjective, by Theorem 3.7, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $D_{(\delta,p)}$-sets in X containing x and y, respectively. Hence by Theorem 3.3, X is a (δ, p)-D_1 space.

4. Sober (δ, p)-R_0 spaces.

Definition 10. Let A be a subset of a topological space (X, τ). The δ-prekernel (resp. prekernel) of A, denoted by $pKer_\delta(A)$ (resp. $pKer(A)$), is defined to be the set $pKer_\delta(A) = \cap\{O \in \delta PO(X, \tau) \mid A \subset O\}$ (resp. $pKer(A)$ $= \cap\{O \in PO(X, \tau) \mid A \subset O\}$).

Lemma 4.1. *Let* (X,τ) *be a topological space and* $x \in X$*. Then* $pKer_\delta(A) = \{x \in X \mid pCl_\delta(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in pKer_\delta(A)$ and suppose $pCl_\delta(\{x\}) \cap A = \emptyset$. Hence $x \notin X \backslash pCl_\delta(\{x\})$ which is a δ-preopen set containing A. This is absurd, since $x \in pKer_\delta(A)$. Consequently, $pCl_\delta(\{x\}) \cap A \neq \emptyset$. Next, let x be such that $pCl_\delta(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin pKer_\delta(A)$. Then, there exists a δ-preopen set D containing A and $x \notin D$. Let $y \in pCl_\delta(\{x\}) \cap A$. Hence, D is a (δ,p)- neighborhood of y which does not containing x. By this contradiction $x \in pKer_\delta(A)$ and the claim is shown.

Definition 11. A topological space (X,τ) is said to be sober (δ,p)-R_0 if $\bigcap_{x\in X} pCl_\delta(\{x\}) = \emptyset$.

Theorem 4.2. *A topological space* (X,τ) *is sober* (δ,p)-R_0 *if and only if* $pKer_\delta(\{x\}) \neq X$ *for every* $x \in X$*.*

Proof. Suppose that the space (X,τ) be sober (δ,p)-R_0. Assume that there is a point y in X such that $pKer_\delta(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every δ-preopen set V containing y and hence $y \in pCl_\delta(\{x\})$ for any $x \in X$. This implies that $y \in \bigcap_{x\in X} pCl_\delta(\{x\})$. But this is a contradiction.

Now assume that $pKer_\delta(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \bigcap_{x\in X} pCl_\delta(\{x\})$, then every δ-preopen set containing y must contain every point of X. This implies that the space X is the unique δ-preopen set containing y. Hence $pKer_\delta(\{y\}) = X$ which is a contradiction. Therefore (X,τ) is sober (δ,p)-R_0.

Definition 12. A function $f : X \to Y$ is called δ-preclosed if the image of every δ-preclosed subset of X is δ-preclosed in Y.

Theorem 4.3. *If $f : X \to Y$ is an injective δ-preclosed function and X is sober (δ,p)-R_0, then Y is sober (δ,p)-R_0.*

Proof. Since X is sober (δ,p)-R_0, $\bigcap_{x\in X} pCl_\delta(\{x\}) = \emptyset$. Since f is a δ-preclosed injection, we have $\emptyset = f(\bigcap_{x\in X} pCl_\delta(\{x\})) = \bigcap_{x\in X} f(pCl_\delta(\{x\})) \supset \bigcap_{x\in X} pCl_\delta(\{f(x)\}) \supset \bigcap_{y\in Y} pCl_\delta(\{y\})$. Therefore, Y is sober (δ,p)-R_0.

Theorem 4.4. *If a topological space X is sober (δ,p)-R_0 and Y is any topological space, then the product $X\times Y$ is sober (δ,p)-R_0.*

Proof. we show that $\bigcap_{(x,y)\in X\times Y} pCl_\delta(\{(x,y)\}) = \emptyset$. We have:

$$\begin{aligned}\bigcap_{(x,y)\in X\times Y} pCl_\delta(\{(x,y)\}) &\subset \bigcap_{(x,y)\in X\times Y} (pCl_\delta(\{x\})\times pCl_\delta(\{y\}))\\ &= \bigcap_{x\in X} pCl_\delta(\{x\}) \times \bigcap_{y\in Y} pCl_\delta(\{y\}) \subset \emptyset\times Y = \emptyset.\end{aligned}$$

5. (δ,p)-R_0 spaces.

Definition 13. A topological space (X,τ) is said to be a (δ,p)-R_0 (resp. pre-R_0 [2]) if every δ-preopen (resp. preopen) set contains the δ-preclosure (resp. preclosure) of each of its singletons.

Theorem 5.1. *A topological space (X,τ) is (δ,p)-R_0 if and only if (X,τ_s) is pre-R_0.*

Proof. By Lemma 2.4, we have the facts that: (i) a subset A of X is δ-preopen in (X,τ) if and only if A is preopen in (X,τ_s); (ii) for any $x\in X$, $pCl_\delta(\{x\}) = \bigcap\{F/x\in F\in \delta PC(X,\tau)\} = \bigcap\{F/x\in F\in PC(X,\tau_s)\} = \tau_s$-$pCl(\{x\})$. The proof follows immediately from the facts stated above.

Theorem 5.2. *A topological space (X,τ) is (δ,p)-R_0 if and only if it is*

(δ, p)-T_1.

Proof. Necessity. Let x and y be any distinct points of X. For any point $x \in X$, $\{x\}$ is preopen or preclosed by Lemma 2.5.

(i) In case when $\{x\}$ is preopen, let $V = \{x\}$ then $x \in V$, $y \notin V$ and $V \in \delta PO(X, \tau)$. Moreover, since (X, τ) is $(\delta, p) - R_0$, we have $pCl_\delta(\{x\}) \subset V$. Hence $x \notin X \backslash V$, $y \in X \backslash V$ and $X \backslash V \in \delta PO(X, \tau)$. (ii) in case when $\{x\}$ is preclosed, $y \in X \backslash \{x\}$ and $X \backslash \{x\} \in \delta PO(X, \tau)$. Hence since (X, τ) is (δ, p)-R_0, $pCl_\delta(\{y\}) \subset X \backslash \{x\}$. Now, let $V = X \backslash pCl_\delta(\{y\})$, then $x \in V$, $y \notin V$ and $V \in \delta PO(X, \tau)$. Then, we obtain that (X, τ) is (δ, p)-T_1.

Sufficiency. Let V be any δ-preopen set of (X, τ) and $x \in V$. For each $y \in X \backslash V$, there exists $V_y \in \delta PO(X, \tau)$ such that $x \notin V_y$ and $y \in V_y$. Therefore, we have $pCl_\delta(\{x\}) \bigcap V_y = \emptyset$ for each $y \in X \backslash V$ and hence $pCl_\delta(\{x\}) \bigcap (\bigcup_{y \in X \backslash V} V_y) = \emptyset$. Since $y \in V_y$, $X \backslash V \subset \bigcup_{y \in X \backslash V} V_y$ and $pCl_\delta(\{x\}) \bigcap (X \backslash V) = \emptyset$. This implies that $pCl_\delta(\{x\}) \subset V$. Hence (X, τ) is (δ, p)-R_0.

Corollary 5.3. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) *is* (δ, p)-R_0*;*

(2) (X, τ) *is* (δ, p)-T_1*;*

(3) (X, τ) *is* (δ, p)*-symmetric.*

Proof. This is an immediate consequence of Theorems 3.5 and 5.2.

Remark 5.4. Observe that by using Theorem 3.5 and Corollary 5.3, we have: (X, τ) is (δ, p)-Ro if and only if for each $x \in X$, $\{x\}$ is δ-preclosed.

Corollary 5.5. *Every (δ, p)-R_0 space (X, τ) is sober (δ, p)-R_0, where we suppose that X has at least two points.*

Proof. Let x and y be any distinct points of X. Since (X,τ) is (δ,p)-R_0, by Theorem 5.2 it is (δ,p)-T_1. Hence by Theorem 3.5 $pCl_\delta(\{x\}) = \{x\}$ and $pCl_\delta(\{y\})=\{y\}$ and hence we obtain $\bigcap_{p\in X} pCl_\delta(\{p\})\subset pCl_\delta(\{x\})\bigcap pCl_\delta(\{y\}) = \{x\}\bigcap\{y\} = \emptyset$. This shows that (X,τ) is sober (δ,p)-R_0.

As we have stated above, the following properties hold:

$\delta PO(X,\tau) = PO(X,\tau_s)$, $\delta PC(X,\tau) = PC(X,\tau_s)$, $pCl_\delta(\{x\}) = \tau_s$-$pCl(\{x\})$ and $pKer_\delta(\{x\}) = \tau_s$-$pKer(\{x\})$ for each point x of a topological space (X,τ). Therefore, we obtain the following important characterizations of (δ,p)-R_0 spaces which are modifications of theorems obtained in [2].

Question 5.6. *Is there any example showing that the converse of Corollary* 5.5 *is not true*?

Theorem 5.7. *For a topological space (X,τ), the following properties are equivalent :*

(1) *(X,τ) is a (δ,p)-R_0 space;*
(2) *For any nonempty set A and $G\in\delta PO(X,\tau)$ such that $A\cap G\neq\emptyset$, there exists $F\in\delta PC(X,\tau)$ such that $A\cap F\neq\emptyset$ and $F\subset G$;*
(3) *Any $G\in\delta PO(X,\tau)$, $G=\cup\{F\in\delta PC(X,\tau)\mid F\subset G\}$;*
(4) *Any $F\in\delta PC(X,\tau)$, $F=\cap\{G\in\delta PO(X,\tau)\mid F\subset G\}$;*
(5) *For any $x\in X, pCl_\delta(\{x\})\subset pKer_\delta(\{x\})$.*

Theorem 5.8. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is a (δ,p)-R_0 space;*
(2) *If F is δ-preclosed, then $F=pKer_\delta(F)$;*
(3) *If F is δ-preclosed and $x\in F$, then $pKer_\delta(\{x\})\subset F$;*
(4) *If $x\in X$, then $pKer_\delta(\{x\})\subset pCl_\delta(\{x\})$.*

6. (δ, p)-R_1 spaces.

Definition 14. A topological space (X, τ) is said to be (δ, p)-R_1 if for x, y in X with $pCl_\delta(\{x\}) \neq pCl_\delta(\{y\})$, there exist disjoint δ-preopen sets U and V such that $pCl_\delta(\{x\})$ is a subset of U and $pCl_\delta(\{y\})$ is a subset of V.

Definition 15. A topological space (X, τ) is said to be pre-R_1 [2] if for x, y in X with $pCl(\{x\}) \neq pCl(\{y\})$, there exist disjoint preopen sets U and V such that $pCl(\{x\})$ is a subset of U and $pCl(\{y\})$ is a subset of V.

Theorem 6.1. *A topological space (X, τ) is (δ, p)-R_1 if and only if (X, τ_s) is pre-R_1.*

Proof. This follows from the facts that $\delta PO(X, \tau) = PO(X, \tau_s)$ and $pCl_\delta(\{x\}) = \tau_s$-$pCl(\{x\})$ for each $x \in X$.

Theorem 6.2. *A topological space (X, τ) is (δ, p)-R_1 if and only if it is (δ, p)-T_2.*

Proof. Necessity. Let x and y be any distinct points of X. By Lemma 2.5, each point x of X is preopen or preclosed.

(i) In case when $\{x\}$ is preopen, since $\{x\} \bigcap \{y\} = \emptyset$, $\{x\} \bigcap pCl_\delta(\{y\}) \subset \{x\} \bigcap pCl(\{y\}) = \emptyset$, and hence $pCl_\delta(\{x\}) \neq pCl_\delta(\{y\})$.

(ii) In case when $\{x\}$ is preclosed, $pCl_\delta(\{x\}) \bigcap \{y\} \subset pCl(\{x\}) \bigcap \{y\} = \{x\} \bigcap \{y\} = \emptyset$ and hence $pCl_\delta(\{x\}) \neq pCl_\delta(\{y\})$. Since (X, τ) is (δ, p)-R_1, there exist disjoint δ-preopen sets U and V such that $x \in pCl_\delta(\{x\}) \subset U$ and $y \in pCl_\delta(\{y\}) \subset V$. This shows that (X, τ) is (δ, p)-T_2.

Sufficiency. Let x and y be any points of X such that $pCl_\delta(\{x\}) \neq pCl_\delta(\{y\})$. By Remark 3.1, every (δ, p)-T_2 space is (δ, p)-T_1. Therefore, by Theorem 3.5 $pCl_\delta(\{x\}) = \{x\}$ and $pCl_\delta(\{y\}) = \{y\}$ and hence $x \neq y$. Since (X, τ) is (δ, p)-T_2, there exist disjoint δ-preopen sets U and V such that

$pCl_\delta(\{x\}) = \{x\} \subset U$ and $pCl_\delta(\{y\}) = \{y\} \subset V$. This shows that (X, τ) is (δ, p)-R_1.

Corollary 6.3. *Every (δ, p)-R_1 space (X, τ) is (δ, p)-R_0.*

Proof. Since every (δ, p)-T_2 space is (δ, p)-T_1, this is an immediate consequence of Theorems 5.2 and 6.2.

Example 6.4. The converse of Corollary 6.3 need not be true. Let the set X be the union of any infinite set N and two distinct one point sets x_1 and x_2. If we topologize X by taking any subset of N open and taking any set containing x_1 and x_2 open if and only if it contains all but a finite number of points in N, then X is (δ, p)-R_0 but not (δ, p)-R_1.

Also take X to be $R \times R$, where R is the set of real numbers. Let τ consists of $\emptyset$ and all subsets of X whose complements are subsets of a finite number of lines parallel to the x-axis. Then this space is (δ, p)-R_0 but not (δ, p)-R_1.

The following theorem is a modification of Theorem 4.3 in [2].

Theorem 6.5. *For a topological space (X, τ), the following statements are equivalent:*

(1) *(X, τ) is a (δ, p)-R_1;*

(2) *If $x, y \in X$ such that $pCl_\delta(\{x\}) \neq pCl_\delta(\{y\})$, then there exists δ-preclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \bigcup F_2$.*

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