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ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{\alpha_1 x_n + \cdots + \alpha_k x_{n-k+1}}{A + f(x_n, \ldots, x_{n-k+1})}$

BY

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Abstract. We investigate the behavior of solutions of the difference equation

$$x_{n+1} = \frac{\alpha_1 x_n + \cdots + \alpha_k x_{n-k+1}}{A + f(x_n, \ldots, x_{n-k+1})}, \quad n \geq k-1,\ k \in \mathbf{N},$$

where the parameters α_i, $i = 1, \ldots, k$, and initial conditions $x_0, x_1, \ldots, x_{k-1}$ are nonegative real numbers, $A > 0$ and where the function f satisfies certain additional conditions. The main result in this note solves and generelizes an open problem in [11]. In the proof of the main result we do not use smoothness of the function f.

1. Introduction. In this note we investigate the behavior of solutions of the difference equation

$$x_{n+1} = \frac{\alpha_1 x_n + \cdots + \alpha_k x_{n-k+1}}{A + f(x_n, \ldots, x_{n-k+1})}, \quad n \geq k-1,\ k \in \mathbf{N}, \tag{1}$$

where the parameters α_i, $i = 1, \ldots, k$, and initial conditions $x_0, x_1, \ldots, x_{k-1}$ are nonegative real numbers. This equation appears in a class of Mathematical Biology models, for example: Discrete delay logistic difference equation [12] and Mosquito population equations [9].

In [11] the author posed the following open problem:

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Open problem 5.2.3. Consider the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + x_{n-1}}$$

where β, γ, $A \in (0, \infty)$. For each of the following statements, obtain necessary and sufficient conditions on β, γ and A so the statement holds.

(a) Every positive solution is bounded.

(b) Every positive solution converges to zero.

The same problems are also posed for the equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + x_n}.$$

For this equation complete answers to the above statements are provided in [10].

In this paper we consider the equation (1) and prove some results which concern the above mentioned questions. For closely related results see, for example, [4, 7, 9, 10, 19, 20, 21, 22, 23, 24, 25] and the references therein.

2. Auxiliary results. In order to prove our main results we need some auxiliary results. The following theorem was established in [17].

Theorem A. *Let $\varphi(y_1, y_2, \dots, y_k)$ be a continuous real function on $\mathbf{R}^k$ which is nondecreasing in each variable and increasing in the first one and $\varphi(x, x, \dots, x) \le x$, for every $x \in \mathbf{R}$. If (a_n) is a bounded sequence which satisfies the inequality*

$$a_{n+k} \le \varphi(a_{n+k-1}, a_{n+k-2}, \dots, a_n) \quad \textit{for} \quad n \in \mathbf{N} \cup \{0\},$$

then it must be convergent.

This theorem generalizes the main results in [2] and [5]. For closely related results see also [3, 13, 16, 18]. Actually we need only the following corollary of Theorem A:

Corollary 1. *Let $k_1>0$, $k_i\ge 0$, $i=2,\ldots,m$, $k_1+\cdots+k_m=1$, and the real sequence (a_n) satisfy the inequality*

$$a_{n+m}\le \sum_{i=1}^{m} k_i a_{n+m-i}, \qquad \textit{for} \qquad n\in \mathbf{N}\cup\{0\}.$$

If (a_n) is bounded, then it converges.

The following lemma was proved in [15].

Lemma A. *Let (a_n) be a sequence of positive numbers which satisfies the inequality*

$$a_{n+k}\le A\max\{a_{n+k-1},a_{n+k-2},\ldots,a_n\} \qquad \textit{for} \qquad n\in\mathbf{N},$$

where $A\in(0,1)$ and $k\in\mathbf{N}$ are fixed. Then there exist $L\in\mathbf{R}_+$ such that

$$a_{km+r}\le L\,A^m \quad \textit{for all} \quad m\in\mathbf{N}\cup\{0\} \quad \textit{and} \quad 1\le r\le k.$$

Corollary 2. *Let (a_n) be the sequence of positive numbers in Lemma A. Then there exists $M>0$ such that*

$$a_n\le M\left(\sqrt[k]{A}\right)^n.$$

3. Main results. In this section we prove the main results in this paper.

Theorem 1. *Consider the difference equation* (1), *where* $\alpha_1 > 0$ *and function* f *satisfies the following conditions*

(a) $f \in C([0,\infty)^k, [0,\infty))$;

(b) $f(0,\dots,0) = 0$;

(c) $f(x,\dots,x) > 0$ *for* $x \in (0,\infty)$;

(d) f *in nondecreasing in each of its variables and strictly increasing in at least one of its variables.*

Then every positive solution of Eq.(1) *converges to zero if and only if* $\sum_{i=1}^k \alpha_i \le A$.

Proof. *Case* 1. Let $\sum_{i=1}^k \alpha_i = A$. From (1) we have

$$x_{n+1} = \frac{\alpha_1 x_n + \cdots + \alpha_k x_{n-k+1}}{A + f(x_n,\dots,x_{n-k+1})} \le \sum_{i=1}^k \frac{\alpha_i}{A} x_{n-i+1}.$$

Thus by Corollary 1 we have that the sequence (x_n) converges, for example to $x \ge 0$. Letting $n \to \infty$ in (1) we obtain $x = \frac{x\sum_{i=1}^k \alpha_i}{A+f(x,\dots,x)}$. Hence $x = 0$.

Case 2. If $\sum_{i=1}^k \alpha_i < A$, then from (1) it follows

$$x_{n+1} \le \frac{\sum_{i=1}^k \alpha_i}{A} \max\{x_n, x_{n-1}, \dots, x_{n-k+1}\}.$$

By Corollary 2 we obtain that the sequence (x_n) geometrically converges to zero.

Case 3. Let $\sum_{i=1}^k \alpha_i > A$. Suppose that there exists a positive solution (x_n) of Eq.(1) such that $x_n \to 0$ as $n \to \infty$. Then for every $\varepsilon > 0$ there exists n_0 such that $0 \le f(x_n,\dots,x_{n-k+1}) < \varepsilon$ for all $n \ge n_0$. Let $0 < \varepsilon < \sum_{i=1}^k \alpha_i - A$. From (1) we have

$$x_{n+1} \ge \sum_{i=1}^k \frac{\alpha_i}{A+\varepsilon} x_{n-i+1}, \quad \text{for} \quad n \ge n_0.$$

Let the sequence (z_n) satisfies the following difference equation

$$z_{n+1}=\sum_{i=1}^{k}\frac{\alpha_i}{A+\varepsilon}z_{n-i+1}, \tag{2}$$

where $z_i=x_i$, $i=n_0,\ldots,n_0+k-1$.

It is easy to show by induction that $z_n\le x_n$ for all $n\ge n_0$. The characteristic polynomial of (2) is

$$P_k(\lambda)=\lambda^k-\sum_{i=1}^{k}\frac{\alpha_i}{A+\varepsilon}\lambda^{k-i}.$$

Since $P_k(1)=(A+\varepsilon-\sum_{i=1}^{k}\alpha_i)/(A+\varepsilon)<0$ we obtain that P_k has a positive characteristic root which belong to the interval $(1,\infty)$. On the other hand we have

$$P_k(\lambda)=\frac{(A+\varepsilon)\lambda^k-\sum_{i=1}^{k}\alpha_i\lambda^{k-i}}{A+\varepsilon}<\lambda^k\frac{A+\varepsilon-\sum_{i=1}^{k}\alpha_i}{A+\varepsilon}<0,$$

when $\lambda\in(0,1)$. Hence all positive characteristic roots of P_k have modulus greater than one.

Let λ_i, $i=1,\ldots,m$, denote all characteristic roots of P_k, such that $\lambda_i\ne\lambda_j$, for $i\ne j$. By the well known representation for solutions of the difference equation (2) we have

$$z_n=(c_1^{(1)}+\cdots+c_{j_1}^{(1)}n^{j_1})\lambda_1^n+\cdots(c_1^{(m)}+\cdots+c_{j_m}^{(m)}n^{j_m})\lambda_m^n$$

where $c_{j_i}^{(i)}\ne 0$, $i=1,\ldots,m$.

Since the sequence z_n is positive the characteristic root with the largest modulus must be positive as well as its first nonzero coefficient $c_{j_i}^{(i)}$. Hence $z_n\to\infty$ as $n\to\infty$, and consequently $x_n\to\infty$ as $n\to\infty$, arriving at a contradiction.

Remark 1. Note that in the conditions of Theorem 1 we do not use smoothness of the function f.

Remark 2. We can prove that $z_n \to \infty$ as $n \to \infty$ also by the following lemma.

Lemma 1. *Consider the difference equation*

$$z_{n+1} = \sum_{i=1}^{k} p_i z_{n-i+1} \tag{3}$$

where the parameters p_i, $i = 1, \ldots, k$, *are nonnegative real numbers such that* $\sum_{i=1}^{k} p_i > 1$ *and initial conditions* $z_0, \ldots, z_{k-1}$ *are positive real numbers. Then*

$$z_{jk+r} \geq (\sum_{i=1}^{k} p_i)^j \min\{z_0, \ldots, z_{k-1}\},$$

for all $j \in \mathbf{N} \cup \{0\}$ *and* $r \in \{0, 1, \ldots, k-1\}$.

Proof. We prove the lemma by induction. Let $j = 1$. From (3) we obtain

$$z_k \geq \sum_{i=1}^{k} p_i \min\{z_0, \ldots, z_{k-1}\} > \min\{z_0, \ldots, z_{k-1}\}.$$

From this we obtain

$$\begin{aligned} z_{k+1} &\geq \sum_{i=1}^{k} p_i \min\{z_1, \ldots, z_k\} \geq \sum_{i=1}^{k} p_i \min\{z_0, \ldots, z_{k-1}\} \\ &> \min\{z_0, \ldots, z_{k-1}\}. \end{aligned}$$

Similarly

$$\begin{aligned} z_{k+r} &\geq \sum_{i=1}^{k} p_i \min\{z_r, \ldots, z_{k+r-1}\} \geq \sum_{i=1}^{k} p_i \min\{z_0, \ldots, z_{k-1}\} \\ &> \min\{z_0, \ldots, z_{k-1}\}, \end{aligned}$$

for $r\in\{2,\ldots,k-1\}$.

Suppose that the result holds for some $j\in\mathbf{N}$ and $0\le r\le k-1$. By (3) and the induction hypothesis we have

$$\begin{aligned} z_{(j+1)k} &\ge (\sum_{i=1}^k p_i)^j \min\{z_{jk},\ldots,z_{(j+1)k-1}\}\ge(\sum_{i=1}^k p_i)^{j+1}\min\{z_0,\ldots,z_{k-1}\}\\ &> (\sum_{i=1}^k p_i)^j\min\{z_0,\ldots,z_{k-1}\}. \end{aligned}$$

Using this we obtain

$$\begin{aligned} z_{(j+1)k+r} &\ge (\sum_{i=1}^k p_i)^j \min\{z_{jk+r},\ldots,z_{(j+1)k+r-1}\}\\ &\ge (\sum_{i=1}^k p_i)^{j+1}\min\{z_0,\ldots,z_{k-1}\}, \end{aligned}$$

for $r\in\{1,\ldots,k-1\}$, as desired.

In Theorem 1 $\sum_{i=1}^k p_i=\sum_{i=1}^k\frac{\alpha_i}{A+\varepsilon}>1$ from which the result follows.

From the proof of Theorem 1 we see that the following statements hold.

Theorem 2. *Consider the difference equation* (1), *where* $\sum_{i=1}^k\alpha_i>A$ *and the function* f *satisfies the following conditions*

(a) $f\in C(\mathbf{R}^k)$;

(b) $\sup_{(y_1,\ldots,y_k)\in[0,\infty)^k} f(y_1,\ldots,y_n)<\sum_{i=1}^k\alpha_i-A$;

(c) $\inf_{(y_1,\ldots,y_k)\in[0,\infty)^k} f(y_1,\ldots,y_n)>-A$.

Then there is no positive solution of Eq.(1) *which converges to zero.*

Example 1. Consider the difference equation

$$x_{n+1}=\frac{\alpha_1x_n+\cdots+\alpha_kx_{n-k+1}}{A+\sin(x_n\cdot x_{n+1}\cdots x_{n-k+1})},\quad n\ge k-1,\ k\in\mathbf{N},$$

where the parameters α_i, $i = 1, \ldots, k$, and initial conditions $x_0, x_1, \ldots, x_{k-1}$ are nonegative real numbers and where $A > 1$ and $\sum_{i=1}^{k} \alpha_i > A + 1$. Then there is no positive solution of this equation which converges to zero.

Theorem 3. *Consider the difference equation*

$$x_{n+1} = \sum_{i=1}^{k} \frac{\alpha_i x_{n-i+1}}{A_i + f_i(x_n, \ldots, x_{n-k+1})}, \quad n \geq k-1, \ k \in \mathbf{N}, \tag{4}$$

where the parameters α_i, $i = 2, \ldots, k$ *and initial conditions* $x_0, x_1, \ldots, x_{k-1}$ *are nonegative real numbers,* α_1 *and* A_i, $i = 1, \ldots, k$, *are positive real numbers,* $f_i \in C([0,\infty)^k, [0,\infty))$, $i = 1, \ldots, k$, *and there is at least an* $i_0 \in \{1, \ldots, n\}$ *such that the function* f_{i_0} *satisfies conditions* (a)-(d) *in Theorem* 1 *and* $\alpha_{i_0} > 0$. *Then every positive solution of Eq.*(4) *converges to zero if and only if* $\sum_{i=1}^{k} \frac{\alpha_i}{A_i} \leq 1$.

Equations (4) generalizes equation (1) and appears in a large class of mathematical biology models for example in Generalized Bedington-Holt stock recruitment model [1, 15], Flour beetle population model [6] and in a special case of Perenial grass model [8].

Remark 3. In [18] we proved the following theorem:

Theorem B. *Let* $\varphi(y_1, y_2, \ldots, y_k)$ *be a continuous real function on* $\mathbf{R}^k$ *where*

(a) $\varphi(x, x, \ldots, x) \leq x$, *for every* $x \in \mathbf{R}$;

(b) $\varphi \in C(\mathbf{R}^k, \mathbf{R})$ *is nondecreasing in each of its arguments;*

(c) $\varphi(y_1, y_2, \ldots, y_k)$ *is strictly increasing in at least two of its arguments* y_i *and* y_j, *where* i *and* j *are relatively prime.*

If (a_n) *is a sequence which satisfies the inequality*

$$a_{n+k} \leq \varphi(a_{n+k-1}, a_{n+k-2}, \ldots, a_n) \quad \textit{for} \quad n \in \mathbf{N} \cup \{0\},$$

then it converges or tends to minus infinity.

Using this theorem we see that if instead of the condition $\alpha_1 > 0$ we assume that there are $i,\ j \in \{1, \ldots, k\}$ relatively prime such that $\alpha_i > 0$ and $\alpha_j > 0$, and if all other conditions of Theorem 3 are satisfied then every positive solution of Eq.(4) converges to zero if and only if $\sum_{i=1}^{k} \frac{\alpha_i}{A_i} \le 1$.

The following theorem concerns with the boundedness of the solutions of equation (4).

Theorem 4. *Consider the difference equation* (4). *Let* $\alpha_{i_0} < A_{i_0}$ *for some* $i_0 \in \{1, \ldots, k\}$, $f_i \in C([0, \infty)^k, [0, \infty))$ *for* $i = 1, \ldots, k$ *and*

$$g_j(y_1, \ldots, y_k) = \frac{y_j}{A_j + f_j(y_1, \ldots, y_k)}$$

are bounded for all $j \ne i_0$.

Then every nonnegative solution of Eq.(4) *is bounded.*

Proof. Let $M_j = \sup_{y_i \ge 0, i=1,\ldots,k} g_j(y_1, \ldots, y_n)$, $M = \sum_{i=1, i \ne i_0}^{k} M_i$ and $q = \frac{\alpha_{i_0}}{A}$, then from (4) we obtain

$$0 \le x_{n+1} \le q x_{n-i_0+1} + M$$

or equivalently

$$0 \le x_{(m+1)i_0+k} \le q x_{mi_0+k} + M$$

for all $m \in \mathbf{N}$ and $k \in \{0, 1, \ldots, i_0 - 1\}$.

Applying the change $z_m^{(k)} = x_{mi_0+k}$ we have

$$z_{m+1}^{(k)} \le q z_m^{(k)} + M \quad \text{for} \quad m \in \mathbf{N} \cup \{0\}.$$

By induction we can prove that

$$z_m^{(k)} \leq M(1+q+\cdots+q^{m-1}) + z_0^{(k)} q^m \quad \text{for} \quad m \in \mathbf{N} \cup \{0\}.$$

Since $q \in (0,1)$ we obtain

$$z_m^{(k)} \leq \frac{M}{1-q} + z_0^{(k)},$$

as desired.

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