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GEOMETRIC MECHANICS ON THE HEISENBERG GROUP

BY

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Abstract. We give detailed discussion of subRiemannian geometry which arised from the sub-Laplacian Δ_H on the Heisenberg group. In particular, we calculate the subRiemannian distances along the geodesics. We also find the complex action function and the volume element on the group. Using this action function and the volume element, we obtain the fundamental solution and the heat kernel for the operator Δ_H.

The Heisenberg group and its sub-Laplacian are at the cross-roads of many domains of analysis and geometry: nilpotent Lie group theory, hypoelliptic second order partial differential equations, strongly pseudoconvex domains in complex analysis, probability theory of degenerate diffusion process, subRiemannian geometry, control theory and semiclassical analysis of quantum mechanics, see *e.g.,* [5], [6], [16], [17], and [22].

Here we give a detailed discussion of the behavior of the subRiemannian geodesics on Heisenberg group, which is the paradigm of the theory. This article is one of a series (see [7], [8], [9], [10], [12] and [13]), whose aim is

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to study the subRiemannian geometry induced by the sub-Laplacian and its analytic consequences. The paper is based on lectures given by the second and the third author during the "Fourth Workshops in Several Complex Variables" which was held at the Mathematical Institute of the Academia Sinica, December 29, 2003. The authors take great pleasure in expressing their thanks Professor Hsuan-Pei Lee and Professor Chin-Huei Chang for organizing this activity and to Professor Tai-Ping Liu for his strong support. The authors would also like to thank many colleagues at the Academia Sinica for their warm hospitality they have received while they visited Taiwan.

1. Definitions for the Heisenberg group. The Heisenberg group is a nilpotent Lie group of step 2 which makes this case very special in the class of subRiemannian manifolds. There are a few ways to introduce the Heisenberg group. In this section we shall show that all these are equivalent. Let $\mathbf{G}$ be a noncommutative group. If $h, k \in \mathbf{G}$ are two elements, define the commutator of h and k by $[h,k] = hkh^{-1}k^{-1} = hk(kh)^{-1}$. If $[h,k] = e$, we say that h and k commute (e denotes the unit element of $\mathbf{G}$). The set of elements which commute with all other elements is called the center of the group $Z(\mathbf{G}) = \{g \in \mathbf{G}; [g,k] = e\}$. If $\mathbf{K} \lhd \mathbf{G}$ is a subgroup of $\mathbf{G}$ then let $[\mathbf{K}, \mathbf{G}]$ be the group generated by all commutators $[k,g]$ with $k \in \mathbf{K}$ and $g \in \mathbf{G}$. When $\mathbf{K} = \mathbf{G}$, then $[\mathbf{G}, \mathbf{G}]$ is called the commutator subgroup of $\mathbf{G}$.

Definition 1.1. Let $\mathbf{G}$ be a group. Define the sequence of groups $(\Gamma(\mathbf{G}))_{n\geq 1}$ by $\Gamma_0(\mathbf{G}) = \mathbf{G}$, $\Gamma_{n+1}(\mathbf{G}) = [\Gamma_n(\mathbf{G}), \mathbf{G}]$. $\mathbf{G}$ is called nilpotent if there is $n \in \mathbb{N}$ such that $\Gamma_n(\mathbf{G}) = e$. The smallest integer n with the above property is called the class of nilpotence.

The subset of $\mathcal{M}_3(\mathbb{R})$ given by

$$\mathbf{G} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}; a, b, c \in \mathbb{R} \right\}$$

define a noncommutative group with the usual matrix multiplication. Consider the matrices

$$A = \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & b_1 & b_3 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & a_1 + b_1 & a_3 + b_3 + a_1 b_2 \\ 0 & 1 & a_2 + b_2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} 1 & -a_1 & a_1 a_2 - a_3 \\ 0 & 1 & -a_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & -b_1 & b_1 b_2 - b_3 \\ 0 & 1 & -b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

The commutator

$$[A, B] = ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 & a_1 b_2 - b_1 a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence the commutator subgroup

$$\Gamma_1(\mathbf{G}) = [\mathbf{G}, \mathbf{G}] = \langle [A, B]; A, B \in \mathbf{G} \rangle = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; k \in \mathbb{R} \right\}.$$

Let

$$C = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$AC = \begin{pmatrix} 1 & a & c+k \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = CA,$$

and therefore $[A, C] = AC(AC)^{-1} = I_3$. Hence $\Gamma_2(\mathbf{G}) = [\Gamma_1(\mathbf{G}), \mathbf{G}] = I_2 = e$, and the group $\mathbf{G}$ is nilpotent of class 2. $\mathbf{G}$ is called the Heisenberg group with 3 parameters.

The nilpotence class measures the noncommutativity of the group. In the following we shall associate with this group a noncommutative geometry of step 2. This geometry will have the Heisenberg uncertainty principle built in.

The bijection $\phi : \mathbb{R}^3 \to \mathcal{M}_3(\mathbb{R})$,

$$\phi(x_1, x_2, t) = \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

induces a noncommutative group law structure on $\mathbb{R}^3$

$$(x_1, x_2, t) \circ (x_1', x_2', t') = (x_1 + x_1', x_2 + x_2', t + t' + x_1 x_2'). \tag{1}$$

The zero element is $e = (0, 0, 0)$ and the inverse of (x_1, x_2, t) is $(-x_1, -x_2, x_1x_2 - t)$. $\mathbb{R}^3$ together with the above group law will be called the *nonsymmetric* 1-*dimensional Heisenberg group*. This group can be regarded also as a Lie group. The left translation $L_a : \mathbf{G} \to \mathbf{G}$, $L_a g = ag$, for all $g \in \mathbf{G}$ is an analytic diffeomorphism with inverse $L_a^{-1} = L_{a^{-1}}$. A vector field X on $\mathbf{G}$ is called left invariant if

$$(L_a)_*(X_g) = X_{ag}, \qquad \text{for all} \quad a, g \in \mathbf{G}.$$

The set of all left invariant vector fields form the Lie algebra of $\mathbf{G}$, denoted by $L(\mathbf{G})$. The Lie algebra of $\mathbf{G}$ has the same dimension as $\mathbf{G}$ and is isomorphic to the tangent space $T_e\mathbf{G}$. We shall use this result in the following proposition in order to compute a basis for the Lie algebra of the Heisenberg group.

Proposition 1.2. *The vector fields*

$$X = \partial_{x_1}, \quad Y = \partial_{x_2} + x_1\partial_t, \quad T = \partial_t$$

are left invariant with respect to the Lie group law (1) *on* $\mathbb{R}^3$.

Proof. Consider the notation $x_3 = t$. In this case the left translation is

$$L_{(a_1,a_2,a_3)}(x_1, x_2, x_3) = (a_1 + x_1, a_2 + x_2, a_3 + x_3 + a_1x_2).$$

Let X be a left invariant vector field. Then for all $a = (a_1, a_2, a_3) \in \mathbf{G}$, $X_a = (L_a)_*X_e$. In local coordinates $X_a = \sum_i X_a^i \partial_{x_i}$. The components are

$$X_a^i = (L_a)_*X_e(x_i) = X_e(x_i \circ L_a), \tag{2}$$

where x_i is the i-th coordinate and X_e is the value of the vector field X at origin. Let $b = (b_1, b_2, b_3) \in \mathbf{G}$. Then

$$\begin{aligned}
(x_1 \circ L_a)(b) &= x_1(L_ab) = x_1(ab) = a_1 + b_1 = x_1(a) + x_1(b),\\
(x_2 \circ L_a)(b) &= x_2(L_ab) = x_2(ab) = a_2 + b_2 = x_2(a) + x_2(b),\\
(x_3 \circ L_a)(b) &= x_3(L_ab) = x_3(ab) = a_3 + b_3 + a_1b_2\\
&= x_3(a) + x_3(b) + x_1(a)x_2(b).
\end{aligned}$$

Dropping b,

$$\begin{aligned} x_1 \circ L_a &= x_1(a) + x_1, \\ x_2 \circ L_a &= x_2(a) + x_2, \\ x_3 \circ L_a &= x_3(a) + x_3 + x_1(a)x_2. \end{aligned}$$

Substituting in equation (2) and using $X_e = \xi^1 \partial_{x_1} + \xi^2 \partial_{x_2} + \xi^3 \partial_{x_3}$, yields

$$\begin{aligned} X_a^1 &= X_e(x_1(a) + x_1) = \xi^1, \\ X_a^2 &= X_e(x_2(a) + x_2) = \xi^2, \\ X_a^3 &= X_e(x_3(a) + x_3 + x_1(a)x_2) = \xi^3 + x_1(a)\xi^2. \end{aligned}$$

Hence, the left invariant vector field X depends on the parameters ξ^i

$$\begin{aligned} X &= \xi^1 \partial_{x_1} + \xi^2 \partial_{x_2} + (\xi^3 + x_1 \xi^2)\partial_{x_3} \\ &= \xi^1 \partial_{x_1} + \xi^2(\partial_{x_2} + x_1 \partial_{x_3}) + \xi^3 \partial_{x_3} \\ &= \xi^1 X + \xi^2 Y + \xi^3 T, \end{aligned}$$

and the Lie algebra is generated by the linear independent vector fields X, Y and T.

In the following we shall show that the nonsymmetric model can be always reduced to a symmetric model, using a coordinate transformation.

Proposition 1.3. *Under the change of coordinates*

$$y_1 = x_1, \quad y_2 = x_2, \quad \tau = 4t - 2x_1 x_2,$$

the vector fields

$$X = \partial_{x_1}, \quad Y = \partial_{x_2} + x_1 \partial_t, \quad T = \partial_t,$$

are transformed into

$$X = \partial_{y_1} - 2y_2\partial_\tau, \quad Y = \partial_{y_2} + 2y_1\partial_\tau, \quad T = 4\partial_\tau.$$

Proof. The proof follows from the following relations

$$\begin{aligned} \partial_t &= 4\partial_\tau, \\ \partial_{x_2} &= \partial_{y_2} - 2y_1\partial_\tau, \\ \partial_{x_1} &= \partial_{y_1} - 2y_2\partial_\tau. \end{aligned}$$

Consider the vector fields $X_1 = \partial_{x_1} - 2x_2\partial_t$, $X_2 = \partial_{x_2} + 2x_1\partial_t$, $X_3 = \partial_t$ on $\mathbb{R}^3 = \mathbb{R}^2_x \times \mathbb{R}_t$. We are interested in a Lie group law on $\mathbb{R}^3$ such that X_1, X_2 and X_3 are left invariant. This shall be done using the Campbell-Hausdorff formula. The constants of structure are denoted by c_{ij}^k and are defined by

$$[X_i, X_j] = \sum_{k=1}^{3} c_{ij}^k X_k.$$

Form $[X_1, X_2] = -4\partial_t$ and $[X_1, \partial_t] = [X_2, \partial_t] = 0$, the constants of structure

$$c_{12}^1 = c_{12}^2 = 0, \qquad c_{12}^3 = -4,$$

$$c_{13}^j = c_{23}^j = 0, \qquad j = 1, 2, 3.$$

If $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, a locally a Lie group structure is given by the Campbell-Hausdorff formula:

$$(x \circ y)_i = x_i + y_i + \frac{1}{2}\sum_{j,k} c_{jk}^i \, x_j \, y_k + \frac{1}{12}\sum_{k,s,p,j} x_p \, y_j \, x_s \, c_{pj}^k \, c_{ks}^i + \cdots$$

In our case, we obtain a globally defined group structure

$$\begin{aligned}(x\circ y)_1 &= x_1+y_1,\\ (x\circ y)_2 &= x_2+y_2,\\ (x\circ y)_3 &= x_3+y_3+\frac{1}{2}(-4)(x_1y_2-x_2y_1),\end{aligned}$$

because the term $x_p\, y_j\, x_s\, c^k_{pj}\, c^3_{ks} = 0$.

Proposition 1.4. *The vector fields* $X_1 = \partial_{x_1} - 2x_2\partial_t$, $X_2 = \partial_{x_2} + 2x_1\partial_t$ *are left invariant with respect to the Lie group law on* $\mathbb{R}^3$

$$(x_1, x_2, t)\circ(x_1', x_2', t') = (x_1 + x_1', x_2 + x_2', t + t' - 2(x_1x_2' - x_2x_1')).$$

The Lie group $\mathbf{H}_1 = (\mathbb{R}^3, \circ)$ is called the symmetric one dimensional *Heisenberg group.* The unit element is $e=(0,0,0)$ and the inverse $(x_1, x_2, t)^{-1} = (-x_1, -x_2, -t)$.

We shall study the subRiemannian geometry associated to this model. The geometry comes with the Heisenberg uncertainty principle

$$[X_1, X_2] = -4\partial_t.$$

This brings the hope, that the Heisenberg manifolds (step 2 subRiemannian manifolds) will play a role for quantum mechanics in the future, similar to the role played by the Riemannian manifolds for classical mechanics. In quantum mechanics, the states of a quantum particle (position, momentum) are described by differential operators. It is known that two states which cannot be measured simultaneously, correspond to operators which do not commute. For instance, if $\mathbf{x}$ and $\mathbf{p}$ are the position and the momentum for a particle, then one cannot measure them simultaneously and hence, we write $[\mathbf{x}, \mathbf{p}] \neq 0$. The state of the particle is measured using a radiation beam sent towards the particle. The radiation is reflected partially back.

Using the variation of frequency between the sent and reflected beams, the Doppler-Fizeau formula will provide the speed of the particle. This method will provide accurate results if the radiation will not significantly change the speed of the particle *i.e.,* its kinetic energy $K = mv^2/2$. This means the radiation has a low energy and hence, a low frequency, because $E_{radiation} = h\nu$. Therefore the wave length of the radiation $\lambda = 1/\nu$ will be large. In this case the position of the particle cannot be measured accurate.

In order to measure the position accurately, the radiation wave length has to be as small as possible. In this case the frequency ν is large as will be the energy $E_{\text{radiation}}$. This will change the kinetic energy of the particle and hence its velocity. Hence, one cannot measure accurately both the position and the speed of the particle. The Heisenberg uncertainty principle, fundamental in the study of quantum particles, can be found also in other examples at the large scale structure. Let's assume that you are watching high-street traffic from an airplane. You will notice the position of the cars but you cannot say too much about their speed. They look like they are not moving at all. A policeman on the road will see the picture completely differently. For him, the speed of the cars will make more sense than their position. The latter is changing too fast to be noticed accurately.

2. The horizontal distribution. Unlike on Riemannian manifolds, where one may measure the velocity and distances in all directions, on Heisenberg manifolds there are directions where we cannot say anything using direct methods. On the Heisenberg group, an important role is played by the distribution generated by the linearly independent vector fields X_1 and X_2:

$$\mathcal{H} : x \to \mathcal{H}_x = \text{span}_x\{X_1, X_2\}.$$

As $[X_1, X_2] \notin \mathcal{H}$, the horizontal distribution $\mathcal{H}$ is not involutive, and hence, by Frobenius theorem, it is not integrable, *i.e.,* there is no surface locally tangent to it. A vector field V is called *horizontal* if and only if $V_x \in \mathcal{H}_x$, for all x. A curve $c : [0,1] \to \mathbb{R}^3$ is called horizontal if the velocity vector $\dot{c}(s)$ is a horizontal vector fields along $c(s)$. Horizontality is a constraint on the velocities and hence, it is also called in the literature nonholonomic constraint.

In this paper, we shall construct many horizontal objects, *i.e.,* a geometric objects which can be constructed directly from the horizontal distribution and the subRiemannian metric defined on it. The main goal is to recover the external structure of the space, such as the missing direction ∂_t by means of horizontal objects.

Proposition 2.1. *A curve $c = (x_1, x_2, t)$ is horizontal if and only if*

$$\dot{t} = 2(\dot{x}_1 x_2 - x_1 \dot{x}_2). \tag{3}$$

Proof. The velocity vector can be written

$$\begin{aligned} \dot{c} &= \dot{x}_1 \partial_{x_1} + \dot{x}_2 \partial_{x_2} + \dot{t}\partial_t \\ &= \dot{x}_1(\partial_{x_1} + 2x_2\partial_t) - 2\dot{x}_1 x_2 \partial_t + \dot{x}_2(\partial_{x_2} - 2x_1\partial_t) + 2x_1\dot{x}_2\partial_t + \dot{t}\partial_t \\ &= \dot{x}_1 X_1 + \dot{x}_2 X_2 + (\dot{t} + 2x_1\dot{x}_2 - 2x_2\dot{x}_1)\partial_t. \end{aligned}$$

Hence, $\dot{c} \in \mathcal{H}$ if and only if the coefficient of ∂_t vanishes.

Corollary 2.2. *A curve $c = (x_1, x_2, t)$ is horizontal if and only if*

$$\dot{c} = \dot{x}_1 X_1 + \dot{x}_2 X_2. \tag{4}$$

In the following we shall give a geometrical interpretation for the t component of a horizontal curve. This will be used in the proof of the connectivity theorem later. Using polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, equation (3) yields

$$\dot{t} = 2(\dot{x}_1 x_2 - x_1 \dot{x}_2) = -2r^2\dot{\phi}(\sin^2\phi + \cos^2\phi) = -2r^2\dot{\phi}.$$

In differential notation

$$dt = -2r^2\, d\phi. \tag{5}$$

Let $r = r(\phi)$ be the equation in polar coordinates of the projection of the horizontal curve on the x-plane. The area of an infinitesimal triangle with vertices at origin, $(r(\phi), \phi)$ and $(r(\phi + d\phi), \phi + d\phi)$ is $\frac{1}{2}r(\phi)r(\phi + d\phi)d\phi \approx \frac{1}{2}r^2(\phi)d\phi$. Integrating, we obtain the area swept by the vectorial radius between the initial angle ϕ_0 and ϕ (see Figure 1),

$$\mathcal{A} = \frac{1}{2}\int_{\phi_0}^{\phi} r^2(\phi)\, d\phi.$$

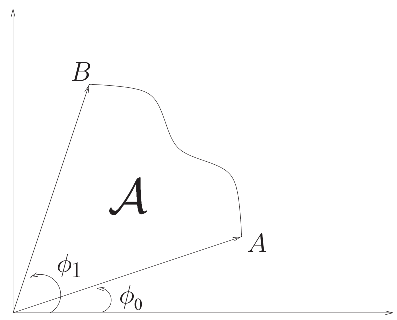

Figure 1. The area swept by the vectorial radius between two points in the plane.

Taking the derivative,

$$\frac{d\mathcal{A}}{d\phi} = \frac{1}{2}r^2(\phi),$$

or

$$d\mathcal{A} = \frac{1}{2}r^2\, d\phi. \tag{6}$$

Dividing the equations (5) and (6)

$$\frac{dt}{d\mathcal{A}} = -4, \tag{7}$$

which says that the t component is roughly the area swept by the vectorial radius on the x-plane, up to a multiplication factor. The negative sign in (7) shows that when t is increasing, the rotation in the x-plane is clock-wise. The equation (7) is valid only if t is not constant.

The areal velocity is defined as

$$\alpha = \frac{d\mathcal{A}}{ds}.$$

From Kepler's first law, all the planets have plane trajectory, which are ellipses with the sun in one of the focuses. This is similar to the second Kepler's law which says that the areal velocity α is constant along the motion.

Theorem 2.3. *A curve c in $\mathbb{R}^3$ is horizontal if and only if the rate of change of the t-component is equal to 4α, i.e., $\dot{t} = 4\alpha$.*

Proof. If $c = (x_1, x_2, t)$ is a horizontal curve,

$$\dot{t} = 2(\dot{x}_1 x_2 - x_1 \dot{x}_2).$$

Using polar coordinates,

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = -r^2 \dot{\phi},$$

and hence

$$\alpha = \frac{1}{2}(x_1 \dot{x}_2 - \dot{x}_1 x_2).$$

The characterization of horizontal curves with t constant is given in the following result.

Proposition 2.4. *A smooth curve $c(s)$ is horizontal with $t(s) = t$ constant if and only if $c(s) = (as, bs, t)$, with $a, b \in \mathbb{R}$, $a^2 + b^2 \neq 0$.*

Proof. If $c(s)$ is horizontal with t constant, the equation $\dot{t} = -2r^2\dot{\phi}$ yields ϕ =constant. Hence the projection on the x-space is a line which passes through the origin. It follows that $(x_1(s), x_2(s)) = (as, bs)$. If $c(s) = (as, bs, t)$, t constant, then $\dot{t} = 0$. On the other hand,

$$2(\dot{x}_1 x_2 - x_1 \dot{x}_2) = 2(abs - abs) = 0,$$

and hence the horizontality condition (3) holds.

The following proposition shows that the left translation of a horizontal curve is horizontal.

Proposition 2.5. *If $c(s)$ is a horizontal curve, then $\overline{c}(s) = L_a c(s)$ is a horizontal curve, for any $a \in \mathbf{H}_1$.*

Proof. If $c = (c_1, c_2, c_3)$ and $\overline{c} = (\overline{c}_1, \overline{c}_2, \overline{c}_3)$, then

$$\begin{aligned} \overline{c}_1 &= a_1 + c_1 \Longrightarrow \dot{\overline{c}}_1 = \dot{c}_1 \\ \overline{c}_2 &= a_2 + c_2 \Longrightarrow \dot{\overline{c}}_2 = \dot{c}_2 \\ \overline{c}_3 &= a_3 + c_3 - 2(a_1 c_2 - a_2 c_1) \Longrightarrow \dot{\overline{c}}_3 = \dot{c}_3 - 2(a_1 \dot{c}_2 - a_2 \dot{c}_1) \end{aligned} \tag{8}$$

Using that c is horizontal, equation (3) yields

$$\begin{aligned} \dot{\overline{c}}_3 &= \dot{c}_3 - 2(a_1 \dot{c}_2 - a_2 \dot{c}_1) \\ &= 2(\dot{c}_1 c_2 - c_1 \dot{c}_2) - 2(a_1 \dot{c}_2 - a_2 \dot{c}_1) \\ &= 2(\dot{c}_1 (a_2 + c_2) - \dot{c}_2 (a_1 + c_1)) \\ &= 2(\dot{\overline{c}}_1 \overline{c}_2 - \dot{\overline{c}}_2 \overline{c}_1). \end{aligned}$$

From equation (3) it follows that $\overline{c}(s)$ is horizontal.

Corollary 2.6. *The velocity of the horizontal curve* $\overline{c}(s) = L_a c(s)$ *is*

$$\dot{\overline{c}}(s) = (L_a)_* \dot{c}(s) = \dot{c}_1(s) X_1|_{\overline{c}(s)} + \dot{c}_2(s) X_2|_{\overline{c}(s)}. \tag{9}$$

Proof. As $\overline{c}(s)$ is horizontal, from equations (4) and (8)

$$\begin{aligned}
\dot{\overline{c}}(s) &= \dot{\overline{c}}_1(s) X_1|_{\overline{c}(s)} + \dot{\overline{c}}_2(s) X_2|_{\overline{c}(s)} \\
&= \dot{c}_1(s) X_1|_{\overline{c}(s)} + \dot{c}_2(s) X_2|_{\overline{c}(s)} \\
&= \dot{c}_1(s) (L_a)_* X_1|_{c(s)} + \dot{c}_2(s) (L_a)_* X_2|_{c(s)} \\
&= (L_a)_* (\dot{c}_1(s) X_1|_{c(s)} + \dot{c}_2(s) X_2|_{c(s)}) \\
&= (L_a)_* \dot{c}(s).
\end{aligned}$$

• Horizontal connectivity theorem

On the Heisenberg group $\mathbf{H}_1$, the vector fields X_1, X_2 and $[X_1, X_2]$ generate the tangent space of $\mathbb{R}^3$ at every point. A such subRiemannian manifold is called step 2. In the case of a general subRiemannian manifold, the number of brackets needed to generate all directions $+1$ is called the step of the manifold. The higher the step, the more noncommutative the geometry will be and harder to study. The step 1 corresponds to Riemannian geometry, which is the commutative case. The step condition was used independently by Chow [15] and Hörmander [19] to study connectivity of subRiemannian geodesics and hypoellipticity of subelliptic operators, respectively. Using Hörmander's theorem, the sub-Laplacian $\Delta_H = \frac{1}{2}(X_1^2 + X_2^2)$ is hypoelliptic, *i.e.*, $\Delta_H u = f \in C^\infty \Longrightarrow u \in C^\infty$.

In the following we shall prove Chow's connectivity theorem in the particular case of Heisenberg group.

Proposition 2.7. *Any two points in* $\mathbf{H}_1$ *can be joined by a piecewise horizontal curve, i.e., a curve tangent to the horizontal distribution.*

Proof. Let P and Q be two points in $\mathbb{R}^3$. Let t_P and t_Q be the t-coordinates of P and Q. We distinguish between the following two cases:

Case (i): $\boldsymbol{t_P \neq t_Q}$ Consider the number $\alpha = t_P - t_Q \neq 0$. Let P_1 and Q_1 be the projections on the x-plane of the points P and Q. Consider in x-plane a curve $\bar{\phi} : [0,1] \to \mathbb{R}^2$ which joins P_1 and Q_1, such that the area situated between the graph of $\bar{\phi}$ and the segments OP_1 and OQ_1 is equal to $\alpha/4$. The area will be considered positive in the case of a counter clock-wise rotation of the curve $\bar{\phi}$ between P_1 and Q_1. If $\bar{\phi}(s) = (x_1(s), x_2(s))$, then consider the function

$$t(s) = t_P + 2\int_0^s \Big(x_2(u)\dot{x}_1(u) - x_1(u)\dot{x}_2(u)\Big)\,du. \tag{10}$$

We claim that $\phi : [0,1] \to \mathbb{R}^3$ defined as $\phi(s) = (\bar{\phi}(s), t(s))$ is a horizontal curve between P and Q. Differentiating in (10) we obtain the horizontality condition $\dot{t}(s) = 2(x_2(s)\dot{x}_1(s) - x_1(s)\dot{x}_2(s))$ and hence, $\phi(s)$ is a horizontal curve. We shall check that ϕ joins the points P and Q.

$$\phi(0) = (\bar{\phi}(0), t(0)) = (x(P_1), t_P) = P.$$

Using $t(0) = t_P$, integrating between 0 and 1 in equation (7) yields

$$\begin{aligned} t(1) &= t(0) - 4(\mathcal{A}(1) - \mathcal{A}(0)) \\ &= t(0) - \alpha = t_P - (t_P - t_Q) \\ &= t_Q, \end{aligned}$$

and hence, $\phi(1) = Q$.

Case (ii): $\boldsymbol{t_P = t_Q = t}$ Let $R = (0,0,t)$. From Proposition 2.4 the curves $c_i : [0,1] \to \mathbb{R}^3$

$$c_1(s) = (sx_1(P), sx_2(P), t),$$

$$c_2(s) = (sx_1(Q), sx_2(Q), t)$$

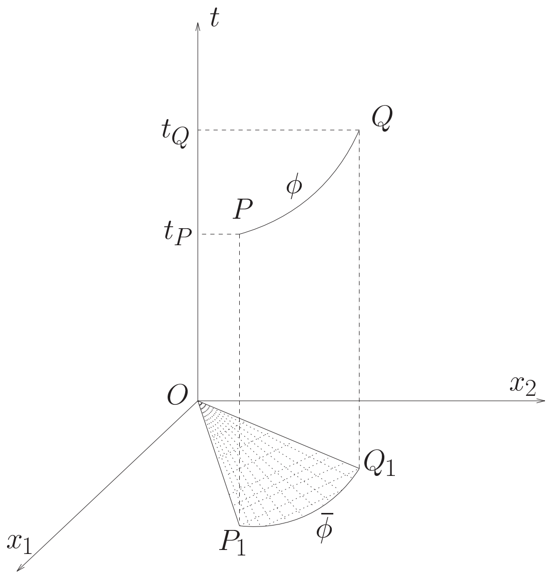

Figure 2. The projection of a horizontal curve.

are horizontal and join the points R with P and R with Q, respectively. Then the piecewise defined curve

$$\phi(s) = \begin{cases} c_1(\frac{1}{2} - s) & 0 \leq s \leq \frac{1}{2} \\ c_2(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

is horizontal with $\phi(0) = c_1(\frac{1}{2}) = P$ and $\phi(1) = c_2(1) = Q$.

The condition *piecewise* in the above proposition can be dropped. The proof can be modified in such that any two given points can be connected by a horizontal *smooth* curve. In order to do that, we should take advantage of the group law.

Given the points $P(x_1, y_1, t_1)$ and $Q(x_2, y_2, t_2)$, a translation to the left by $(-x_1, -y_1, -t_1)$ will transform them into $O(0, 0, 0)$ and $S(x', y', t')$, with

$$x' = x_2 - x_1, \quad y' = y_2 - y_1, \quad t' = t_2 - t_1 - 2(y_1 x_2 - x_1 y_2).$$

If $t' \neq 0$, applying case (i) of the previous proposition, we get a smooth horizontal curve $c(s)$ joining O and S.

If $t' = 0$, then $c(s) = (sx', sy', 0)$ is horizontal and joins O and S, see Proposition 2.4. Translating to the left by (x_1, y_1, t_1), the points O and S are sent to P and Q, respectively. Applying Proposition 2.5, the curve $c(s) = (c_1(s), c_2(s), c_3(s))$ is sent into a horizontal smooth curve between P and Q:

$$(x_1, y_1, t_1) \circ c(s) = \Big(x_1 + c_1(s), y_1 + c_2(s), t_1 + c_3(s) - 2(x_1 c_2(s) - y_1 c_1(s))\Big). \tag{11}$$

Corollary 2.8. *By a left translation, the t-axis $c(s) = (0, 0, s)$ is transformed into*

$$L_{(x_1, y_1, t_1)} c(s) = (x_1, y_1, t_1 + s).$$

Proof. It is an obvious consequence of equation (11).

3. Hamiltonian formalism on the Heisenberg group. The Heisenberg group is a good environment to apply the Hamiltonian formalism. Consider the Hamiltonian function $H : T^*_{(x,t)}\mathbb{R}^3 \to \mathbb{R}$ given by

$$H(\xi, \theta, x, t) = \frac{1}{2}(\xi_1 + 2x_2\theta)^2 + \frac{1}{2}(\xi_2 - 2x_1\theta)^2, \tag{12}$$

which is the principal symbol of the sub-Laplacian

$$\Delta_H = \frac{1}{2}(X_1^2 + X_2^2), \tag{13}$$

where $X_1 = \partial_{x_1} + 2x_2\partial_t$, $X_2 = \partial_{x_2} - 2x_1\partial_t$. In quantum mechanics, the procedure of obtaining the operator (13) from the Hamiltonian (12) is called *quantization*.

It is natural to consider the Hamiltonian system

$$(14)\qquad \begin{cases} \dot{x} = \partial H/\partial \xi \\ \dot{t} = \partial H/\partial \theta \\ \dot{\xi} = -\partial H/\partial x \\ \dot{\theta} = -\partial H/\partial t. \end{cases}$$

The solutions $c(s) = (x(s), t(s), \xi(s), \theta(s))$ of the system (14) are called *bicharacteristics.*

Definition 3.1. Given two points $P(x_0, t_0)$, $Q(x_1, t_1) \in \mathbb{R}^3$, a geodesic between P and Q is the projection on the (x, t)-space of a bicharacteristic $c : [0, \tau] \to \mathbb{R}^3$ which satisfies the boundary conditions:

$$(x(0), t(0)) = (x_0, t_0), \qquad (x(\tau), t(\tau)) = (x_1, t_1).$$

In studying of subRiemannian geometry, the most basic questions are:

Question 1. Given any two points, can we join them by a geodesic?

Question 2. How many geodesics are between any two given points?

Proposition 3.2. *Any geodesic is a horizontal curve.*

Proof. Let $c(s) = (x_1(s), x_2(s), t(s))$ be a geodesic. From the Hamiltonian system (14)

$$\dot{x}_1 = \xi_1 + 2x_2\theta, \qquad \dot{x}_2 = \xi_2 - 2x_1\theta,$$

and then

$$\begin{aligned} \dot{t} &= \frac{\partial H}{\partial \theta} \\ &= 2x_2(\xi_1 + 2x_2\theta) - 2x_1(\xi_2 - 2x_1\theta) \\ &= 2x_2\dot{x}_1 - 2x_1\dot{x}_2, \end{aligned}$$

which is the horizontality condition (3). Hence, any geodesic is a horizontal curve.

• Solving the Hamiltonian system

We shall solve the Hamiltonian system explicitly. We start with the observation that H does not depend on t. Then

$$\dot{\theta} = -\frac{\partial H}{\partial t} = 0$$

and hence, the momentum θ = constant along the solution which can be considered as Lagrange multiplier. The equations

$$\dot{x}_1 = \frac{\partial H}{\partial \xi_1}, \qquad \dot{x}_2 = \frac{\partial H}{\partial \xi_2}$$

become

$$\begin{cases} \dot{x}_1 = \xi_1 + 2x_2\theta \\ \dot{x}_2 = \xi_2 - 2x_1\theta. \end{cases} \tag{15}$$

Differentiating, yields

$$\begin{cases} \ddot{x}_1 = \dot{\xi}_1 + 2\dot{x}_2\theta \\ \ddot{x}_2 = \dot{\xi}_2 - 2\dot{x}_1\theta. \end{cases} \tag{16}$$

Using $\dot{\xi} = -\partial H/\partial x$ and the system (15)

$$\begin{cases} \dot{\xi}_1 = 2\theta(\xi_2 - 2x_1\theta) = 2\theta\dot{x}_2 \\ \dot{\xi}_2 = -2\theta(\xi_1 + 2x_2\theta) = -2\theta\dot{x}_1. \end{cases} \tag{17}$$

From systems (16) and (17)

$$\begin{cases} \ddot{x}_1 = 4\theta\dot{x}_2 \\ \ddot{x}_2 = -4\theta\dot{x}_1 \end{cases} \tag{18}$$

with constant θ. The system (18) can be written as

$$\ddot{x}(s) = 4\theta \mathcal{J} \dot{x}(s), \tag{19}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The equation (19) describes the projection of the geodesic on the x-space. We shall show that this is a circle.

With the substitution $\dot{x}(s) = y(s)$ equation (19) becomes

$$\dot{y}(s) = 4\theta \mathcal{J} y(s),$$

with the solution

$$y(s) = e^{4\theta \mathcal{J} s} y(0).$$

Therefore $\dot{x}(s) = e^{4\theta \mathcal{J} s} y(0)$. Integrating and using that $\mathcal{J}$ and $e^{4\theta \mathcal{J} s}$ commute,

$$\begin{aligned} x(s) &= x(0) + \int_0^s e^{4\theta \mathcal{J} u} y(0)\, du \\ &= x(0) + \frac{1}{4\theta} \mathcal{J}^{-1} e^{4\theta \mathcal{J} s} y(0) \Big|_{u=0}^{u=s} \\ &= x(0) - \frac{1}{4\theta} \mathcal{J} e^{4\theta \mathcal{J} s} y(0) + \frac{1}{4\theta} \mathcal{J}^{-1} y(0) \\ &= e^{4\theta \mathcal{J} s} K + C, \end{aligned} \tag{20}$$

where $K = -\mathcal{J} y(0)/(4\theta)$ and $C = x(0) + K$.

Lemma 3.3. *If* $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *then* $e^{4\theta \mathcal{J} s} = R_{4\theta s}$, *where* R_α *denotes the rotation by angle* α *in the* x*-plane.*

Proof.

$$\begin{aligned}
e^{4\theta\mathcal{J}s} &= \sum_{n=0}^{\infty}\frac{(4\theta s)^n\mathcal{J}^n}{n!} = I\sum_{k=0}^{\infty}\frac{(4\theta s)^{4k}}{(4k)!} + \mathcal{J}\sum_{k=0}^{\infty}\frac{(4\theta s)^{4k+1}}{(4k+1)!} \\
&- I\sum_{k=0}^{\infty}\frac{(4\theta s)^{4k+2}}{(4k+2)!} - \mathcal{J}\sum_{k=0}^{\infty}\frac{(4\theta s)^{4k+3}}{(4k+3)!} \\
&= \sum_{k=0}^{\infty}\begin{pmatrix} \frac{(4\theta s)^{4k}}{(4k)!} - \frac{(4\theta s)^{4k+2}}{(4k+2)!} & \frac{(4\theta s)^{4k+1}}{(4k+1)!} - \frac{(4\theta s)^{4k+3}}{(4k+3)!} \\ -\frac{(4\theta s)^{4k+1}}{(4k+1)!} + \frac{(4\theta s)^{4k+3}}{(4k+3)!} & \frac{(4\theta s)^{4k}}{(4k)!} - \frac{(4\theta s)^{4k+2}}{(4k+2)!} \end{pmatrix} \\
&= \begin{pmatrix} \cos(4\theta s) & \sin(4\theta s) \\ -\sin(4\theta s) & \cos(4\theta s) \end{pmatrix} \\
&= R_{4\theta s}.
\end{aligned}$$

Using Lemma 3.3, the equation (20) becomes

$$x(s) = R_{4\theta s}K + C.$$

As $|x(s)-C| = |R_{4\theta s}K| = |K| = \text{constant}$, $x(s)$ will describe a circle centered at C of radius

$$|K| = \left|\frac{-\mathcal{J}y(0)}{4\theta}\right| = \frac{|y(0)|}{4|\theta|} = \frac{|\dot{x}(0)|}{4|\theta|}.$$

Proposition 3.4. *Consider a geodesic which joins the points $P(x_0, t_0)$ and $Q(x_1, t_1)$, with $t_0 \neq t_1$.*

(i) *The projection of the geodesic on the x-space is a circle or a piece of a circle with end points x_0 and x_1.*

(ii) *If the projection is one complete circle, with $x_0 = x_1$, denote its area by σ. Then*

$$\sigma = \frac{|t_1 - t_0|}{4}.$$

Proof. (i) comes from the solution of the Hamiltonian system discussed above.

(ii) By Proposition 3.2, any geodesic is horizontal. From equation (7), the area of the projection on the x-plane and the t-component of a horizontal curve are related by

$$4\,d\mathcal{A} = -dt.$$

Integrating,

$$\begin{aligned} 4\sigma &= \int_0^1 4\,d\mathcal{A} = -\int_0^1 dt \\ &= t_0 - t_1. \end{aligned}$$

Corollary 3.5. *The radius of the projection circle is* $R = \sqrt{\frac{|t_1 - t_0|}{4\pi}}$.

In Proposition 3.2, it is shown that geodesics are horizontal curves. The converse is false.

Proposition 3.6. *There are horizontal curves which are not geodesics.*

Proof. Consider $c(s) = (s^2/2, s, s^3/3)$. The curve is horizontal, because the horizontality condition holds

$$\dot{t}(s) = s^2 = 2(s^2 - s^2/2) = 2(\dot{x}_1 x_2 - x_1 \dot{x}_2).$$

On the other hand, the system (18) becomes $4\theta = 1$ and $0 = -4\theta s$, which leads to a contradiction.

• **The t-component**

Using the Hamiltonian equation $\dot{t} = \partial H/\partial\theta$,

$$\begin{aligned} \dot{t}(s) &= 2(x_2(s)\dot{x}_1(s) - x_1(s)\dot{x}_2(s)) \\ &= 2\langle x(s), \mathcal{J}(\dot{x}(s))\rangle \\ &= 2\langle e^{4\theta\mathcal{J}s}K + C,\ \mathcal{J}(4\theta\mathcal{J}e^{4\theta\mathcal{J}s}K)\rangle \end{aligned}$$

$$\begin{aligned} &= 2\langle e^{4\theta\mathcal{J}s}K,\ -4\theta e^{4\theta\mathcal{J}s}K\rangle + 2\langle C,\ -4\theta e^{4\theta\mathcal{J}s}K\rangle \\ &= -8\theta|K|^2 - 8\theta\langle C, e^{4\theta\mathcal{J}s}K\rangle. \end{aligned}$$

Integrating

$$t(s) = \int \Big(-8\theta|K|^2 - 8\theta\langle C, e^{4\theta\mathcal{J}s}K\rangle\Big)\, ds.$$

As

$$\begin{aligned} \frac{d}{ds}\langle \mathcal{J}C, e^{4\theta\mathcal{J}s}K\rangle &= \langle \mathcal{J}C, \frac{d}{ds}e^{4\theta\mathcal{J}s}K\rangle = \langle \mathcal{J}C, 4\theta\mathcal{J}e^{4\theta\mathcal{J}s}K\rangle \\ &= 4\theta\langle \mathcal{J}^T\mathcal{J}C, e^{4\theta\mathcal{J}s}K\rangle = 4\theta\langle C, e^{4\theta\mathcal{J}s}K\rangle, \end{aligned}$$

then

$$\int \langle C, e^{4\theta\mathcal{J}s}K\rangle\, ds = \frac{1}{4\theta}\langle \mathcal{J}C, e^{4\theta\mathcal{J}s}K\rangle + \text{constant}.$$

Hence,

$$t(s) = -8\theta|K|^2 s - 2\langle \mathcal{J}C, e^{4\theta\mathcal{J}s}K\rangle + C_1$$

where $C_1 = t(0) + 2\langle \mathcal{J}C, K\rangle$.

• **The conservation of energy**

Let $K = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2)$ be the kinetic energy. One may show that the Hamiltonian is equal to K along the geodesics, and hence K is a first integral for the Hamiltonian system. In the following proposition we shall give a direct proof.

Proposition 3.7. The kinetic energy is preserved along the geodesics.

Proof. Using equation (19)

$$\begin{aligned} \frac{dK}{ds} &= \frac{d}{ds}\frac{\dot{x}_1^2 + \dot{x}_2^2}{2} = \dot{x}_1\ddot{x}_1 + \dot{x}_2\ddot{x}_2 \\ &= \langle \dot{x}, \ddot{x}\rangle = 4\theta\,\langle \dot{x}, \mathcal{J}\dot{x}\rangle = 0. \end{aligned}$$

4. The connection form. Let $x \to \mathcal{H}_x = \text{span}_x\{X_1, X_2\}$ be the horizontal distribution on $\mathbb{R}^3$. A connection 1-form is a non-vanishing form $\omega \in T^*\mathbb{R}^3$ such that $\ker_x \omega = \mathcal{H}_x$. The form ω is unique up to a multiplicative factor. In this chapter we shall choose the *standard* 1-form with the property $\omega(\partial_t) = 1$

$$\omega = dt - 2(x_2 dx_1 - x_1 dx_2).$$

Definition 4.1. The curvature 2-form of the distribution $\mathcal{H}$ is defined as $\Omega : \mathcal{H} \times \mathcal{H} \to \mathcal{F}(\mathcal{R}^3)$

$$\Omega(U, V) = d\omega(U, V). \tag{21}$$

In our case $\Omega = 4dx_1 \wedge dx_2$. If the horizontal distribution $\mathcal{H}$ belongs to the intrinsic subRiemannian geometry, the form Ω describes the extrinsic geometry of the Heisenberg group. In general, the 2-form Ω describes the non-integrability of the horizontal distribution.

Definition 4.2. The pair $(\mathbb{R}^3, \omega)$ is called a contact manifold if $\omega \wedge \Omega$ never vanishes.

In our case $\omega \wedge \Omega = 4dt \wedge dx_1 \wedge dx_2$ and hence, the Heisenberg group becomes a contact manifold. The following theorem shows that, locally, all contact manifolds are the same as the Heisenberg group (see Cartan [11]).

Theorem 4.3. (Darboux) *Each point p of a contact manifold admits a local coordinate system t, x_1, x_2 in a neighborhood U such that $\omega = dt - 2(x_2 dx_1 - x_1 dx_2)$.*

• The osculator plane

Let $c(s) = (x_1(s), x_2(s), t(s))$ be a curve. The *osculator* plane at $c(s)$ is defined as $\text{span}\{\dot{c}(s), \ddot{c}(s)\}$.

Proposition 4.4. *Let $c(s)$ be a curve. Then the curve $c(s)$ is horizontal if and only if the osculator plane at $c(s)$ is the horizontal plane $\mathcal{H}_{c(s)}$, for any s.*

Proof. If $\text{span}\{\dot{c}(s), \ddot{c}(s)\} = \mathcal{H}_{c(s)}$, then $\dot{c}(s) \in \mathcal{H}_{c(s)}$. Hence, the curve is horizontal.

If the curve $c(s)$ is horizontal, $\dot{c}(s) \in \mathcal{H}_{c(s)}$. It suffices to show $\ddot{c}(s) \in \mathcal{H}_{c(s)}$. From the horizontality condition,

$$\dot{t} = 2x_2\dot{x}_1 - 2x_1\dot{x}_2. \tag{22}$$

Differentiating in equation (22)

$$\begin{aligned} \ddot{t} &= 2\dot{x}_2\dot{x}_1 + 2x_2\ddot{x}_1 - 2\dot{x}_1\dot{x}_2 - 2x_1\ddot{x}_2 \\ &= 2x_2\ddot{x}_1 - 2x_1\ddot{x}_2. \end{aligned} \tag{23}$$

The acceleration vector along $c(s)$ is

$$\begin{aligned} \ddot{c} &= \ddot{x}_1\partial_{x_1} + \ddot{x}_2\partial_{x_2} + \ddot{t}\partial_t \\ &= \ddot{x}_1(\partial_{x_1} + 2x_2\partial_t) - 2x_2\ddot{x}_1\partial_t + \ddot{x}_2(\partial_{x_2} - 2x_1\partial_t) + 2x_1\ddot{x}_2\partial_t + \ddot{t}\partial_t \\ &= \ddot{x}_1X_1 + \ddot{x}_2X_2 + (\ddot{t} - 2x_2\ddot{x}_1 + 2\ddot{x}_2x_1)\partial_t \\ &= \ddot{x}_1X_1 + \ddot{x}_2X_2, \end{aligned}$$

where we used equation (23). Hence, $\ddot{c} \in \mathcal{H}_c$ and the osculator plane is horizontal.

Corollary 4.5. *For a horizontal curve c*

$$\begin{aligned} \ddot{c} &= \ddot{x}_1X_1 + \ddot{x}_2X_2, \\ \ddot{t} &= 2x_2\ddot{x}_1 - 2\ddot{x}_2x_1. \end{aligned}$$

Definition 4.6. Let $J : \mathcal{H} \to \mathcal{H}$ be defined by $J(X_1) = -X_2$, $J(X_2) = X_1$. J is called the complex structure of the horizontal plane.

We shall use J is order to write the equations for the geodesics on the Heisenberg group. The following result shows the geodesics satisfy a Newton type equation. The left side is the acceleration, while the right side is the force, which keeps the distribution bent. As before, θ is a constant.

Proposition 4.7. *A curve c is a geodesic on the Heisenberg group if and only if*

(i) *c is a horizontal curve and*

(ii) *c satisfies*

$$\ddot{c} = 4\theta J\dot{c}. \tag{24}$$

Proof. If $c(s)$ is a geodesic, by Proposition 3.2, $c(s)$ is horizontal. Using Corollary 4.5 and the system (18),

$$\begin{aligned} \ddot{c} &= \ddot{x}_1 X_1 + \ddot{x}_2 X_2 \\ &= 4\theta\dot{x}_2 X_1 - 4\theta\dot{x}_1 X_2 \\ &= 4\theta\dot{x}_2 J(X_2) + 4\theta\dot{x}_1 J(X_1) \\ &= 4\theta J(\dot{x}_1 X_1 + \dot{x}_2 X_2) \\ &= 4\theta J(\dot{c}). \end{aligned}$$

Let us prove the converse: if (i) and (ii) hold, then c is a geodesic. We shall use Definition 3.1. The horizontality condition (i) can be written as $\dot{t} = \partial H/\partial\theta$, which is the Hamiltonian equation for t. Using a similar computation as in the first part, equation (24) written on components becomes the system (18). Let $x_1(s)$ and $x_2(s)$ be solutions for this system. Define the following curve in the cotangent space

$$\gamma(s) = (x_1(s), x_2(s), t(s), \xi_1(s), \xi_2(s), \theta),$$

where

$$\xi_1 = \dot{x}_1 - 2x_2(s)\theta, \quad \xi_2 = \dot{x}_2 + 2x_1\theta,$$

with θ constant. Then $\gamma(s)$ satisfies the bicharacteristics system (14) for the Hamiltonian (12). Then the projection on the (x, t)-space is a geodesic and hence $c(s)$ is a geodesic.

4.1. The subRiemannian metric.

Definition 4.8. A non-degenerate, positive definite bilinear form $g_x : \mathcal{H}_x \times \mathcal{H}_x \to \mathcal{F}(\mathbb{R})$ at any point $x \in \mathbb{R}^3$, is called a subRiemannian metric.

We will consider that subRiemannian metric in which the vector fields X_1, X_2 are orthonormal. In this way, we relate the intrinsic geometry of the vector fields X_i and the extrinsic geometry of Ω. The subRiemannian metric will become a Kähler metric on the horizontal distribution, as we shall explain later. The following definitions can be found see *e.g.,* Kobayashi and Nomizu [20]).

Definition 4.9. A Hermitian metric on a real vector space V with a complex structure J is a non-degenerate, positive definite inner product h such that

$$h(JX, JY) = h(X, Y) \qquad \text{for } X, Y \in V.$$

We associate to each Hermitian metric of a vector space V a skew-symmetric bilinear form on V.

Definition 4.10. The fundamental 2-form Φ is defined by

$$\Phi(X, Y) = h(X, JY) \quad \text{for all vector fields } X \text{ and } Y.$$

A Hermitian metric on a vector space V with a complex structure J is called a Kähler metric if its fundamental 2-form is closed.

The relationship with the subRiemannian metric is given in the following proposition.

Proposition 4.11. *The subRiemannian metric g in which $\{X_1, X_2\}$ are orthonormal is a Kähler metric on $\mathcal{H}_x$, for any $x \in \mathbb{R}^3$. The fundamental 2-form is $4\Phi = \Omega$. Hence*

$$\Omega(U,V) = 4g(U,JV) \quad \textit{for all horizontal vectors } U \text{ and } V.$$

Proof. Consider $V = \mathcal{H}_x$. We shall show first that g is a Hermitian metric. Let $U = U^1X_1 + U^2X_2$ and $V = V^1X_1 + V^2X_2$ be two horizontal vector fields. Using $JX_1 = -X_2$ and $JX_2 = X_1$, yields

$$JU = -U^1X_2 + U^2X_1$$

$$JV = -V^1X_2 + V^2X_1.$$

Using the orthonormality of X_1 and X_2

$$\begin{aligned} g(JU,JV) &= g(U^2X_1 - U^1X_2, V^2X_1 - V^1X_2) \\ &= U^1V^1 + U^2V^2 \\ &= g(U^1X_1 + U^2X_2, V^1X_1 + V^2X_2) \\ &= g(U,V), \end{aligned}$$

and hence g is invariant by J. The 2-form Ω is closed because it is exact $\Omega = d\omega$.

$$\begin{aligned} \Omega(U,V) &= \Omega(U^1X_1 + U^2X_2, V^1X_1 + V^2X_2) \\ &= (U^1V^2 - U^2V^1)\Omega(X_1,X_2) \end{aligned}$$

$$\begin{aligned}
&= (U^1V^2 - U^2V^1)(X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2])) \\
&= 4(U^1V^2 - U^2V^1)\omega(\partial_t) \\
&= 4(U^1V^2 - U^2V^1) \\
&= 4g(U^1X_1 + U^2X_2, V^2X_1 - V^1X_2) \\
&= 4g(U, JV).
\end{aligned}$$

Using the skew-symmetry of Ω we obtain:

Corollary 4.12.

$$g(U, JU) = 0 \quad \textit{for any horizontal vector } U.$$

The geometrical interpretation of Ω is given below.

Proposition 4.13. *Let $\pi : \mathbb{R}^3_{(x,t)} \to \mathbb{R}^2_x$ be the projection which sends the horizontal plane onto the x-plane*

$$\pi_*(X_1) = \partial_{x_1}, \qquad \pi_*(X_2) = \partial_{x_2}.$$

Then, for any horizontal vectors U and V, the area of the parallelogram generated by $\pi_*(U)$ and $\pi_*(V)$ is equal to $|\Omega(U, V)|/4$.

Proof.

$$\begin{aligned}
\Omega(U, V) &= 4(dx_1 \wedge dx_2)(U, V) = 4\begin{vmatrix} dx_1(U) & dx_1(V) \\ dx_2(U) & dx_2(V) \end{vmatrix} \\
&= 4\begin{vmatrix} U(x_1) & V(x_1) \\ U(x_2) & V(x_2) \end{vmatrix} = 4\begin{vmatrix} U^1 & V^1 \\ U^2 & V^2 \end{vmatrix}.
\end{aligned}$$

Using the interpretation of the determinant as an area, the proof is complete.

Proposition 4.14. *Let $c(s)$ be a geodesic curve. Then*

$$\theta\Omega(U,\dot{c}) = g(U,\ddot{c}) \qquad \text{for any horizontal vector } U.$$

Proof. Using the Kähler property of the metric g and the geodesics equation (24)

$$\theta\Omega(U,\dot{c}) = 4\theta g(U,J\dot{c}) = g(U,\ddot{c}).$$

When $\theta = 0$, then $g(U,\ddot{c}) = 0$ for any horizontal vector U and then $\ddot{c} = 0$. Hence $\ddot{c}_1(s) = \ddot{c}_2(s) = 0$. We obtain the following known result:

Corollary 4.15. *Let $c(s)$ be a geodesic for which the momentum θ vanishes. Then*

$$c(s) = (as, bs, t_0),$$

with t_0 constant.

The following proposition deals with the metric properties of velocity and acceleration.

Proposition 4.16. *Let $c(s)$ be a geodesic. Then*

(i) *The velocity $\dot{c}$ and the acceleration $\ddot{c}$ vector fields are perpendicular in the subRiemannian metric.*

(ii) *The magnitude of $\dot{c}$ in the subRiemannian metric is constant along the geodesic.*

(iii) *The magnitude of $\ddot{c}$ in the subRiemannian metric is constant along the geodesic.*

Proof. (i) Proposition 4.14 yields

$$g(\dot{c},\ddot{c}) = \theta\Omega(\dot{c},\dot{c}) = 0.$$

(ii) Differentiating

$$\frac{d}{ds}g(\dot{c},\dot{c}) = g(\ddot{c},\dot{c}) + g(\dot{c},\ddot{c}) = 0.$$

(iii) The vector field $\ddot{c}$ is horizontal, and using equation (24) and (ii)

$$g(\ddot{c},\ddot{c}) = 16\theta^2 g(\dot{c},\dot{c}) = \text{ constant.}$$

In the classical theory of three dimensional curves, the curvature along a curve $c(s)$ is a function defined by $k(s) = |\dot{T}(s)|$, where $T(s) = \dot{c}(s)/|\dot{c}(s)|$ is the unit tangent vector. If s is the arc length parameter, $|\dot{c}(s)| = 1$ and the curvature $k(s) = |\ddot{c}(s)|$.

Proposition 4.17. *The curvature of a geodesic curve is constant,* $k(s) = 4|\theta|$.

Proof. Consider the geodesic $c(s)$ parametrized by the arc length. Then

$$k(s)^2 = g(\ddot{c},\ddot{c}) = 16\theta^2 g(\dot{c},\dot{c}) = 16\theta^2.$$

The curvature of a geodesic depends on the momentum θ. As we shall show later, θ is a Lagrange multiplier which describes the number of rotations of the geodesics around the t-axis. Hence, we shall be able to prove a Gauss-Bonett type theorem for the geodesic curves.

The following proposition provides the complex structure of the horizontal distribution in function of Ω and the basic horizontal vector fields.

Proposition 4.18. *For any horizontal vector field U we have*

$$J(U) = \frac{1}{4}\Big(\Omega(X_1,U)X_1 + \Omega(X_2,U)X_2\Big).$$

Proof. As both sides are linear, it suffices to check the relation only for the basic vector fields. Using $\Omega(X_1, X_2) = -\Omega(X_2, X_1) = 4$, yields

$$JX_1 = -X_2 = \frac{1}{4}\Omega(X_2, X_1)X_2 = \frac{1}{4}\Big(\Omega(X_1, X_1)X_1 + \Omega(X_2, X_1)X_2\Big)$$

$$JX_2 = X_1 = \frac{1}{4}\Omega(X_1, X_2)X_1 = \frac{1}{4}\Big(\Omega(X_1, X_2)X_1 + \Omega(X_2, X_2)X_2\Big).$$

5. Lagrangian formalism on the Heisenberg group. In this section we deal with finding the subRiemannian geodesics and characterizing their lengths. The horizontality condition is a constraint on velocities. This type of constraints is called non-holonomic. The Lagrangian which describes the geodesics has non-holonomic constraint. This constraint can be expressed using the 1-form ω.

Proposition 5.1. *If $\phi(s)$ is a horizontal curve, then*

$$\int_\phi \omega = 0.$$

Proof. As $\phi^*\omega$ is a 1-form on $\mathbb{R}$, then $\phi^*\omega$ and ds are proportional

$$\phi^*\omega(s) = h(s)\, ds,$$

where the proportionality function $h(s)$ is

$$h(s) = \phi^*_{(s)}\omega\Big(\frac{d}{ds}\Big).$$

Let $\phi(s)$ be defined on $[0, 1]$. Then

$$\begin{aligned}\int_\phi \omega &= \int_0^1 \phi^*\omega = \int_0^1 h(s)\, ds = \int_0^1 \phi^*_{(s)}(\omega)\Big(\frac{d}{ds}\Big)\, ds \\ &= \int_0^1 \omega\big(\phi_*\Big(\frac{d}{ds}\Big)\big)\, ds = \int_0^1 \omega(\dot\phi(s)) = 0,\end{aligned}$$

because $\dot{\phi} \in \mathcal{H}$.

We shall associate a Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ with the Hamiltonian (12). This is done using the Legendre transform in $(\dot{x}, \dot{t})$. The Lagrangian is given by the maximal distance between the hyperplane $< \xi, \dot{x} > +\theta\dot{t}$ and the convex surface given by the Hamiltonian in $\mathbb{R}^6$:

$$\begin{aligned} L(x,t,\dot{x},\dot{t}) &= \max_{\xi,\theta}\Big(\xi_1\dot{x}_1 + \xi_2\dot{x}_2 + \theta\dot{t} - H(\xi,\theta,x,t)\Big) \\ &= \max_{\xi,\theta} F(x,t,\dot{x},\dot{t},\xi,\theta) \end{aligned}$$

When the maximum is reached, the partial derivatives vanish

$$\frac{\partial F}{\partial \xi} = 0, \quad \frac{\partial F}{\partial \theta} = 0. \tag{25}$$

The equations (25) can be written as

$$\dot{x}_i = \frac{\partial H}{\partial \xi_i}, \quad \dot{t} = \frac{\partial H}{\partial \theta} = 0,$$

$$\dot{x}_1 = \xi_1 + 2x_2\theta, \quad \dot{x}_2 = \xi_2 - 2x_1\theta, \quad \dot{t} = 2x_2\dot{x}_1 - 2x_1\dot{x}_2. \tag{26}$$

Using relations (26), the Lagrangian becomes

$$\begin{aligned} L(x,t,\dot{x},\dot{t}) &= \xi_1\dot{x}_1 + \xi_2\dot{x}_2 + \theta\dot{t} - \frac{1}{2}(\xi_1 + 2x_2\theta)^2 - \frac{1}{2}(\xi_2 - 2x_1\theta)^2 \\ &= (\dot{x}_1 - 2x_2\theta)\dot{x}_1 + (\dot{x}_2 + 2x_1\theta)\dot{x}_2 + \theta\dot{t} - \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) \\ &= (\dot{x}_1^2 + \dot{x}_2^2) + \theta(\dot{t} - 2x_2\dot{x}_1 + 2x_1\dot{x}_2) - \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) \\ &= \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \theta(\dot{t} - 2x_2\dot{x}_1 + 2x_1\dot{x}_2). \end{aligned}$$

Using the 1-connection form ω

$$L(c,\dot{c}) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \theta\omega(\dot{c}), \tag{27}$$

where $c = (x_1, x_2, t)$. We are interested to minimize the action integral

$$S(c,\tau) = \int_0^\tau L(c,\dot c)\, ds = \int_0^\tau \frac{1}{2}(\dot x_1^2 + \dot x_2^2)\, ds + \theta \int_c \omega.$$

The action has two parts: the kinetic energy and the non-holonomic constraint. θ is called the Lagrange multiplier. The curves c, which are critical points for the action $S(c,\tau)$, satisfy the Euler-Lagrange system of equations

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot c}\Big) = \frac{\partial L}{\partial c}. \tag{28}$$

Proposition 5.2. *A solution of the Euler-Lagrange system of equations* (28) *is a geodesic if and only if it is a horizontal curve.*

Proof. As any geodesic is a horizontal curve, it suffices to show that a horizontal solution of the system (28) is a geodesic. If $c = (x_1, x_2, t)$, the system (28) becomes

$$\ddot x_1 = 4\theta \dot x_2, \quad \ddot x_2 = -4\theta \dot x_1, \quad \dot\theta = 0.$$

Proposition 4.7 completes the proof.

5.1. Lagrangian symmetries.

Proposition 5.3. *The Lagrangian* (27) *is left invariant with respect to the Heisenberg Lie group structure, i.e.,* $L(\overline{c}, \dot{\overline{c}}) = L(c,\dot c)$, *where* $\overline{c} = L_a(c)$, *for any* $a \in \mathbf{H}_1$.

Proof. If $c = (c_1, c_2, c_3)$, $\overline{c} = (\overline{c}_1, \overline{c}_2, \overline{c}_3)$ and $a = (a_1, a_2, a_3)$, then

$$\begin{aligned}
\overline{c}_1 &= a_1 + c_1 \Longrightarrow \dot{\overline{c}}_1 = \dot c_1\\
\overline{c}_2 &= a_2 + c_2 \Longrightarrow \dot{\overline{c}}_2 = \dot c_2\\
\overline{c}_3 &= a_3 + c_3 - 2(a_1 c_2 - a_2 c_1) \Longrightarrow \dot{\overline{c}}_3 = \dot c_3 - 2(a_1 \dot c_2 - a_2 \dot c_1)
\end{aligned}$$

Then the kinetic energy is left invariant

$$\frac{1}{2}(\dot{c}_1^2+\dot{c}_2^2)=\frac{1}{2}(\dot{\overline{c}}_1{}^2+\dot{\overline{c}}_2{}^2).$$

The horizontal constraint

$$\begin{aligned}\dot{\overline{c}}_3-2\overline{c}_2\dot{\overline{c}}_1+2\overline{c}_1\dot{\overline{c}}_2 &= \dot{c}_3-2(a_1\dot{c}_2-a_2\dot{c}_1)-2(a_2+c_2)\dot{c}_1+2(a_1+c_1)\dot{c}_2\\ &= \dot{c}_3-2c_2\dot{c}_1+2c_1\dot{c}_2.\end{aligned}$$

Hence, the Lagrangian is preserved by left translations.

Corollary 5.4. *The solutions of the Euler-Lagrange equations* (28) *are left invariant by the translations on* $\mathbf{H}_1$.

We shall use Noether's theorem approach to find first integrals of motion for the Lagrangian (27):

$$L(x,t,\dot{x},\dot{t})=\frac{1}{2}(\dot{x}_1^2+\dot{x}_2^2)+\theta(\dot{t}-2x_2\dot{x}_1+2x_1\dot{x}_2).$$

The following theorem can be found in Chapter 4 of Arnold's book [1].

Theorem 5.5. (Noether) *To every one-parameter group of diffeomorphisms* $(h_s)_s$ *of the coordinate space* M *of a Lagrangian system which preserves the Lagrangian, corresponds a first integral of the Euler-Lagrange equations of motion* $I:TM\to\mathbb{R}$

$$I(q,\dot{q})=\Big\langle\frac{\partial L}{\partial\dot{q}},\frac{dh_s(q)}{ds}\Big|_{s=0}\Big\rangle. \tag{29}$$

For the Heisenberg group, $M=\mathbb{R}^3$, $q=(x,t)$ and

$$\frac{\partial L}{\partial\dot{q}}=\Big(\frac{\partial L}{\partial\dot{x}_1},\frac{\partial L}{\partial\dot{x}_2},\frac{\partial L}{\partial\dot{t}}\Big)=\Big(\dot{x}_1-2\theta x_2,\dot{x}_2+2\theta x_1,\theta\Big).$$

There are three independent one-parameter groups of diffeomorphisms

$$h_s(x,t) = L_{a(s)}(x,t) = (a_1(s)+x_1, a_2(s)+x_2, a_3(s)+t-2(a_1(s)x_2-a_2(s)x_1)),$$

with $a(s) \in \{(0,0,s),(0,s,0),(s,0,0)\}$. The associated vector field is

$$\frac{dh_s(q)}{ds}\Big|_{s=0} = \begin{cases} (0,0,1), & \text{for } a(s) = (0,0,s) \\ (0,1,2x_1), & \text{for } a(s) = (0,s,0) \\ (1,0,-2x_2), & \text{for } a(s) = (s,0,0). \end{cases}$$

Hence, formula (29) provides three functional independent first integrals

$$\begin{aligned} I_1 &= \theta = \text{constant}, \\ I_2 &= \dot{x}_2 + 4\theta x_1 = \text{constant}, \\ I_3 &= \dot{x}_1 - 4\theta x_2 = \text{constant}. \end{aligned}$$

Differentiating, we obtain the Euler-Lagrange system (28)

$$\begin{cases} \ddot{x}_1 = 4\theta\dot{x}_2 \\ \ddot{x}_2 = -4\theta\dot{x}_1 \\ \theta = \text{constant}. \end{cases}$$

The rotational symmetry of the Lagrangian will provide a non obvious first integral. The Lagrangian is invariant by the one-parameter group of rotations $h_s(x,t) = (R_s x, t)$. Using $R_s(x) = e^{Js}x$, we have $\frac{dR_s}{ds} = Je^{Js}$ and hence, the vector field generated by the rotation

$$\frac{dh_s(q)}{ds}\Big|_{s=0} = (Jx, 0) = (x_2, -x_1, 0).$$

The first integral associated to the rotation vector fields is the kinetic momentum with respect to the t-axis

$$I = (\dot{x}_1 - 2\theta x_2)x_2 + (\dot{x}_2 + 2\theta x_1)(-x_1)$$

$$= \quad (\dot{x}_1 x_2 - \dot{x}_2 x_1) - 2\theta \|x\|^2.$$

Using the horizontality condition $\dot{t} = 2\dot{x}_1 x_2 - 2\dot{x}_2 x_1$, we get the first integral

$$2I = \dot{t} - 4\theta \|x\|^2 = \text{ constant.}$$

Proposition 5.6. *If c is a geodesic, for any $a \in \mathbf{H}_1$*

(i) *the left translation $\overline{c} = L_a c$ is also a geodesic,*

(ii) *the geodesics c and $\overline{c}$ have the same length.*

Proof. (i) Let c be a geodesic and let $\overline{c} = L_a c$. From Proposition 5.2 the curve c is horizontal and solves the system (28). From Corollary 5.4 and Proposition 2.5, the curve $\overline{c}$ is a solution of the Euler-Lagrange system (28) and it is horizontal. Using Proposition 5.2, we get that $\overline{c}$ is a geodesic.

(ii) From Corollary 2.6

$$\dot{\overline{c}} = \dot{c}_1 X_1|_{\overline{c}} + \dot{c}_2 X_2|_{\overline{c}}$$

and hence $|\dot{c}|^2 = |\dot{\overline{c}}|^2$. Using Proposition 4.16 (ii)

$$\ell(c) = \int_0^1 |\dot{c}|\, ds = |\dot{c}| = |\dot{\overline{c}}| = \int_0^1 |\dot{\overline{c}}|\, ds = \ell(\overline{c}).$$

5.2. Connectivity by geodesics.

Consider a geodesic joining the points P and Q. By a left translation, the point P can be transformed into $(0, 0, 0)$. By Proposition 5.6, the geodesic is transformed into another geodesic of the same length, which starts at origin. Hence, it makes sense to study the metric properties and the connectivity only for geodesics starting from the origin.

Because of the Lagrangian rotational symmetry, it is useful to use polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. The Lagrangian becomes

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \theta(\dot{t} + 2r^2\dot{\phi}). \tag{30}$$

The symmetries for the Lagrangian will provide symmetries for the geodesics. The Lagrangian (30) is invariant under the symmetry

$$S : (r, \phi, t; \theta) \to (r, -\phi, -t; -\theta).$$

If $(r(s), \phi(s), t(s))$ is a geodesic corresponding to θ, then $(r(s), -\phi(s), -t(s))$ is a geodesic corresponding to $-\theta$. Consequently, whatever statement is made for the geodesics joining the origin with the point $(r(s), \phi(s), t(s))$, when $\theta > 0$, it can be also made for the geodesics between origin and $(r(s), -\phi(s), -t(s))$, when $\theta < 0$. This allows us to do the analysis only for the case $\theta > 0$.

• **Euler-Lagrange equations**

A computation shows

$$\begin{aligned} &\frac{d}{ds}\frac{\partial L}{\partial \dot{r}} = \ddot{r}, &&\frac{\partial L}{\partial r} = r\dot{\phi}(\dot{\phi} + 4\theta),\\ &\frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} + 2\theta r^2, &&\frac{\partial L}{\partial \phi} = 0,\\ &\frac{\partial L}{\partial \dot{t}} = \theta, &&\frac{\partial L}{\partial t} = 0, \end{aligned}$$

and hence $r(s)$, $\phi(s)$ and θ satisfy the Euler-Lagrange system

$$\begin{cases} \ddot{r} = r\dot{\phi}(\dot{\phi} + 4\theta) \\ r^2(\dot{\phi} + 2\theta) = C(\text{constant}) \\ \theta = \theta_0 = \text{constant} \end{cases} \tag{31}$$

If the geodesic starts at origin, $r(0) = 0$ and hence $C = 0$. The second equation of (31) yields $\dot{\phi} = -2\theta$. With the assumption $\theta > 0$, the argument angle ϕ will rotate clock-wise. The Euler-Lagrange system becomes

$$\begin{cases} \ddot{r} = -4\theta^2 r \\ \dot{\phi} = -2\theta \\ \theta = \theta_0(\text{constant}). \end{cases} \tag{32}$$

When $\theta = 0$, the system (32) becomes

$$\begin{cases} \ddot{r} = 0 \\ \dot{\phi} = 0 \\ \theta = 0. \end{cases}$$

Proposition 5.7. *Given a point $P(x, 0)$, there is a unique geodesic between the origin and P. It is a straight line in the plane $\{t = 0\}$ of length $\|x\| = \sqrt{x_1^2 + x_2^2}$, and it is obtained for $\theta = 0$.*

Proof. In the case $\theta = 0$, the Hamiltonian is $H(\xi) = \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2$. From Hamilton's equation $\dot{t} = \partial H/\partial\theta = 0$, $t(s)$ is constant, and as $t(0) = 0$, it follows that $t(s) = 0$ and the solution belongs to x-plane. Using the equation $\ddot{r} = 0$, it follows the solution is a straight line.

From now on, unless otherwise stated, we shall assume $\theta > 0$. From the system (32) we can arrive at a first integral of energy.

Proposition 5.8. (i) *The function $I(r, \dot{t}, \theta) = \dot{r}^2 + 4\theta^2 r^2$ is a first integral for the system (32).*

(ii) $\frac{1}{2}I(r, \dot{t}, \theta)$ *is equal to the energy of the system, i.e., $I = \dot{x}_1^2 + \dot{x}_2^2$.*

Proof. (i) Differentiating

$$\frac{d}{ds}I(r, \dot{t}, \theta) = 2\dot{r}(\ddot{r} + 4\theta^2 r) = 0.$$

(ii) In polar coordinates, $\dot{x}_1^2 + \dot{x}_2^2 = \dot{r}^2 + r^2\dot{\phi}^2$. Using $\dot{\phi} = -2\theta$ completes the proof.

Let $c(s) = (x_1(s), x_2(s), t(s))$ be a geodesic. As $c(s)$ is horizontal

$$\dot{c} = \dot{x}_1 X_1 + \dot{x}_2 X_2$$

and if g denotes the subRiemannian metric in which X_1, X_2 are orthonormal, then

$$|\dot{c}(s)|_g^2 = g(\dot{c}(s), \dot{c}(s)) = \dot{x}_1^2(s) + \dot{x}_2^2(s).$$

By the arc length parametrization, *i.e.*, when the parameter s is the arc length along the geodesic, the curve $c(s)$ becomes unit speed:

$$\dot{x}_1^2(s) + \dot{x}_2^2(s) = 1.$$

Proposition 5.8 yields

$$\dot{r}^2(s) + 4\theta^2 r^2(s) = 1.$$

Separating

$$\frac{dr}{\sqrt{1 - 4\theta^2 r^2}} = \pm ds,$$

and integrating

$$\frac{1}{2\theta} \arcsin(2\theta s) = \pm s,$$

yields

$$r(s) = \pm\frac{1}{2\theta}\sin(2\theta s), \tag{33}$$

where we consider

$$+\text{sign for } 2n\pi \leq 2\theta s \leq (2n+1)\pi, \text{ and}$$
$$-\text{sign for } (2n+1)\pi \leq 2\theta s \leq (2n+2)\pi.$$

This yields a circle of diameter $1/(2\theta)$ which passes through origin.

Lemma 5.9. *If $\phi(0) = \phi_0$, then*

$$r(\phi)^2 = \Big(\frac{1}{2\theta}\Big)^2 \sin^2(\phi - \phi_0), \tag{34}$$

$$t(\phi) - t(\phi_0) = \frac{1}{4\theta^2}\frac{\sin 2(\phi - \phi_0) - 2(\phi - \phi_0)}{2}. \tag{35}$$

Proof. From the second equation of the system (32) $\phi(s) - \phi(0) = -2\theta s$. Substituting in (33) yields (34).

One of the Hamiltonian equations yields

$$\dot t = \frac{\partial H}{\partial \theta} = 2(\dot x_1 x_2 - x_1 \dot x_2) = -2r^2\dot\phi.$$

Integrating between ϕ_0 and ϕ and using equation (34)

$$\begin{aligned} t(\phi) - t(\phi_0) &= -2\int_{\phi_0}^{\phi} r^2(\phi)\,d\phi = -2\Big(\frac{1}{2\theta}\Big)^2\int_{\phi_0}^{\phi}\sin^2(\phi-\phi_0)\,d\phi \\ &= -\frac{2}{4\theta^2}\int_0^{\phi-\phi_0}\sin^2 u\,du = \frac{1}{4\theta^2}\frac{\sin 2(\phi-\phi_0) - 2(\phi-\phi_0)}{2}. \end{aligned}$$

• **Boundary conditions**

We are interested in connecting the origin with a point $P(x, t)$ by a geodesic $c : [0, \tau] \to \mathbb{R}^3$. Consider the boundary conditions:

$$x(0) = 0, \quad t(0) = 0, \quad \phi(0) = \phi_0, \tag{36}$$

$$\|x(\tau)\| = R, \quad t(\tau) = t, \quad \phi(\tau) = \phi_1. \tag{37}$$

Because of the rotational invariance around the t-axis, we may choose $\phi_0 = 0$. From $\dot t = -2r^2\dot\phi$ and $\dot\phi = -2\theta$, we get $\dot t = 4\theta r^2 > 0$. Hence $t(s)$ is increasing and if $t(0) = 0$, then $t(\tau) > 0$.

Lemma 5.10. *The following relations take place among the boundary conditions:*

$$\phi_1 = -2\theta\tau, \tag{38}$$

$$\sin(\phi_1)^2 = 4\theta^2 R^2, \tag{39}$$

$$t = \frac{1}{4\theta^2}\frac{\sin(2\phi_1) - 2\phi_1}{2}, \tag{40}$$

$$\frac{t}{R^2} = -\mu(\phi_1) = \mu(2\theta\tau), \tag{41}$$

where

$$\mu(x) = \frac{x}{\sin^2 x} - \cot x. \tag{42}$$

Proof. Integrating in the second equation of the system (32) and using the boundary conditions yields (38). Equations (34) and (35) together with the boundary conditions (36)-(37) yield equations (39) and (40). Eliminating $4\theta^2$ from equations (39) and (40) yields (41).

The behavior of the function μ given by (42) is very important in understanding the subRiemannian geometry of the Heisenberg group. The function μ was first used by Gaveau when he studied heat operator on the Heisenberg group. Later, this function was studied extensively in the papers [2], [4], [7], and [9]. The graph of μ is given in the following figure.

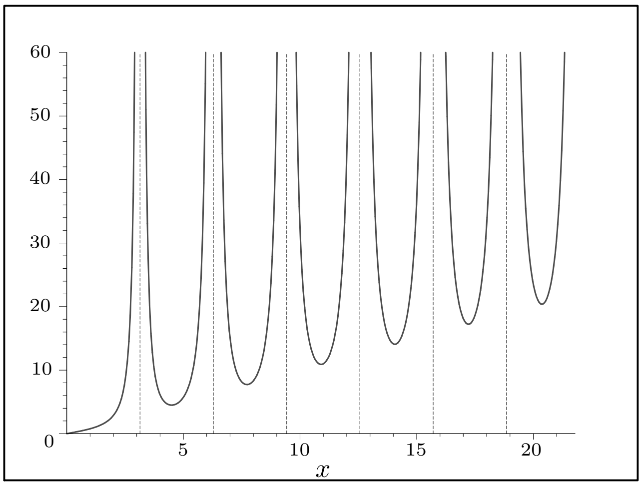

Figure 3. The graph of $\mu(x)$.

Lemma 5.11. *μ is a monotone increasing diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$. On each interval $(m\pi,(m+1)\pi)$, $m=1,2,\ldots$, μ has a unique critical point x_m. On this interval μ decreases strictly from $+\infty$ to $\mu(x_m)$ and then increases strictly from $\mu(x_m)$ to $+\infty$. Moreover*

$$\mu(x_m)+\pi<\mu(x_{m+1}),\quad m=1,2,\ldots$$

$$0<\Big(m+\frac{1}{2}\Big)\pi-x_m<\frac{1}{m\pi}.$$

Proof. As μ is an odd function, it suffices to show that it is a monotone increasing diffeomorphism of the interval $(0,\pi)$ onto $(0,+\infty)$. We note that $\sin x - x\cos x$ vanishes at $x=0$ and it is increasing in $(0,\pi)$. Then

$$\frac{1}{2}\mu'(x)=\frac{\sin x - x\cos x}{\sin^3 x}=\begin{cases} =1/3, & x=0,\\ >1/3, & x\in(0,\pi).\end{cases}$$

The first identity holds as an application of the l'Hospital rule:

$$\lim_{x\to 0}\frac{\sin x - x\cos x}{\sin^3 x} \quad = \quad \lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{3\sin^2 x}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{x}{\sin x} = \frac{1}{3}.$$

The second inequality holds because

$$\frac{1}{2}\mu''(x) = \frac{x + 2x\cos^2 x - 3\cos x \sin x}{\sin^4 x} > 0;$$

The numerator vanishes at $x = 0$, and its derivative is

$$4\sin x(\sin x - x\cos x) > 0, \quad x \in (0, \pi).$$

Therefore μ is a diffeomorphism of the interval $(0, \pi)$ onto $(0, \infty)$. In the interval $(m\pi, (m+1)\pi)$ μ approaches $+\infty$ at the endpoints. In order to find the critical points, we set

$$\frac{1}{2}\mu'(x) = \frac{\sin x - x\cos x}{\sin^3 x} = \frac{1 - x\cot x}{\sin^4 x} = 0.$$

Hence the critical point x_m is the solution of the equation $x = \tan x$ on the interval $(m\pi, (m+1)\pi)$. Note that

$$\begin{aligned} \mu(x+\pi) &= \frac{x+\pi}{\sin^2(x+\pi)} - \cot(x+\pi) \\ &= \frac{x}{\sin^2(x+\pi)} - \cot(x+\pi) + \frac{\pi}{\sin^2 x} \\ &= \mu(x) + \frac{\pi}{\sin^2 x}, \end{aligned}$$

so the successive minimum values increase by more than π. From Figure 3 we have

$$m\pi < x_m < m\pi + \frac{\pi}{2} = (m + \frac{1}{2})\pi.$$

Using $x_m = \tan x_m$, yields

$$\cot(x_m) = \frac{1}{x_m} < \frac{1}{m\pi}. \tag{43}$$

Let $f(x) = \cot x$. As $f'(x) = -\frac{1}{\sin^2 x} < -1$, there is a ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x - y) < -(x - y).$$

Hence $x - y < f(y) - f(x)$. Choosing $x = m\pi + \frac{\pi}{2}$, $y = x_m$ and using

$$f(m\pi + \frac{\pi}{2}) = \frac{\cos(m\pi + \frac{\pi}{2})}{\sin(m\pi + \frac{\pi}{2})} = 0,$$

and (43) yields

$$0 < (m + \frac{1}{2})\pi - x_m < \cot x_m < \frac{1}{m\pi}.$$

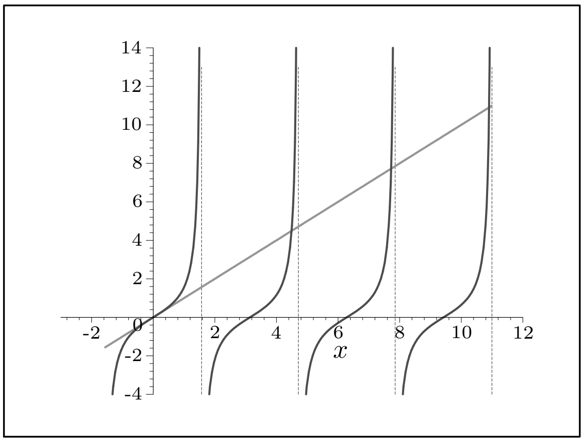

Figure 4. Critical points of μ are solutions of $\tan x = x$.

The number of geodesics which join the origin with a arbitrary given point is given in the following theorems.

Theorem 5.12. *There are finitely many geodesics that join the origin to* (x, t) *if and only if* $x \neq 0$. *These geodesics are parametrized by the solutions* ζ *of*

$$\frac{|t|}{\|x\|^2} = \mu(\zeta). \tag{44}$$

There is exactly one such geodesic if and only if

$$|t| < \mu(x_1)\|x\|^2, \tag{45}$$

where x_1 *is the first critical point of* μ. *The number of geodesics increase without bound as* $|t|/\|x\|^2 \to \infty$. *Let* $\zeta_1 < \zeta_2 < \cdots < \zeta_N$ *be the solutions of* (44). *The square of the length associated to the solution* ζ_m *is*

$$s_m^2 = \Big(\frac{\zeta_m}{\sin\zeta_m}\Big)^2\|x\|^2, \qquad m = 1, \dots, N. \tag{46}$$

Proof. The enumeration of geodesics follow from Lemma 5.11. The line $y = |t|/\|x\|^2$ intersects the graph of μ finitely many times. There is only one intersection if and only if inequality (45) holds. See the graph of μ.

As the geodesic was parametrized by arc length, its length is given by the value of the parameter τ. Dividing the square of the equation (38) by equation (39) yields the square of the length

$$\tau^2 = \Big(\frac{\phi_1}{\sin\phi_1}\Big)^2 R^2.$$

ϕ_1 satisfies equation (41). For each $\phi_1 = -\zeta_m$ we get the length described by formula (46).

The natural dilations of the sub-Laplacian operator

$$\frac{1}{2}\Big(\partial_{x_1} + 2x_2\partial_t\Big)^2 + \frac{1}{2}\Big(\partial_{x_2} - 2x_1\partial_t\Big)^2$$

are

$$(x_1, x_2, t) \to (\lambda x_1, \lambda x_2, \lambda^2 t), \ \ \lambda > 0.$$

We are looking for a formula for the length of geodesics, different than (46), such that τ is homogeneous of degree 1 with respect to the above dilations.

Theorem 5.13. *Consider the geodesics joining the origin with the point* (x,t)*,* $x \neq 0$*. Let* $\zeta_1, \ldots, \zeta_N$ *be the solutions of equation* (44)*. Then the square of the lengths is given by*

$$s_m^2 = \nu(\zeta_m)\Big(|t| + \|x\|^2\Big), \tag{47}$$

where

$$\nu(x) = \frac{x^2}{x + \sin^2 x - \sin x \cos x}.$$

Proof. Equation (44) yields

$$|t| + \|x\|^2 = \mu(\zeta_m)\|x\|^2 + \|x\|^2 = (1 + \mu(\zeta_m))\|x\|^2,$$

and hence

$$\|x\|^2 = \frac{1}{1 + \mu(\zeta_m)}\Big(|t| + \|x\|^2\Big).$$

Using equation (46) and the definition of μ given in (42)

$$\begin{aligned} s_m^2 &= \Big(\frac{\zeta_m}{\sin \zeta_m}\Big)^2 \|x\|^2 = \Big(\frac{\zeta_m^2}{\sin^2 \zeta_m} \cdot \frac{1}{1 + \mu(\zeta_m)}\Big)\Big(|t| + \|x\|^2\Big) \\ &= \frac{\zeta_m^2}{\sin^2 \zeta_m + \zeta_m - \sin^2 \zeta_m \cot \zeta_m}\Big(|t| + \|x\|^2\Big) = \nu(\zeta_m)\Big(|t| + \|x\|^2\Big). \end{aligned}$$

Proposition 5.14. *The projection of the geodesics joining the origin and* (x,t)*,* $x \neq 0$ *are circles or arcs of circle with diameters of at least* $\|x\|$

$$\frac{1}{2\theta_m} = \frac{\|x\|}{|\sin \zeta_m|} \geq \|x\|. \tag{48}$$

Proof. From the first equation of Lemma 5.9, the diameter of the circle is $1/2\theta$. Using relation (39) yields

$$\frac{1}{2\theta} = \frac{R}{|\sin \phi_1|}.$$

Replacing R by $\|x\|$ and ϕ_1 by ζ_m, we arrive at relation (48).

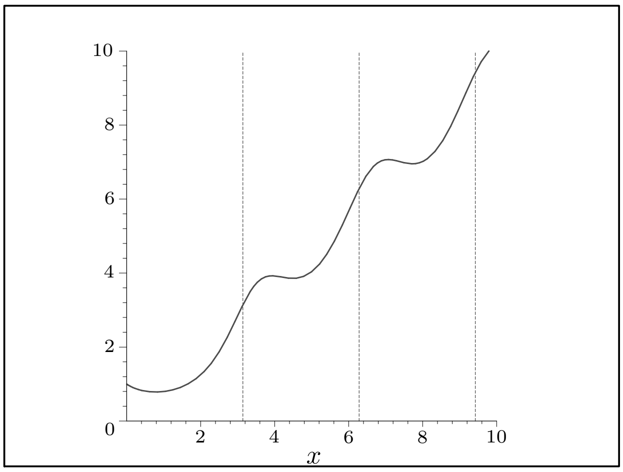

Figure 5. The graph of $\nu(x)$.

• **Geodesics between the origin and points on the t-axis**

We have seen that the number of geodesics increases without bound as $|t|/\|x\|^2 \to \infty$. The following theorem deals with the limit case $\|x\| = 0$.

Theorem 5.15. *The geodesics that join the origin to a point* $(0, 0, t)$ *have lengths* $s_1, s_2 \ldots$, *where*

$$s_m^2 = m\pi|t|.$$

For each length s_m, *the geodesics of that length are parametrized by the circle* $\mathbb{S}^1$.

Proof. We shall treat the problem as a limit case $\|x\| \to 0$ of Theorem 5.13. Then the solutions $\zeta_m \to m\pi$. Relation (47) becomes

$$s_m^2 = \nu(m\pi)|t|,$$

with the coefficient given by

$$\nu(m\pi) = \frac{m^2\pi^2}{m\pi + \sin^2 m\pi - \sin m\pi \, \cos m\pi} = m\pi.$$

The above proof is using Theorem 5.13. In the following we shall provide a direct proof. The approach will use polar coordinates and the fact that the solution is given by

$$r(\phi) = r_{max}|\sin(\phi - \phi_0)|.$$

Consider the solution γ parameterized by $[0,1]$, with the end points $\gamma(0) = (0,0,0)$ and $\gamma(1) = (0,0,t)$. Using that $|\dot\gamma|$ is constant along the solution, we have identity in Cauchy's inequality

$$\ell(\gamma) = \int_0^1 |\dot\gamma(s)|\, ds = \Big(\int_0^1 ds\Big)^{1/2}\Big(\int_0^1 |\dot\gamma(s)|^2\Big)^{1/2} = \sqrt{2H_0},$$

where H_0 is the constant value of the Hamiltonian along the solution

$$H_0 = \frac{1}{2}|\dot\gamma| = \frac{1}{2}(\dot x_1^2 + \dot x_2^2).$$

• **Quantization of θ**

The parameter θ is constant along the solution and it is an eigenvalue for the following boundary value problem

$$\begin{cases} \ddot r = -4\theta^2 r \\ r(0) = 0, \ r(1) = 0. \end{cases}$$

The solution is $r(s) = A\sin(2\theta s)$. Using the boundary condition yields $2\theta = m\pi$, m integer. Each value of θ will determinate a certain length.

• The Hamiltonian and the t component

From the boundary conditions $t(1) = t$. One has $\dot{\phi} = -2\theta$ and then $\phi(s) = \phi_0 - 2\theta s$, where $\phi_0 = \phi(0)$ and $\phi_1 = \phi(1) = \phi_0 - 2\theta$. Using the equation $\dot{t} = -2r^2\dot{\phi}$ one may integrate between ϕ_0 and $\phi_1 = \phi_0 - m\pi$.

$$\begin{aligned} t &= -2\int_{\phi_0}^{\phi_0 - m\pi} r^2(\phi)\, d\phi = -2\, r^2_{max} \int_{\phi_0}^{\phi_0-m\pi} \sin^2(\phi - \phi_0)\, d\phi \\ &= -2\, r^2_{max}\int_0^{-m\pi} \sin^2 u\, du = 2\, r^2_{max}\int_0^{m\pi}\sin^2 u\, du \\ &= 2\, r^2_{max}(\frac{1}{2}u - \frac{1}{4}\sin 2u)\Big|_0^{m\pi} = r^2_{max} m\pi = 2\theta r^2_{max}. \end{aligned}$$

From the conservation of energy

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{1}{2}(\dot{r}^2 + 4r^2\theta^2) = H_0.$$

When $r = r^2_{max}$, then $\dot{r} = 0$. Hence

$$r^2_{max} = \frac{H_0}{2\theta^2},$$

and using $t = 2\theta r^2_{max}$, one obtains

$$t = \frac{H_0}{\theta}.$$

The lengths are

$$(\ell_m)^2 = 2H_0 = 2\theta t = m\pi t\ , \quad m = 1, 2, 3\ldots$$

Proposition 5.16. *The equations of the geodesics starting at origin are*

$$\begin{aligned} r_m(\phi) &= \sqrt{\frac{|t|}{m\pi}}\sin(\phi - \phi_0) \\ t_m(\phi) &= \frac{|t|}{2m\pi}\Big(\sin(2(\phi-\phi_0)) - 2(\phi - \phi_0)\Big),\ \ m \geq 1. \end{aligned}$$

Proof. Using that

$$r_{max} = \sqrt{\frac{H_0}{2\theta^2}} = \sqrt{\frac{|t|}{2\theta}} = \sqrt{\frac{|t|}{m\pi}}$$

then

$$r_m(\phi) = \sqrt{\frac{|t|}{m\pi}}\,\sin(\phi - \phi_0).$$

For computing $t_m(\phi)$ one integrates $dt = -2r^2\,d\phi$ between ϕ_0 and ϕ

$$t(\phi) = -2r_{max}^2 \int_{\phi_0}^{\phi} \sin^2(\phi - \phi_0)\,d\phi = -2\frac{|t|}{m\pi}\int_0^{\phi-\phi_0} \sin^2 u\,du.$$

Hence

$$t_m(\phi) = \frac{|t|}{2m\pi}\Big(\sin(2(\phi - \phi_0)) - 2(\phi - \phi_0)\Big).$$

Corollary 5.17. *The projection of the m-th geodesic on the x-plane is a circle of radius*

$$R_m = \frac{1}{2}\sqrt{\frac{|t|}{m\pi}}$$

and area

$$\sigma_m = \frac{|t|}{4m}.$$

We note that r_m and t_m depend on ϕ_0 but the corresponding length $\ell_m = \sqrt{m\pi|t|}$ does not.

Let us return to the equation $\dot{t} = -2r^2\dot{\phi}$. As before, one may integrate between ϕ_0 and $\phi_1 = \phi_0 - \pi$ and obtain

$$t(\phi_0 - \pi) - t(\phi_0) = -2\int_{\phi_0}^{\phi_0-\pi} r^2(\phi)d\phi = \pi r_{max}^2 = \pi R^2.$$

Here R is the radius of the projection of the geodesic with $m = 1$ (since we just need to consider the first complete circle). Denote

$$T = \Big| t(\phi_0 - \pi) - t(\phi_0) \Big|.$$

Then $T = \pi R^2$. Here T is the period, *i.e.,* the time needed for the particle to come back to the origin. Hence

$$\sqrt{T} = \sqrt{\pi} \cdot R$$

is an analog of the third law of Kepler.

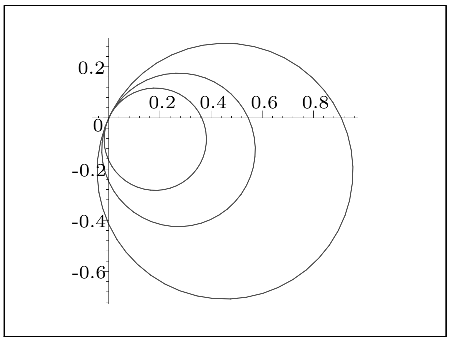

Figure 6. Projection of geodesics through the origin.

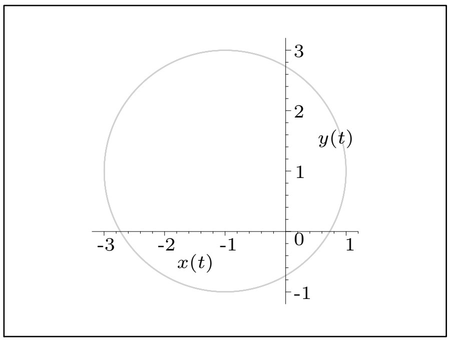

Figure 7. Projection of a geodesic outside the origin.

• A Milnor-type property

In the classical theory of three dimensional curves one defines the total curvature of a curve $c : [0, \tau] \to \mathbb{R}^3$ as $\int_0^\tau k(u)\, du$, where $k(s)$ is the curvature along the curve c. Milnor's theorem (see Millman and Parker [21]) states

Theorem 5.18. *The total curvature of a closed curve in* $\mathbb{R}^3$ *is* $2\pi\chi$, *where* χ *is an integer.*

It is known that χ represents the number of the knots of the curve c. In the case of a plane curve, $\chi = 1$.

We shall show a similar property for the geodesics between origin and points on the t-axis. We shall prove the following:

Proposition 5.19. *The total curvature of a geodesic between the origin and the point* $(0, 0, t)$ *is* $2\pi m$, *where* $m = 1, 2, \dots$ *is an integer which gives the number of rotations of the geodesic.*

Proof. Consider the curve $c : [0, \tau] \to \mathbb{R}^3$ joining $(0, 0, 0)$ and $(0, 0, t)$, parametrized by arc length. Then $|\dot{c}(u)| = 1$, and by Proposition 4.17, the curvature is $k(u) = |\ddot{c}(u)| = 4|\theta|$. Hence, the total curvature

$$\int_0^\tau k(u)\, du = 4|\theta|\tau.$$

From the boundary value problem

$$\begin{cases} \ddot{r} = -4\theta^2 r \\ r(0) = 0, \; r(\tau) = 0. \end{cases}$$

it follows that $2\theta\tau = m\pi$. Therefore, the total curvature is equal to $2\pi m$, $m = 1, 2, \dots$

5.3. Geodesics between any two arbitrary points.

Given two points P_1 and P_2, we shall study the connectivity by geodesics in the cases $(x_1, y_1) = (x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$. Making a left translation by $(-x_1, -y_1, -t_1)$, the points P_1 and P_2 will become the origin and the point R, respectively, where R has coordinates

$$(x_2 - x_1, y_2 - y_1, t_2 - t_1 - 2(y_1 x_2 - x_1 y_2)).$$

By Proposition 5.6 the left translation of a geodesic is a geodesic of the same length. Then Theorems 5.12 and 5.13 become:

Theorem 5.20. *Given the points $P_1(x_1, y_1, t_1)$ and $P_2(x_2, y_2, t_2)$, there are finitely many geodesics between P_1 and P_2 if and only if $(x_1, y_1) \neq (x_2, y_2)$. These geodesics are parametrized by the solutions ζ of*

$$\frac{|t_2 - t_1 - 2(y_1 x_2 - x_1 y_2)|}{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \mu(\zeta). \tag{49}$$

There is exactly one such geodesic if and only if

$$|t_2 - t_1 - 2(y_1 x_2 - x_1 y_2)| < \mu(c_1)[(x_2 - x_1)^2 + (y_2 - y_1)^2],$$

where c_1 is the first critical point of μ. The number of geodesics increase without bound as

$$\frac{|t_2 - t_1 - 2(y_1 x_2 - x_1 y_2)|}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]} \to \infty.$$

Let $\zeta_1 < \zeta_2 < \cdots < \zeta_N$ be the solutions of (49). *The square of the length associated to the solution ζ_m is*

$$\begin{aligned} s_m^2 &= \Big(\frac{\zeta_m}{\sin \zeta_m}\Big)^2 [(x_2 - x_1)^2 + (y_2 - y_1)^2] \\ &= \nu(\zeta_m)\Big(|t_2 - t_1 - 2(y_1 x_2 - x_1 y_2)| + [(x_2 - x_1)^2 + (y_2 - y_1)^2]\Big), \end{aligned}$$

where

$$\nu(x) = \frac{x^2}{x + \sin^2 x - \sin x \cos x}.$$

Now, Theorem 5.13 becomes:

Theorem 5.21. *Given the points $P_1(x_1, y_1, t_1)$ and $P_2(x_2, y_2, t_2)$, with $x_1 = x_2$ and $y_1 = y_2$, the geodesics that join P_1 and P_2 have lengths*

$$s_m^2 = m\pi|t_2 - t_1|.$$

6. Carnot-Carathéodory distance. Recall that a curve $c(s) = (x_1(s), x_2(s), t(s))$ is called horizontal if $\dot{c}(s) \in \mathcal{H}_{c(s)}$, *i.e.*,

$$\omega = \dot{t} - 2x_2 dx_1 + 2x_1 dx_2.$$

By Chow's theorem, any two points P and Q can be joined by a horizontal curve. We have shown in Proposition 2.6 that the curved can be considered smooth. Hence

$$S = \{c; c(0) = P, c(1) = Q, c \text{ horizontal curve}\} \neq \emptyset.$$

The length of a horizontal curve c is

$$\ell(c) = \int_0^1 \sqrt{g(\dot{c}(s), \dot{c}(s))}\, ds,$$

where $g : \mathcal{H} \times \mathcal{H} \to \mathcal{F}$ is the subRiemannian metric. The Carnot-Carathéodory distance is defined as $d_C : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$,

$$d_C(P, Q) = \inf\{\ell(c); c \in S\}.$$

One may verify that d_C has the distance properties:

(i) $d_C(P, P) = 0$,
(ii) $d_C(P, Q) = d(Q, P)$,
(iii) $d_C(P, R) \leq d_C(P, Q) + d(Q, R)$.

The main result of this section is to show that the Carnot-Carathéodory distance d_C is complete, i.e any Cauchy sequence with respect to d_C is convergent. We shall prove this using the equivalence between the geodesic completeness and the completeness as a metric space.

Definition 6.1. If for any point P, any geodesic $c(t)$ emanating from P is defined for all $t \in \mathbb{R}$, the geometry is called geodesically complete.

The following theorem can be found in Strichartz [23]:

Theorem 6.2. *Let M be a connected step 2 subRiemannian manifold.*

(a) *If M is complete, then any two points can be joined by a geodesic.*
(b) *If there exists a point P such that every geodesic from P can be indefinitely extended, then M is complete.*
(c) *Every nonconstant geodesic is locally a unique length minimizing curve.*
(d) *Every length minimizing curve is a geodesic.*

It is known that the geodesics starting from the origin on the Heisenberg group are infinitely extendable. Using (b) of Theorem 6.2 we get

Theorem 6.3. *The Carnot-Carathéodory metric d_C is complete.*

• Computing the Carnot-Carathéodory distance

In the following we shall describe two ways to obtain the Carnot-Carathéodory distance. From (c) and (d) of Theorem 6.2 we get a way to compute the Carnot-Carathéodory distance

$$d_C(P,Q) = \{\ell(c),\ \text{where } c \text{ is the shortest geodesic joinning P and Q}\}.$$

Using Theorem 5.13 with $m = 1$, the Carnot-Carathéodory distance from origin

$$d_C(0,(x,t)) = \nu(\zeta_1)(|t| + \|x\|^2).$$

In particular, if the points are on the same vertical line, the Carnot-Carathéodory distance squared is proportional with the Euclidian distance

$$d_C((x,t),(x,t')) = \pi|t'-t| = \pi d_E((x,t),(x,t')).$$

Another way to obtain the Carnot-Carathéodory distance is using the complex action (see [3]). Consider the modified complex action function

$$f(x,t,\tau) = \tau g(x,t,\tau) = -i\tau t + \tau\coth(2\tau)\|x\|^2,$$

where

$$\begin{aligned} g(x,t,\tau) &= -it + \int_0^\tau \{\langle \dot{x},\xi\rangle - H\}\,ds \\ &= -it + \coth(2\tau)\|x\|^2 \end{aligned}$$

is the complex action. Like the classical action, the complex action g satisfies the Hamilton-Jacobi equation:

$$\frac{\partial g}{\partial \tau} + H\Big(x, \frac{\partial g}{\partial x}\Big) = 0.$$

Using Theorem 1.66 of [4], there is a unique critical point with respect to τ of the modified complex action function f in the strip $\{|\mathrm{Im}(\tau)| < \pi/2\}$ given by $\tau_c(x,t) = i\theta_c(x,t)$, where θ_c is the solution of

$$t = \mu(2\tau\theta)\|x\|^2$$

in this interval. At the critical point

$$f(x,t,\tau_c) = \frac{1}{2} d_C(0,(x,t)).$$

This works only in the case $x \neq 0$.

7. Complex Hamiltonian mechanics on $\mathbf{H}_1$.

7.1. The harmonic oscillator and the Heisenberg group.

We shall show that the harmonic oscillator leads to the Heisenberg operator by certain complex quantization. Consider a unit mass particle under the influence of force $F(x) = x$. The Newton's equation is $\ddot{x} = x$. This is the equation which describes the dynamics of an inverse pendulum in an unstable equilibrium, for small angle x, see Figure 8.

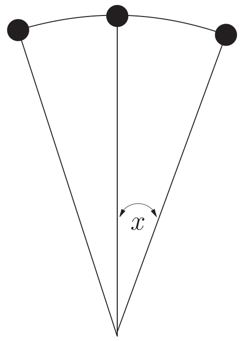

Figure 8. The inverse pendulum problem.

The potential energy

$$U(x) = -\int_0^x F(u)\,du = -\frac{x^2}{2}.$$

The Lagrangian $L : T\mathbb{R} \to \mathbb{R}$ is the difference between the kinetic and the potential energy

$$L(x, \dot{x}) = K - U = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2.$$

The momentum $p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$ and the Hamiltonian associated with the above Lagrangian is obtained using the Legendre transform: $H : T^*\mathbb{R} \to \mathbb{R}$

$$H(x, p) = p\dot{x} - L(x, \dot{x}) = p^2 - \frac{1}{2}p^2 - \frac{1}{2}x^2 = \frac{1}{2}p^2 - \frac{1}{2}x^2.$$

Using the ideas in Xavier and de Augiar [24], [25], we consider the following complexification

$$x = x_1 + ip_2, \quad p = p_1 + ix_2.$$

Hence $H : T^*\mathbb{C} \to \mathbb{C}$ and

$$\begin{aligned} H(x,p) &= \frac{1}{2}p^2 - \frac{1}{2}x^2 \\ &= \frac{1}{2}(p_1 + ix_2)^2 - \frac{1}{2}(x_1 + ip_2)^2 \\ &= \frac{1}{2}(p_1 + ix_2)^2 + \frac{1}{2}i^2(x_1 + ip_2)^2 \\ &= \frac{1}{2}(p_1 + ix_2)^2 + \frac{1}{2}(ix_1 - p_2)^2 \\ &= \frac{1}{2}(p_1 + ix_2)^2 + \frac{1}{2}(p_2 - ix_1)^2. \end{aligned}$$

Replacing $\theta = -i$,

$$H(x,p;\theta) = \frac{1}{2}(p_1 - \theta x_2)^2 + \frac{1}{2}(p_2 + \theta x_1)^2.$$

Quantizing, $p_1 \to \partial_{x_1}$, $p_2 \to \partial_{x_2}$, $\theta \to \partial_t$ and hence $H \to \Delta_X$, where

$$\Delta_X = \frac{1}{2}(\partial_{x_1} - x_2\partial_t)^2 + \frac{1}{2}(\partial_{x_2} + x_1\partial_t)^2$$

is the Heisenberg operator.

8. Fundamental solution and the heat kernel for sub-Laplacian. We plan to construct the fundamental solution of the operator

$$\begin{aligned} \Delta_H &= X_1^2 + X_2^2 \\ &= \Big(\partial_{x_1} + 2x_2\partial_t\Big)^2 + \Big(\partial_{x_2} - 2x_1\partial_t\Big)^2 \\ &= \partial_{x_1}^2 + \partial_{x_2}^2 + 4\partial_t\Big(x_2\partial_{x_1} - x_1\partial_{x_2}\Big) + 4(x_1^2 + x_2^2)\partial_t^2. \end{aligned}$$

It is expected that the fundamental solution has the following form (see [2] and [8]):

$$K(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}} \frac{E(\mathbf{x},\mathbf{y},\tau)v(\mathbf{x},\mathbf{y},\tau)}{g(\mathbf{x},\mathbf{y},\tau)} d\tau \tag{50}$$

which has a simple geometric interpretation. Here $\mathbf{x} = (x,t) = (x_1,x_2,t)$ and $\mathbf{y} = (y,s) = (y_1,y_2,s)$. It is easy to see that the operator Δ_H has a characteristic variety in the cotangent bundle $T^*\mathbf{H}_1$ given by $H = 0$. Over every point $\mathbf{x} \in \mathbf{H}_1$, this is a line, parametrized by $\theta \in (-\infty,\infty)$,

$$\xi_1 = -2x_2\theta, \qquad \xi_2 = 2x_1\theta.$$

Consequently, K may be thought of as the $(\text{action})^{-1}$ summed over the characteristic variety with measure Ev. When Δ_H is elliptic, its characteristic variety is the zero section, so we do get simply 1/distance, as expected. When Δ_H is sub-elliptic, τg behaves like the square of a distance function, even though it is complex.

Obviously, the operator Δ_H is left-invariant with respect to the Heisenberg translations. Thus we may set $\mathbf{y} = \mathbf{0}$ in $K(\mathbf{x},\mathbf{y})$. From Section 7, we know that

$$g(\mathbf{x},\mathbf{0},\tau) = g(\mathbf{x},\tau) = \coth(2\tau)\|x\|^2 - it.$$

In this case, the volume element v is the solution of the following transport equation

$$\frac{\partial v}{\partial \tau} + \sum_{j=1}^{2}(X_j g)(X_j v) + (\Delta_H g)v = 0. \tag{51}$$

Moreover, the energy E and the volume element v can be calculated explicitly:

$$E(\mathbf{x},\tau) = -\frac{\partial g}{\partial \tau} = \frac{2\|x\|^2}{\sinh^2(2\tau)} \qquad \text{and} \qquad v(\mathbf{x},\tau) = -\frac{1}{4\pi^2}\frac{\sinh(2\tau)}{\|x\|^2}.$$

Change the variable of integration τ to g:

$$K(\mathbf{x},\mathbf{0}) = \int_{\mathbb{R}} \frac{E(\mathbf{x},\tau)v(\mathbf{x},\tau)}{g(\mathbf{x},\tau)} d\tau = \int_{\mathbb{R}} \frac{v}{g}\left(-\frac{\partial g}{\partial \tau}\right) d\tau = -\int_{g_-}^{g_+} v\frac{dg}{g}, \tag{52}$$

where

$$g_\pm = \lim_{\tau\to\pm\infty} g = \pm(x_1^2+x_2^2) - it = \pm\|x\|^2 - it.$$

In this notation

$$v = -\frac{e^{i\pi/2}}{4\pi^2}\frac{1}{\sqrt{(g_+-g)(g-g_-)}}.$$

Thus

$$K(\mathbf{x},\mathbf{0}) = \int_{\mathcal{C}_-} v\frac{dg}{g},$$

where we made a branch cut

$$\mathcal{C} = (-\infty - it, g_-] \cup [g_+, -it+\infty),$$

with upper and lower directed edges denoted by $\mathcal{C}_+$ and $\mathcal{C}_-$, respectively. Integrals of v/g vanish on upper and lower semicircles as their radii increase without bound. Therefore we can integrate on a "dumbbell" with waist at $g_\pm$. Inside this domain v/g has a simple pole at $g = 0$. Hence

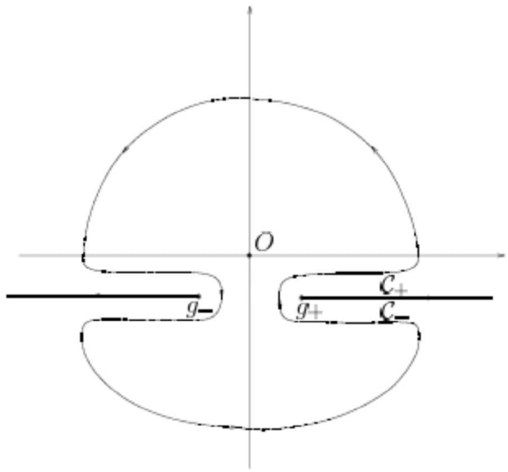

Figure 9. The "dumbbell" with waist at $g_\pm$.

$$-\frac{e^{i\pi/2}}{4\pi^2}(g_+)^{-1/2}(-g_-)^{-1/2} = \mathrm{Res}|_{g=0}\,\frac{v(g)}{g} = -\frac{1}{2\pi i}\int_{\mathcal{C}_-\cup\mathcal{C}_+} v\frac{dg}{g}.$$

Now

$$\int_{\mathcal{C}_+} v\frac{dg}{g} = \left(e^{2\pi i}\right)^{-\frac{1}{2}}\int_{-\mathcal{C}_-} v\frac{dg}{g} = -\int_{\mathcal{C}_-} v\frac{dg}{g},$$

hence

$$\begin{aligned} v|_{g=0} &= -\frac{1}{2\pi i}\int_{\mathcal{C}_-\cup\mathcal{C}_+} v\frac{dg}{g} \\ &= -\frac{1}{2\pi i}\int_{\mathcal{C}_-} v\frac{dg}{g} + \frac{1}{2\pi i}\int_{\mathcal{C}_+} v\frac{dg}{g} \\ &= -\frac{1}{2\pi i}\int_{\mathcal{C}_-} v\frac{dg}{g} - \frac{1}{2\pi i}\int_{\mathcal{C}_-} v\frac{dg}{g} \\ &= -\frac{1}{\pi i}\int_{\mathcal{C}_-} v\frac{dg}{g} = \frac{i}{\pi}K(\mathbf{x},\mathbf{0}), \end{aligned}$$

and therefore we obtain a closed form for (52)

$$K(\mathbf{x},\mathbf{0}) = -\frac{1}{8\pi}\frac{1}{\sqrt{\|x\|^4+t^2}}.$$

This is the Folland-Stein formula of [16] on the Heisenberg group. Unfortunately this simple formula is just a lucky coincidence of too much symmetry, and on general non-isotropic Heisenberg groups the fundamental solutions are given in the form (50). The reason is that the fundamental solution must include all the distances, which necessitates the use of g, and the summation over all the distances means integration on τ. Note that the change of variables $\tau \to g$ is reminiscent of classical calculations in action-angle coordinates.

Next we write the heat kernel associated to the sub-Laplacian Δ_H on the Heisenberg group in terms of the "distance function" $f = \tau g$ as follows (see [2], [3], [4], and [14]):

$$e^{-\Delta_H u}\psi(x,t,u) = \int_{\mathbf{H}_1} P_u((y,s)^{-1}\circ(x,t))\psi(y,s)dyds,$$

where

$$P_u(x,t) = \frac{1}{(2\pi u)^2} \int_{\mathbb{R}} e^{-\frac{f(x,t,\tau)}{u}} V(\tau) d\tau.$$

Here

$$f(x,t,\tau) = \tau g(x,t,\tau) = -i\tau t + \tau \coth(2\tau)\|x\|^2$$

is the complex action and

$$V(\tau) = \frac{2\tau}{\sinh(2\tau)}$$

is the Van Vleck determinant. Using the modified complex action function $f(x,t,\tau)$, one may discuss the small time behavior of the heat kernel. Then we have the following theorems.

Theorem 8.1. *Given a fixed point (x,t), $x \neq 0$, let θ_c denote the solution of equation (44) in the interval $[0, \pi/2)$. Then the heat kernel on $\mathbf{H}_1$ has the following small time behavior:*

$$P_u(x,t) = \frac{1}{(2\pi u)^2} e^{-\frac{d_c^2(x,t)}{2u}} \left\{ \Theta(x,t)\sqrt{2\pi u} + \mathcal{O}(u) \right\}, \qquad u \to 0^+,$$

where

$$\Theta(x,t) = \frac{\theta_c}{\|x\|\sqrt{[1 - 2\theta_c \coth(2\theta_c)]}},$$

and d_c denotes the Carnot-Carathéodory distance.

Theorem 8.2. *At point $(0,t)$, $t \neq 0$, we have*

$$P_u(x,t) = \frac{1}{(2u)^2} e^{-\frac{d_c^2(0,t)}{2u}} \left\{ 1 + \mathcal{O}(u) \right\}, \qquad u \to 0^+.$$

One also has

Theorem 8.3. *The heat kernel $P_u(x,t)$ on $\mathbf{H}_1$ has the following sharp* *upper bound:*

$$|P_u(x,t)| \le \frac{C}{u^2} e^{-\frac{d_c^2(x,t)}{2u}} \cdot \min\left\{1, \sqrt{\frac{u}{\|x\| d_c(x,t)}}\right\}.$$

For the proofs of Theorems 8.2 and 8.3, see [4]. There exist no explicit heat kernel for a higher step heat operator as yet. For the examples of this paper we are looking for a heat kernel of the form

$$P(\mathbf{x};u) = \frac{1}{u^2}\int_{\mathbb{R}} e^{-\frac{f}{u}} V\left(-\frac{\partial f}{\partial \tau} d\tau\right) = -\frac{1}{u^2}\int_{f_-}^{f_+} e^{-\frac{f}{u}} V(f) df,$$

where $f = \tau g$ and $f_\pm = \lim_{\tau\to\pm\infty} f$. $\frac{\partial f}{\partial \tau}$ turns out to be a constant of motion, just like $\frac{\partial g}{\partial \tau} = -E$ is; *i.e.,* a constant on the bicharacteristics. Then V is a solution of

$$\tau(T + \Delta_H g)\frac{\partial V}{\partial \tau} - \frac{\partial f}{\partial \tau}\Delta_H V = 0, \tag{53}$$

where

$$T = \frac{\partial}{\partial \tau} + (X_1 g)X_1 + (X_2 g)X_2$$

is derivation along the bicharacteristic curve. The equation (53) may be put in the following form:

$$\tau\left[(T + \Delta_H g)\frac{\partial V}{\partial \tau} - \frac{\partial g}{\partial \tau}\Delta_H V\right] = g\Delta_H V. \tag{54}$$

This should be compared to the equation for the volume element v of (51) which is a solution of

$$(T + \Delta_H g)\frac{\partial v}{\partial \tau} - \frac{\partial g}{\partial \tau}\Delta_H v = 0. \tag{55}$$

As we mentioned earlier, the above equation may be reduced to an Euler-Poisson-Darboux equation by a clever choice of coordinates. To find a higher step heat kernel we need a solution of equation (54). Equation (55) suggests that one may try to find such a solution as a perturbation of the volume element of the fundamental solution.

Remark. We also can apply this method to the higher step case, *i.e.*, the vector X_1 and X_2 are defined as follows:

$$\begin{aligned} X_1 &= \partial_{x_1} + 2kx_2(x_1^2+x_2^2)^{k-1}\partial_t \\ X_2 &= \partial_{x_2} - 2kx_1(x_1^2+x_2^2)^{k-1}\partial_t \end{aligned}$$

with $k > 1$. In this case, there is no group structure and the complex bicharacteristics run between two arbitrary points $\mathbf{y}$ and $\mathbf{x}$. We obtain 2 invariants of the motion, the energy E and the angular momentum Ω. One cannot calculate them explicitly, but we know their analytic properties, and g (see Theorem 3.3) and v may be found in terms of E and Ω. We state the result as follows.

Theorem 8.4. *For $k > 1$, the fundamental solution $K(x,y,t-s)$ of Δ_H has the following invariant represention,*

$$K(x,y,t-s) = \int_{\mathbb{R}} \frac{E(x,y,t-s;\tau)v(x,y,t-s;\tau)}{g(x,y,t-s;\tau)} d\tau,$$

where the second order transport equation (51) *may be reduced to an Euler-Poisson-Darboux equation and solved explicitly as a function of E and Ω. Namely,*

$$v = -\frac{e^{i\pi/2}}{2\pi^3 k}(\mathcal{A}_+ - g)^{-\frac{1}{2}}(\mathcal{A}_- - g)^{-\frac{1}{2}}F(\mathcal{P}_+, \mathcal{P}_-),$$

where

$$\mathcal{A}_+ = \overline{\mathcal{A}}_- = \|x\|^{2k} + \|y\|^{2k} - i(t-s) = \frac{\Omega_+}{k} + g_+,$$

and

$$\mathcal{P}_+ = \overline{\mathcal{P}}_- = \frac{2^{1/k}(x_1+ix_2)(y_1-iy_2)}{\mathcal{A}_+^{1/k}} = \left(1 + \frac{g_+}{\Omega_+/k}\right)^{-1/k},$$

with

$$\Omega_\pm = \lim_{\tau\to\pm\infty} \Omega.$$

Here F is a hypergeometric function of 2 variables,

$$F(\mathcal{P}_+,\mathcal{P}_-) = \frac{2}{\pi}\int_0^1\int_0^1 \Big\{ \sqrt{\frac{s_+s_-}{(1-s_+)(1-s_-)}} \frac{1-\mathcal{P}_+\mathcal{P}_-(s_+s_-)^{1/k}}{(1-\mathcal{P}_+s_+^{1/k})(1-\mathcal{P}_-s_-^{1/k})(1-(\mathcal{P}_+\mathcal{P}_-)^k s_+s_-)} \Big\} \frac{ds_+}{s_+}\frac{ds_-}{s_-}.$$

Readers may consult the paper [2] and the forthcoming book [10] for detailed discussions. When $k=2$, the fundamental solution $K(x,y,t-s)$ has the following simple form:

$$K(x,y,t-s) = \frac{i}{2\pi^2 d}\log h,$$

where

$$h(\mathcal{P},\overline{\mathcal{P}}) = \frac{|1-\mathcal{P}^2| - i(\mathcal{P}+\overline{\mathcal{P}})}{1+|\mathcal{P}|^2},$$

with

$$\mathcal{P} = \frac{(x_1y_1+x_2y_2)+i(x_1y_2-x_2y_1)}{\mathcal{A}^{1/2}}, \qquad \mathcal{A} = \frac{1}{2}\Big(\|x\|^4+\|y\|^4+i(t-s)\Big).$$

This formula first appeared in [2] and [18].

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