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TOTALLY GEODESIC COMPLEX EXTENSORS IN INDEFINITE COMPLEX EUCLIDEAN SPACES

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Abstract. The notion of complex extensors in complex Euclidean spaces was first introduced in [2]. The notion was extended to complex extensors in indefinite complex Euclidean spaces in [3]. In this article we classify totally geodesic complex extensors in indefinite complex Euclidean spaces.

1. Complex extensors. Let $\mathbf{E}_k^m$ denote the pseudo-Euclidean m-space endowed with pseudo-Euclidean metric g_k with index k given by $g_k = -\sum_{j=1}^k dx_j^2 + \sum_{\ell=k+1}^m dx_\ell^2$.

The complex number m-space $\mathbf{C}^m$ with complex coordinates $z_1, \ldots, z_m$ endowed with $g_{m,k}$: the real part of the Hermitian form

$$b_{m,k}(z,w) = -\sum_{k=1}^{k} \bar{z}_k w_k + \sum_{j=k+1}^{m} \bar{z}_j w_j, \quad z, w \in \mathbf{C}^m, \tag{1.1}$$

is a flat indefinite complex space with complex index k. We simply denote the pair $(\mathbf{C}^n, g_{m,k})$ by $\mathbf{C}_k^m$ which is called an indefinite complex Euclidean m-space.

We recall the definition of complex extensors as follows:

Let $G : M_t^{n-1} \to \mathbf{E}_k^m - \{0\}$ be an isometric immersion of a pseudo-

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Riemannian $(n-1)$-manifold with index t into $\mathbf{E}_k^m - \{0\}$ and let $F : I \to \mathbf{C}^*$ be a unit speed curve in the punctured complex plane $\mathbf{C}^* := \mathbf{C} - \{0\}$. Then we may extend the immersion $G : M_t^{n-1} \to \mathbf{E}_k^m$ to a map of $I \times M_t^{n-1}$ into $\mathbf{C}_k^m = \mathbf{C} \otimes \mathbf{E}_k^m$ by

$$\phi = F \otimes G : I \times M_t^{n-1} \to \mathbf{C}_k^m, \tag{1.2}$$

where $F \otimes G$ is the tensor product of F and G defined by

$$(F \otimes G)(s, p) = F(s) \otimes G(p), \quad s \in I,\ p \in M_t^{n-1}. \tag{1.3}$$

If $\phi = F \otimes G$ is an immersion, we call such an extension $F \otimes G$ of the immersion G a *complex extensor* of G via F (or of the submanifold M_t^{n-1} via F).

2. Totally geodesic complex extensors. The following result classifies totally geodesic complex extensors in indefinite complex Euclidean spaces.

Theorem 1. *Let $G : M_t^{n-1} \to \mathbf{E}_k^m - \{0\}$ be an isometric immersion of a pseudo-Riemannian $(n-1)$-manifold into $\mathbf{E}_k^m - \{0\}$ and $F : I \to \mathbf{C}^*$ be a unit speed curve. Then the complex extensor $\phi = F \otimes G : I \times M_t^{n-1} \to \mathbf{C}_k^m$ is totally geodesic* (*with respect to the induced metric*) *if and only if one of the following two cases occurs:*

(a) *$G : M_t^{n-1} \to \mathbf{E}_k^m$ is either totally geodesic or contained in an affine n-subspace of $\mathbf{E}_k^m$, and $F(s) = (s + a)c$ for some real number a and some unitary complex number c.*

(b) *$n = 2$, $F : I \to \mathbf{C}^*$ is an arbitrary unit speed curve, and G is an open portion of either a space-like line or a time-like line through the origin $o \in \mathbf{E}_k^m$.*

Proof. If $G : M_t^{n-1} \to \mathbf{E}_k^m - \{0\}$ is an isometric immersion of a pseudo-Riemannian $(n-1)$-manifold with index t into $\mathbf{E}_k^m - \{0\}$, then each normal space of G is definite.

For a given unit speed curve $F : I \to \mathbf{C}^*$, the complex extensor $\phi = F \otimes G : I \times M_t^{n-1} \to \mathbf{C}_k^m$ satisfies

$$\phi_s = F'(s) \otimes G, \quad Y\phi = F \otimes Y, \tag{2.1}$$

$$\phi_{ss} = F''(s) \otimes G, \quad Y\phi_s = F'(s) \otimes Y, \tag{2.2}$$

$$YZ\phi = F \otimes \nabla_Y Z + F \otimes h_G(Y, Z), \tag{2.3}$$

where $\phi_s = \partial\phi/\partial s$, $\phi_{ss} = \partial^2\phi/\partial s^2$, ∇ denotes the Levi-Civita connection of M_t^{n-1}, Y and Z vectors fields tangent to the second component of $I \times M_t^{n-1}$, and h_G the second fundamental form of the isometric immersion $G : M_t^{n-1} \to \mathbf{E}_k^m$.

We shall regard tangent vectors of M_t^{n-1} also as tangent vectors of the product manifold $I \times M_t^{n-1}$ in a natural way. We shall also identify each tangent vector of a submanifold with its image via the differential of the immersion.

Since $F : I \to \mathbf{C}$ is a unit speed curve in $\mathbf{C}$, we have $F''(s) = i\kappa(s)F'(s)$, where κ is the curvature function of F. Thus, for any unit normal vector ξ of M_t^{n-1} in $\mathbf{E}_k^m$, the vector $F''(s) \otimes \xi$ is normal to $I \times M_t^{n-1}$ in $\mathbf{C}_k^m$.

If the complex extensor $\phi = F \otimes G$ is totally geodesic with respect to its induced metric, then $\phi_{ss}, Y\phi_s$ and $YZ\phi$ are tangent vectors of $I \times M_t^{n-1}$ in $\mathbf{C}_k^m$ by definition. So, it follows from (2.1) and (2.3) that $F \otimes h_G(Y, Z)$ is tangent to $I \times M_t^{n-1}$. Hence we have

$$\big\langle F''(s) \otimes \xi, F(s) \otimes h_G(Y, Z)\big\rangle = 0 \tag{2.4}$$

for any $s \in I$, any tangent vectors Y, Z of M_t^{n-1}, and any normal vector ξ of G. From (2.4) we obtain $\langle\langle F''(s), F(s)\rangle\rangle \langle \xi, h_G(Y, Z)\rangle = 0$ identically, where $\langle\langle\ ,\ \rangle\rangle$ is the canonical inner product of the complex plane. Since each normal space of G is non-degenerate, $\langle\langle F''(s), F(s)\rangle\rangle \langle \xi, h_G(Y, Z)\rangle = 0$ implies

(i) G is non-totally geodesic at each point and $\langle\langle F'', F\rangle\rangle = 0$ identically, or

(ii) G is a totally geodesic immersion.

Case (i): *G is non-totally geodesic at each point and $\langle\langle F'', F\rangle\rangle = 0$ identically.* In this case, since F is a unit speed curve in $\mathbf{C}^*$, we have $F(s) = \alpha(s)F'(s)$ for some nonzero real-valued function α defined on the open subset $I_1 = \{s \in I : \kappa(s) \neq 0\}$. So, after applying the Frenet formula, we have $F''(s) = 0$ for $s \in I_1$. On $I - I_1$, we have $F'' = 0$ trivially. Hence, we have $F''(s) = 0$ identically on I. Thus, by using the fact that F is unit speed, we obtain $F(s) = sc + b$ for some $c, b \in \mathbf{C}$ with $|c| = 1$. Therefore, we obtain from (2.1), (2.2) and (2.3) that

$$\phi_s = c \otimes G, \quad Y\phi = (sc+b) \otimes Y, \tag{2.5}$$

$$\phi_{ss} = 0, \quad Y\phi_s = c \otimes Y, \tag{2.6}$$

$$YZ\phi = (sc+b) \otimes \nabla_Y Z + (sc+b) \otimes h_G(Y,Z) \tag{2.7}$$

for Y, Z tangent to M_t^{n-1}.

Since ϕ is totally geodesic, (2.5) and (2.6) imply that, for each tangent vector Y of M_t^{n-1}, there is a tangent vector Z of M_t^{n-1} such that

$$c \otimes (Y - \alpha G) = (sc+b) \otimes \beta Z \tag{2.8}$$

for some real-valued functions α and β. But this is impossible unless $b = ac$ for some real number a. Therefore we have $F(s) = (s+a)c$. Consequently, (2.5), (2.6) and (2.7) reduce to

$$\phi_s = c \otimes G, \ \ Y\phi = c \otimes (s+a)Y, \ \ \phi_{ss} = 0, \tag{2.9}$$

$$Y\phi_s = c \otimes Y, \ \ YZ\phi = (\nabla_Y Z)\phi + c \otimes (s+a)h_G(Y,Z). \tag{2.10}$$

Because ϕ is totally geodesic, (2.9) and (2.10) imply that, for any given tangent vectors Y, Z of M_t^{n-1}, we have

$$h_G(Y, Z) = \gamma G + \delta W \tag{2.11}$$

for some tangent vector W of M_t^{n-1} and real-valued functions γ and δ. So, if we denote by $G^{\perp}$ the normal component of G in $\mathbf{E}_k^m$, then (2.11) implies that $h_G(Y, Z)$ is in the direction of $G^{\perp}$. Thus, the first normal space of G at each point $p \in M_t^{n-1}$ is one-dimensional, since G is non-totally geodesic at p by assumption. Furthermore, by taking the covariant derivative of (2.11) with respect to the normal connection and by applying (2.11) again, we also know that the first normal bundle is parallel in the normal bundle (with respect to the normal connection). Consequently, the reduction theorem implies that M_t^{n-1} is immersed into some affine n-subspace of $\mathbf{E}_k^m$. Hence we obtain Case (a) of the theorem.

Case (ii): *G is a totally geodesic immersion.* In this case, M_t^{n-1} is immersed into an affine $(n-1)$-subspace, say E, of $\mathbf{E}_k^m$. Also, it follows from (2.3) that, for Y, Z tangent to M_t^{n-1}, $YZ\phi$ is a tangent vector of the complex extensor ϕ.

Case (ii-1): $n \geq 3$. In this case, for each $p \in G(M_t^{n-1})$ with $G(p) \neq o$, there exists a nonzero vector $Y \in T_pM_t^{n-1}$ perpendicular to $G(p)$. For such Y, (2.1) and (2.2) imply that $Y\phi_s$ is parallel to $Y\phi$. Hence, for each $s \in I$, $F(s)$ and $F'(s)$ are parallel. So, $F(s) = \alpha(s)F'(s)$ for some real-valued function α on I. Therefore, by applying the Frenet formula we obtain $F''(s) = 0$ and $\alpha(s) = s + b$ for some real number b.

It follows from $F''(s) = 0$ that the unit speed curve F satisfies $F(s) = as + c$ for some $a, c \in \mathbf{C}$ with $|a| = 1$. Hence, by using $\alpha(s) = s + b$ and $F(s) = \alpha(s)F'(s)$, we obtain $F(s) = (s + b)a$. Consequently, we obtain Case (a) as well.

Case (ii-2): $n = 2$. In this case, G is an open portion of a line, say L, in $\mathbf{E}_k^m$.

If line L passes through the origin o, we have Case (b) of the theorem.

If L does not pass through o, then, for a unit tangent vector Y_p of L at any given point p with $G(p) \neq o$, the vectors Y_p and $G(p)$ are independent. Thus, it follows from (2.1) and (2.2) that, for each $s \in I$, $F(s)$ and $F'(s)$ are parallel. Consequently, the same method as Case (ii-1) yields Case (a) of the theorem.

To prove the converse, first let us assume that Case (a) of the theorem occurs. In this case, after applying a suitable translation in s, we have $F(s) = cs$. Therefore, (2.1), (2.2) and (2.3) become

$$\phi_s = c \otimes G, \;\; Y\phi = cs \otimes Y, \;\; \phi_{ss} = 0, \tag{2.12}$$

$$Y\phi_s = c \otimes Y, \;\; YZ\phi = cs \otimes \nabla_Y Z + cs \otimes h_G(Y, Z), \tag{2.13}$$

Since the complex extensor is an immersion, (2.12) shows that G is transversal to submanifold $G : M_t^{n-1} \to \mathbf{E}_k^m$ at each point. Moreover, we know from (2.12) and (2.13) that in order to prove the totally geodesicity of the complex extensor, it suffices to show the following condition holds:

(A) *For any point $p \in M_t^{n-1}$ and vectors $Y, Z \in T_p(M_t^{n-1})$, $h_G(Y, Z)$ lies in the vector subspace of $\mathbf{E}_k^m$ spanned by $G(p)$ and $G_*(T_p(M_t^{n-1}))$.*

When G is totally geodesic, we have $h_G = 0$. So condition (A) holds trivially. If $G(M_t^{n-1})$ is contained in an affine n-space of $\mathbf{E}_k^m$, condition (A) follows from the fact that G is transversal to its image at each point. Consequently, the complex extensor is totally geodesic in either cases.

Next, let us assume that Case (b) of the theorem holds. In this case, we may assume that $G = tv$ for some space-like or time-like unit vector $v \in \mathbf{E}_k^m$. Thus, (2.1), (2.2) and (2.3) reduce to

$$\phi_s = tF'(s) \otimes v, \quad \phi_t = F \otimes v, \quad \phi_{ss} = tF''(s) \otimes v, \tag{2.14}$$

$$\phi_{st} = F'(s) \otimes v, \;\; \phi_{tt} = 0. \tag{2.15}$$

Clearly, it follows from (2.14) and (2.15) that, in order to show totally geodesicity of ϕ, it suffices to prove that ϕ_{ss} is a tangent vector of ϕ for each $s \in I$.

If the curvature function κ is nonzero at some $s_o \in I$, then $F''(s_o)$ and $F'(s_o)$ are linearly independent over $\mathbf{R}$, since $F'' = i\kappa F'$. In this case, we may put $F''(s_o) = c_1 F'(s_o) + c_2 F(s_o)$ for some real numbers c_1 and c_2. Thus, according to (2.14) and (2.15), $\phi_{ss}(s_o)$ is a tangent vector of the complex extensor.

If $\kappa(s_1) = 0$ for some $s_1 \in I$, then (2.14) yields $\phi_{ss}(s_1) = 0$. Therefore, it follows from these that ϕ_{ss} is always a tangent vector of the complex extensor. Consequently, the complex extensor is totally geodesic.

Remark 1. The result of this article has been stated in [3] in somewhat different form without proof.

Remark 2. When M_t^{n-1} and $\mathbf{E}_k^m$ are Riemannian, Theorem 1 reduces to Proposition 2.2 of [2]. Its proof in [2] contains an obscurity; so it shall be replaced by the one given above under the condition: $t = k = 0$.

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