BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 33, Number 2, June 2005

TENSOR PRODUCT SURFACES OF A LORENTZIAN SPACE CURVE AND A LORENTZIAN PLANE CURVE

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Abstract. In this paper, we study the tensor product surfaces of a Lorentzian space curve and a Lorentzian plane curve. In particular, we classify all minimal, totally real and complex tensor product surfaces of such curves.

1. Preliminaries. The Lorentzian 3-space (or Minkowski 3-space) $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$.

Since g is indefinite metric, recall that a vector $v \in \mathbb{E}^3_1$ can have one of three Lorentzian causal characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}^3_1$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike) [12].

Let c be a fixed point in $\mathbb{E}^3_1$, and $r > 0$ be a constant. The pseudosphere

Received by the editors January 27, 2004 and in revised form July 22, 2004.

1991 Mathematics Subject Classification: 53C50, 53C40.

Key words and phrases: Lorentzian curve, tensor product surfaces, minimal surface, totally real and complex tensor product surface.

with center c and of radius r is defined by

$$\mathbb{S}^2_1(c,r)=\{x\in\mathbb{E}^3_1|\ g(x-c,x-c)=r^2\},$$

and the pseudohyperbolical space with center c and of radius r is defined by

$$\mathbb{H}^2_0(c,r)=\{x\in\mathbb{E}^3_1|\ g(x-c,x-c)=-r^2\}.$$

2. Tensor product immersions. B. Y. Chen initiated the study of the *tensor product immersion* of two immersions of a given Riemannian manifold ([4]). This concept originated from the investigation of the quadratic representation of submanifold. Inspired by Chen's definition, F. Decruyenaere, F. Dillen, L. Verstraelen and L. Vrancken studied in [5] the tensor product of two immersions of, in general, different manifolds. Under some conditions, this realizes an immersion of the product manifold.

Let M and N be two differentiable manifolds and assume that

$$f\ :\ M\to E^m$$

and

$$g\ :\ N\to E^n$$

are two immersions. Then the direct sum and tensor product maps are defined respectively by

$$\begin{aligned} & f\oplus h: M\times N\to E^{m+n}:(p,q)\to f(p)\oplus h(q)\\ = \ & (f^1(p),\dots,f^m(p),h^1(q),\dots,h^n(q))\end{aligned}$$

and

$$\begin{aligned} & f\otimes h: M\times N\to E^{mn}:(p,q)\to f(p)\otimes h(q)\\ = \ & (f^1(p)h^1(q),\dots,f^1(p)h^n(q),\dots,f^m(p)h^n(q))\end{aligned}$$

Necessary and sufficient conditions for $f \otimes h$ to be an immersion were obtained in [5]. It is also proved there that the pairing $(\oplus, \otimes)$ determines a structure of a semiring on the set of classes of differentiable manifolds transversally immersed in Euclidean spaces, modulo orthogonal transformations. Some semirings were studied in [6].

A classification of minimal tensor product surfaces of two Euclidean curves is obtained in [1, 8, 9]. For the tensor product surfaces of two Lorentzian plane curves we refer to [11, 13]. In particular, the tensor product surfaces of a Lorentzian curve and a Euclidean curve, were studied in [7, 10].

In this paper, we study the tensor product surfaces of a Lorentzian space curve and a Lorentzian plane curve. Moreover, we classify all minimal, totally real and complex tensor product surfaces of such curves.

3. Tensor Product surfaces of a lorentzian space curve and a Lorentzian plane curve. Let $\alpha : \mathbb{R} \to \mathbb{E}^3_1$ and $\beta : \mathbb{R} \to \mathbb{E}^2_1$ be a Lorentzian space curve and a Lorentzian plane curve respectively. Denote $g_1 = -dx_1^2 + dx_2^2 + dx_3^2$ and $g_2 = -dx_1^2 + dx_2^2$ metric tensors on $\mathbb{E}^3_1$ and $\mathbb{E}^2_1$ respectively. Let us put $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and $\beta(s) = (\beta_1(s), \beta_2(s))$. Then their tensor product is given by

$$\begin{aligned} f &= \alpha \otimes \beta : \mathbb{R}^2 \to \mathbb{E}^6_3 \\ f(t,s) &= (\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), \alpha_2(t)\beta_1(s), \alpha_2(t)\beta_2(s), \\ &\quad \alpha_3(t)\beta_1(s), \alpha_3(t)\beta_2(s)). \end{aligned} \tag{1}$$

The metric tensor on $\mathbb{E}^6_3$ is given by

$$g = dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2 - dx_5^2 + dx_6^2.$$

Therefore, the coefficients of the pseudo-Riemannian metric g, induced on Im f by the pseudo-Riemannian metric of $\mathbb{E}^6_3$, are

$$\begin{aligned} g_{11} &= g_1(\dot\alpha,\dot\alpha)g_2(\beta,\beta), \\ g_{12} &= g_1(\alpha,\dot\alpha)g_2(\beta,\dot\beta), \\ g_{22} &= g_1(\alpha,\alpha)g_2(\dot\beta,\dot\beta). \end{aligned}$$

In the following, assume that $\alpha(t)$ and $\beta(s)$ are spacelike or timelike curves with spacelike or timelike position vectors. Then $g_1(\alpha,\alpha) \neq 0 \neq g_1(\dot\alpha,\dot\alpha)$ and $g_2(\beta,\beta) \neq 0 \neq g_2(\dot\beta,\dot\beta)$ and hence $g_{11} \neq 0 \neq g_{22}$. Consequently, an orthonormal basis for the tangent space is given by

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{|g_{11}|}}\frac{\partial f}{\partial t}, \\ e_2 &= \frac{1}{\sqrt{|g_{11}(g_{11}g_{22}-g_{12}^2)|}}(g_{11}\frac{\partial f}{\partial s} - g_{12}\frac{\partial f}{\partial t}). \end{aligned}$$

Recall that the mean curvature vector field H is defined by

$$H = \frac{1}{2}(\epsilon_1 h(e_1,e_1) + \epsilon_2 h(e_2,e_2)),$$

where h is second fundamental form of $\mathrm{Im}(\alpha\otimes\beta)$ and $\epsilon_i = g(e_i,e_i)$, $i = 1,2$. In particular, by *Beltrami's* formula we have

$$H = -\frac{1}{2}\Delta f.$$

Next recall that a surface M in $\mathbb{E}^6_3$ is said to be minimal if its mean curvature vector field H vanishes identically. Since a basis $\{n_1,n_2,n_3,n_4\}$ of the normal space can be calculated and is given by

$$\begin{aligned} n_1 &= (\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s), \alpha_1(t)\beta_2(s), \alpha_1(t)\beta_1(s), 0, 0), \\ n_2 &= (0,0,-\alpha_3(t)\beta_2(s), -\alpha_3(t)\beta_1(s), \alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s)), \\ n_3 &= (\dot\alpha_2(t)\dot\beta_2(s), \dot\alpha_2(t)\dot\beta_1(s), \dot\alpha_1(t)\dot\beta_2(s), \dot\alpha_1(t)\dot\beta_1(s), 0, 0), \\ n_4 &= (0,0,-\dot\alpha_3(t)\dot\beta_2(s), -\dot\alpha_3(t)\dot\beta_1(s), \dot\alpha_2(t)\dot\beta_2(s), \dot\alpha_2(t)\dot\beta_1(s)), \end{aligned} \tag{2}$$

it follows that a surface Imf is minimal in $\mathbb{E}^6_3$ if and only if

$$g(H, n_i) = 0, \qquad i = 1, 2, 3, 4.$$

On the other hand, by using *Beltrami's* formula, a surface Imf is minimal in $\mathbb{E}^6_3$ if and only if

$$g(\Delta f, n_i) = 0, \quad i = 1, 2, 3, 4. \tag{3}$$

Since the Laplacian of f is given by

$$\Delta f = g^{11}\frac{\partial^2 f}{\partial t^2} + 2g^{12}\frac{\partial^2 f}{\partial t \partial s} + g^{22}\frac{\partial^2 f}{\partial s^2},$$

the minimality conditions (3) become

$$g(g_{11}\frac{\partial^2 f}{\partial s^2} + 2g_{12}\frac{\partial^2 f}{\partial t \partial s} + g_{22}\frac{\partial^2 f}{\partial t^2}, n_i) = 0, \quad i = 1, 2, 3, 4. \tag{4}$$

In the first theorem, we classify all minimal tensor product surfaces in $\mathbb{E}^6_3$.

Theorem 1. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian space curve $\alpha : \mathbb{R} \to \mathbb{E}^3_1$ and a Lorentzian plane curve $\beta : \mathbb{R} \to \mathbb{E}^2_1$, is a minimal surface in $\mathbb{E}^6_3$ if and only if :*

(i) *either α or β is a straight line through the origin;*

(ii) *α is the circle centered at the origin, given by $\alpha(t) = (0, \cos(t), \sin(t))$ and β is either the circle with the equation*

$$\beta(s) = \frac{c}{\sqrt{\cosh(2s)}}(\cosh(s), \sinh(s)), \quad c \in \mathbb{R}^+, \tag{5}$$

or an orthogonal hyperbola with the equation

$$\beta(s) = \frac{d}{\sqrt{|\sinh(2s)|}}(\cosh(s), \sinh(s)), \quad d \in \mathbb{R}^+; \tag{6}$$

(iii) *α is the orthogonal hyperbola centered at the origin given by* $\alpha(t) = (\cosh(t), \sinh(t), 0)$, *and β is the curve given either by* (5) *or* (6);

(iv) *β is given either by* (5) *or* (6), *and α is either the circle given by* $\alpha(t) = A\cos(t) + B\sin(t)$, *where $A, B \in \mathbb{E}^3_1$ are mutually orthogonal unit spacelike vectors, or else the orthogonal hyperbola given by* $\alpha(t) = C\cosh(t) + D\sinh(t)$, *where $C, D \in \mathbb{E}^3_1$ are the unit mutually orthogonal spacelike and timelike vectors;*

(v) *β is the orthogonal hyperbola given by* $\beta(s) = (a\sinh(s), a\cosh(s))$, *or else by* $\beta(s) = (a\cosh(s), a\sinh(s))$, $a \in \mathbb{R}^+$ *and α is either the orthogonal hyperbola given by* $\alpha(t) = \frac{c}{\sqrt{|\cos(2t)|}}(0, \sin(t), \cos(t))$, $c \in \mathbb{R}^+$ *or else by* $\alpha(t) = \frac{d}{\sqrt{|\sinh(2t)|}}(\cosh(t), \sinh(t), 0)$, $d \in \mathbb{R}^+$ *or else the circle given by* $\alpha(t) = \frac{e}{\sqrt{\cosh(2t)}}(\cosh(t), \sinh(t), 0)$, $e \in \mathbb{R}^+$;

(vi) *β is the orthogonal hyperbola given by* $\beta(s) = (a\sinh(s), a\cosh(s))$ *or else by* $\beta(s) = (a\cosh(s), a\sinh(s))$, $a \in \mathbb{R}^+$ *and α is either the orthogonal hyperbola given by* $\alpha(t) = C\cosh(t) + D\sinh(t)$, *where $C, D \in \mathbb{E}^3_1$ are both spacelike or timelike vectors, such that* $||C|| = ||D||$, *or else the circle given by* $\alpha(t) = A\cos(t) + B\sin(t)$, *where $A, B \in \mathbb{E}^3_1$ are the spacelike and the timelike vectors such that* $||A|| = ||B||$.

Proof. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a Lorentzian space curve and $\beta(s) = (\beta_1(s), \beta_2(s))$ be a Lorentzian plane curve. Then their tensor product $f = \alpha \otimes \beta$ is given by (1). First suppose that Imf is a minimal surface in $\mathbb{E}^6_3$. Then relation (4) holds. From (1) we have

$$\frac{\partial^2 f}{\partial t^2} = (\ddot{\alpha}_1(t)\beta_1(s), \ddot{\alpha}_1(t)\beta_2(s), \ddot{\alpha}_2(t)\beta_1(s), \ddot{\alpha}_2(t)\beta_2(s), \ddot{\alpha}_3(t)\beta_1(s), \ddot{\alpha}_3(t)\beta_2(s)),$$
$$\frac{\partial^2 f}{\partial s^2} = (\alpha_1(t)\ddot{\beta}_1(s), \alpha_1(t)\ddot{\beta}_2(s), \alpha_2(t)\ddot{\beta}_1(s), \alpha_2(t)\ddot{\beta}_2(s), \alpha_3(t)\ddot{\beta}_1(s), \alpha_3(t)\ddot{\beta}_2(s)),$$
$$\frac{\partial^2 f}{\partial t \partial s} = (\dot{\alpha}_1(t)\dot{\beta}_1(s), \dot{\alpha}_1(t)\dot{\beta}_2(s), \dot{\alpha}_2(t)\dot{\beta}_1(s), \dot{\alpha}_2(t)\dot{\beta}_2(s), \dot{\alpha}_3(t)\dot{\beta}_1(s), \dot{\alpha}_3(t)\dot{\beta}_2(s)).$$

Since the normal space is spanned by vectors $\{n_1, n_2, n_3, n_4\}$ given by (2), a straightforward computations yield

$$g(\frac{\partial^2 f}{\partial t^2}, n_i) = g(\frac{\partial^2 f}{\partial s^2}, n_i) = g(\frac{\partial^2 f}{\partial t \partial s}, n_j) = 0, \quad i = 1, 2, \quad j = 3, 4, \tag{7}$$

$$g(\frac{\partial^2 f}{\partial t \partial s}, n_1) = (\dot{\alpha}_1 \alpha_2 - \alpha_1 \dot{\alpha}_2)(\dot{\beta}_1 \beta_2 - \beta_1 \dot{\beta}_2), \tag{8}$$

$$g(\frac{\partial^2 f}{\partial t \partial s}, n_2) = (\dot{\alpha}_2 \alpha_3 - \alpha_2 \dot{\alpha}_3)(\dot{\beta}_1 \beta_2 - \beta_1 \dot{\beta}_2), \tag{9}$$

and

$$\begin{aligned} g(\frac{\partial^2 f}{\partial t^2}, n_3) &= (\beta_1 \dot{\beta}_2 - \dot{\beta}_1 \beta_2)(\ddot{\alpha}_1 \dot{\alpha}_2 - \dot{\alpha}_1 \ddot{\alpha}_2), \\ g(\frac{\partial^2 f}{\partial s^2}, n_3) &= (\ddot{\beta}_1 \dot{\beta}_2 - \dot{\beta}_1 \ddot{\beta}_2)(\alpha_1 \dot{\alpha}_2 - \dot{\alpha}_1 \alpha_2), \\ g(\frac{\partial^2 f}{\partial t^2}, n_4) &= (\beta_1 \dot{\beta}_2 - \dot{\beta}_1 \beta_2)(\ddot{\alpha}_2 \dot{\alpha}_3 - \dot{\alpha}_2 \ddot{\alpha}_3), \\ g(\frac{\partial^2 f}{\partial s^2}, n_4) &= (\ddot{\beta}_1 \dot{\beta}_2 - \dot{\beta}_1 \ddot{\beta}_2)(\alpha_2 \dot{\alpha}_3 - \dot{\alpha}_2 \alpha_3). \end{aligned} \tag{10}$$

Then relations (4) and (7) imply that there hold the minimality conditions

$$g(\frac{\partial^2 f}{\partial t \partial s}, n_i) = 0, \quad g(g_{22} \frac{\partial^2 f}{\partial t^2} + g_{11} \frac{\partial^2 f}{\partial s^2}, n_j) = 0, \quad i = 1, 2, \quad j = 3, 4, \tag{11}$$

or else

$$g_{12} = 0, \quad g(g_{22} \frac{\partial^2 f}{\partial t^2} + g_{11} \frac{\partial^2 f}{\partial s^2}, n_j) = 0, \quad j = 3, 4. \tag{12}$$

Therefore, we distinguish the following two cases:

Case I. Assume that (11) holds. Then relations (8) and (9) imply that

$$\begin{aligned} (\dot{\alpha}_1 \alpha_2 - \alpha_1 \dot{\alpha}_2)(\dot{\beta}_1 \beta_2 - \beta_1 \dot{\beta}_2) &= 0, \\ (\dot{\alpha}_2 \alpha_3 - \alpha_2 \dot{\alpha}_3)(\dot{\beta}_1 \beta_2 - \beta_1 \dot{\beta}_2) &= 0. \end{aligned}$$

Now we consider two subcases:

Case I.1. If $\dot{\beta}_1\beta_2 - \beta_1\dot{\beta}_2 = 0$, it follows that β is a straight line in $\mathbb{E}^2_1$ passing through the origin and relation (11) is satisfied for any non-null curve α.

Case I.2. If $\dot{\alpha}_1\alpha_2 - \alpha_1\dot{\alpha}_2 = 0$ and $\dot{\alpha}_2\alpha_3 - \alpha_2\dot{\alpha}_3 = 0$, it follows that α is a straight line in $\mathbb{E}^3_1$ passing through the origin, and relation (11) is satisfied for any non-null curve β. Thus we proved statement (i) in theorem.

Case II. Assume that relation (12) holds. Since $g_{12} = g_1(\alpha,\dot{\alpha})g_2(\beta,\dot{\beta}) = 0$, we distinguish the following two subcases:

Case II.1. If $g_1(\alpha,\dot{\alpha}) = 0$, then $-\alpha_1\dot{\alpha}_1 + \alpha_2\dot{\alpha}_2 + \alpha_3\dot{\alpha}_3 = 0$ and hence $-\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \text{constant} = c$, which means that α lies in pseudosphere $\mathbb{S}^2_1$ if $c > 0$ or α lies in pseudohyperbolical space $\mathbb{H}^2_0$ if $c < 0$. Further, the equation of an arbitrary curve $\beta \in \mathbb{E}^2_1$ can be written as $\beta(s) = \rho(s)(\sinh(s), \cosh(s))$ or $\beta(s) = \rho(s)(\cosh(s), \sinh(s))$, depending on its causal character. Without loss of generality, assume that β has the equation $\beta(s) = \rho(s)(\cosh(s), \sinh(s))$.

Then from the equation $g(g_{22}\dfrac{\partial^2 f}{\partial t^2} + g_{11}\dfrac{\partial^2 f}{\partial s^2}, n_j) = 0, \quad j = 3, 4$, we have

$$
\begin{aligned}
&g_1(\alpha,\alpha)g_2(\dot{\beta},\dot{\beta})(\beta_1\dot{\beta}_2 - \dot{\beta}_1\beta_2)(\ddot{\alpha}_1\dot{\alpha}_2 - \dot{\alpha}_1\ddot{\alpha}_2)\\
&+g_1(\dot{\alpha},\dot{\alpha})g_2(\beta,\beta)(\ddot{\beta}_1\dot{\beta}_2 - \dot{\beta}_1\ddot{\beta}_2)(\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2) = 0\\
&g_1(\alpha,\alpha)g_2(\dot{\beta},\dot{\beta})(\beta_1\dot{\beta}_2 - \dot{\beta}_1\beta_2)(\ddot{\alpha}_2\dot{\alpha}_3 - \dot{\alpha}_2\ddot{\alpha}_3)\\
&+g_1(\dot{\alpha},\dot{\alpha})g_2(\beta,\beta)(\ddot{\beta}_1\dot{\beta}_2 - \dot{\beta}_1\ddot{\beta}_2)(\alpha_2\dot{\alpha}_3 - \dot{\alpha}_2\alpha_3) = 0
\end{aligned} \tag{13}
$$

If α has any constant component, we have distinguish two cases:

Case II.1.1 Assume that $\alpha_1 = \text{constant}$. From (13) we find that $\alpha_1 = 0$. Then α lies in a spacelike plane with the equation $x_1 = 0$. Therefore, $g_1(\alpha,\alpha) > 0$ and $g_1(\dot{\alpha},\dot{\alpha}) > 0$. Now we may assume that

$g_1(\alpha,\alpha) = g_1(\dot\alpha,\dot\alpha) = 1$. It follows that α is the circle with the equation $\alpha(t) = (0, \cos(t), \sin(t))$. Then, we have

$$
\begin{aligned}
&g_{11} = -\rho^2, \quad g_{22} = \rho^2 - \dot\rho^2,\\
&\beta_1\dot\beta_2 - \dot\beta_1\beta_2 = \rho^2,\\
&\ddot\beta_1\dot\beta_2 - \dot\beta_1\ddot\beta_2 = \ddot\rho\rho - 2\dot\rho^2 + \rho^2,\\
&\ddot\alpha_2\dot\alpha_3 - \dot\alpha_2\ddot\alpha_3 = -1,\\
&\alpha_2\dot\alpha_3 - \dot\alpha_2\alpha_3 = 1.
\end{aligned} \tag{14}
$$

Substituting (14) in (13), we obtain

$$\ddot\rho\rho - 3\dot\rho^2 + 2\rho^2 = 0. \tag{15}$$

Therefore,

$$\frac{\dot\rho^2 - \rho\ddot\rho}{\rho^2} = 2(1 - (\frac{\dot\rho}{\rho})^2). \tag{16}$$

Putting $w = \dfrac{\dot\rho}{\rho}$, it follows that $\dot w = \dfrac{\rho\ddot\rho - \dot\rho^2}{\rho^2}$ and the equation (15) becomes

$$\dot w = 2(w^2 - 1). \tag{17}$$

If $w^2 = 1$, then $g_2(\dot\beta,\dot\beta) = \rho^2 - \dot\rho^2 = 0$, which is a contradiction. Thus $w^2 \neq 1$ and we consider two subcases:

Case II.1.1.1. $|w| < 1$. Integrating (17) we get

$$\int \frac{dw}{w^2 - 1} = 2\int ds,$$

and consequently

$$-\tanh^{-1}(w) = 2s + c_1, \quad c_1 \in \mathbb{R}.$$

Next we have

$$-w = -\frac{\dot{\rho}}{\rho} = \tanh(2s + c_1),$$

so after integration we obtain

$$\rho(s) = \frac{c_2}{\sqrt{\cosh(2s + c_1)}}, \quad c_1 \in \mathbb{R},\ c_2 \in \mathbb{R}^+.$$

If we take $c_1 = 0$, it follows that β is the circle centered at the origin, with the equation $\beta(s) = \dfrac{c_2}{\sqrt{\cosh(2s)}}(\cosh(s), \sinh(s))$, $c_2 \in \mathbb{R}^+$.

Case II.1.1.2. $|w| > 1$. In this subcase, integrating (17) we obtain

$$\int \frac{dw}{w^2 - 1} = 2 \int ds,$$

and therefore

$$-\coth^{-1}(w) = 2s + d_1, \quad d_1 \in \mathbb{R}.$$

Next we find

$$-w = -\frac{\dot{\rho}}{\rho} = \coth(2s + d_1).$$

After one more integration, we obtain

$$\rho(s) = \frac{d_2}{\sqrt{|\sinh(2s + d_1)|}}, \quad d_1 \in \mathbb{R},\ d_2 \in \mathbb{R}^+.$$

If we take $d_1 = 0$, it follows that β is the orthogonal hyperbole centered at the origin, given by $\beta(s) = \dfrac{d_2}{\sqrt{|\sinh(2s)|}}(\cosh(s), \sinh(s))$, $d_2 \in \mathbb{R}^+$. Consequently, statement (ii) is proved.

Case II.1.2. Assume that $\alpha_3 =$ constant. From (13) we find that $\alpha_3 = 0$. Then α lies in a timelike plane with the equation $x_3 = 0$. Since $g_1(\alpha, \alpha) = c$, $c > 0$ or $c < 0$, and $g_1(\alpha, \dot{\alpha}) = 0$, it follows that if α is a spacelike then $\dot{\alpha}$ is a timelike or vice versa. Assume that $g_1(\alpha, \alpha) = -1$

and $g_1(\dot{\alpha}, \dot{\alpha}) = 1$. It follows that $\alpha(t) = (\cosh(t), \sinh(t), 0)$, i.e. α is an orthogonal hyperbola, centered at the origin. Then we have

$$
\begin{aligned}
&g_{11} = -\rho^2, \quad g_{22} = -(\rho^2 - \dot{\rho}^2),\\
&\beta_1\dot{\beta}_2 - \dot{\beta}_1\beta_2 = \rho^2,\\
&\ddot{\beta}_1\dot{\beta}_2 - \dot{\beta}_1\ddot{\beta}_2 = \ddot{\rho}\rho - 2\dot{\rho}^2 + \rho^2,\\
&\ddot{\alpha}_1\dot{\alpha}_2 - \dot{\alpha}_1\ddot{\alpha}_2 = 1,\\
&\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2 = 1.
\end{aligned} \tag{18}
$$

Substituting (18) in (13), we obtain the equation

$$\ddot{\rho}\rho - 3\dot{\rho}^2 + 2\rho^2 = 0. \tag{19}$$

Thus we have the same equations for β as in the cases (II.1.1.1.) and (II.1.1.2.). Consequently, statement (iii) is proved.

Further, if α has no constant components, we distinguish the following two subcases:

Case II.1.3. In this case, we assume that $g_1(\alpha, \alpha) = g_1(\dot{\alpha}, \dot{\alpha}) = \pm 1$. Then relation (13) becomes

$$\frac{\ddot{\rho}\rho + 2\dot{\rho}^2 - \rho^2}{\rho^2 - \dot{\rho}^2} = \frac{\ddot{\alpha}_1\dot{\alpha}_2 - \dot{\alpha}_1\ddot{\alpha}_2}{\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2}, \quad \frac{\ddot{\rho}\rho + 2\dot{\rho}^2 - \rho^2}{\rho^2 - \dot{\rho}^2} = \frac{\ddot{\alpha}_1\dot{\alpha}_3 - \dot{\alpha}_1\ddot{\alpha}_3}{\alpha_1\dot{\alpha}_3 - \dot{\alpha}_1\alpha_3}.$$

Since the left side terms are functions of the parameter s and the right side terms are functions of the parameter t, it follows that

$$\frac{\ddot{\rho}\rho - 2\dot{\rho}^2 + \rho^2}{\rho^2 - \dot{\rho}^2} = \frac{\ddot{\alpha}_1\dot{\alpha}_2 - \dot{\alpha}_1\ddot{\alpha}_2}{\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2} = \frac{\ddot{\alpha}_1\dot{\alpha}_3 - \dot{\alpha}_1\ddot{\alpha}_3}{\alpha_1\dot{\alpha}_3 - \dot{\alpha}_1\alpha_3} = k, \quad k \in \mathbb{R}^+. \tag{20}$$

From (20) we get

$$\ddot{\rho}\rho + 2\dot{\rho}^2 - \rho^2 \quad = \quad k(\rho^2 - \dot{\rho}^2),$$

$$
\begin{aligned}
\dot{\alpha}_2(\ddot{\alpha}_1 - k\alpha_1) &= \dot{\alpha}_1(\ddot{\alpha}_2 - k\alpha_2), \\
\dot{\alpha}_3(\ddot{\alpha}_2 - k\alpha_2) &= \dot{\alpha}_2(\ddot{\alpha}_3 - k\alpha_3),
\end{aligned}
$$

and consequently

$$
\frac{\ddot{\alpha}_1 - k\alpha_1}{\dot{\alpha}_1} = \frac{\ddot{\alpha}_2 - k\alpha_2}{\dot{\alpha}_2} = \frac{\ddot{\alpha}_3 - k\alpha_3}{\dot{\alpha}_3} = m(t), \tag{21}
$$

where $m = m(t)$ is some differentiable function of t. Next (21) implies that

$$
\begin{aligned}
\ddot{\alpha}_1 &= k\alpha_1 + m\dot{\alpha}_1, \\
\ddot{\alpha}_2 &= k\alpha_2 + m\dot{\alpha}_2, \\
\ddot{\alpha}_3 &= k\alpha_3 + m\dot{\alpha}_3.
\end{aligned} \tag{22}
$$

Taking the derivative with respect to t of the equation $g_1(\dot{\alpha}, \dot{\alpha}) = \pm 1$, we find $-\dot{\alpha}_1\ddot{\alpha}_1 + \dot{\alpha}_2\ddot{\alpha}_2 + \dot{\alpha}_3\ddot{\alpha}_3 = 0$. Substituting (22) in the previous equation, we get $m = m(t) = 0$, so (22) becomes

$$
\begin{aligned}
\ddot{\alpha}_1 &= k\alpha_1, \\
\ddot{\alpha}_2 &= k\alpha_2, \\
\ddot{\alpha}_3 &= k\alpha_3.
\end{aligned} \tag{23}
$$

Then $k = -l^2$, $l \in \mathbb{R}^+$. Thus the solutions of the differential equations (23) are given by

$$
\begin{aligned}
\alpha_1(t) = A_1 e^{lt} + B_1 e^{-lt}, \\
\alpha_2(t) = A_2 e^{lt} + B_2 e^{-lt}, \\
\alpha_3(t) = A_3 e^{lt} + B_3 e^{-lt}.
\end{aligned}
$$

Let $A_i + B_i = C_i$ and $A_i - B_i = D_i$, $i = 1, 2, 3$. Then equation of α reads $\alpha(t) = C\cosh(lt) + D\sinh(lt)$, so α lies in the plane spanned by $\{C, D\}$. Consequently, after some coordinate transformations, it follows that α is an orthogonal hyperbola, centered at the origin. Knowing that $g_1(\alpha,\alpha)=\pm1$ and using the linear independence of functions $\sinh(x)$ and $\cosh(x)$, we find $g_1(C,C)=-g_1(D,D)=1, g_1(C,D)=0$. Moreover from $g_1(\dot\alpha,\dot\alpha)=1$, we get $l^2=1$. Next, substituting $k=-l^2=-1$ in (20) gives equation (15). Then subcases (II.1.1.) and (II.1.2.) imply that β is either a circle or an orthogonal hyperbola.

Case II.1.4. In this case, we assume that $g_1(\alpha,\alpha)=-g_1(\dot\alpha,\dot\alpha)=\pm1$. Then relation (13) becomes

$$\frac{-\ddot\rho\rho+2\dot\rho^2-\rho^2}{\rho^2-\dot\rho^2}=\frac{\ddot\alpha_1\dot\alpha_2-\dot\alpha_1\ddot\alpha_2}{\dot\alpha_1\alpha_2-\alpha_1\dot\alpha_2},\quad \frac{-\ddot\rho\rho+2\dot\rho^2-\rho^2}{\rho^2-\dot\rho^2}=\frac{\ddot\alpha_2\dot\alpha_3-\dot\alpha_2\ddot\alpha_3}{\alpha_3\dot\alpha_2-\dot\alpha_3\alpha_2}.$$

It follows that

$$\frac{-\ddot\rho\rho+2\dot\rho^2-\rho^2}{\rho^2-\dot\rho^2}=\frac{\ddot\alpha_1\dot\alpha_2-\dot\alpha_1\ddot\alpha_2}{\alpha_2\dot\alpha_1-\dot\alpha_2\alpha_1}=\frac{\ddot\alpha_2\dot\alpha_3-\dot\alpha_2\ddot\alpha_3}{\alpha_3\dot\alpha_2-\dot\alpha_3\alpha_2}=k,\quad k\in\mathbb{R}^+. \tag{24}$$

From relation (20) we obtain

$$\begin{aligned}-\ddot\rho\rho+2\dot\rho^2-\rho^2 &= k(\rho^2-\dot\rho^2),\\ \dot\alpha_2(\ddot\alpha_1+k\alpha_1) &= \dot\alpha_1(\ddot\alpha_2+k\alpha_2),\\ \dot\alpha_3(\ddot\alpha_2+k\alpha_2) &= \dot\alpha_2(\ddot\alpha_3+k\alpha_3),\end{aligned}$$

and hence

$$\frac{\ddot\alpha_1+k\alpha_1}{\dot\alpha_1}=\frac{\ddot\alpha_2+k\alpha_2}{\dot\alpha_2}=\frac{\ddot\alpha_3+k\alpha_3}{\dot\alpha_3}=m(t), \tag{25}$$

where $m=m(t)$ is some differentiable function of t. Next (25) implies that

$$\begin{aligned}\ddot\alpha_1&=m\dot\alpha_1-k\alpha_1,\\ \ddot\alpha_2&=m\dot\alpha_2-k\alpha_2,\\ \ddot\alpha_3&=m\dot\alpha_3-k\alpha_3.\end{aligned} \tag{26}$$

Taking the derivative with respect to t of the equation $g_1(\dot\alpha,\dot\alpha)=\mp1$, we find $-\dot{\alpha}_1\ddot{\alpha}_1+\dot{\alpha}_2\ddot{\alpha}_2+\dot{\alpha}_3\ddot{\alpha}_3=0$. Substituting (26) in the previous equation, we get $m=m(t)=0$, so (26) becomes

$$\begin{aligned}\ddot{\alpha}_1&=-k\alpha_1,\\ \ddot{\alpha}_2&=-k\alpha_2,\\ \ddot{\alpha}_3&=-k\alpha_3.\end{aligned}\tag{27}$$

Then $k=l^2,\quad l\in\mathbb{R}^+$. Thus the solutions of the differential equations (27) are given by

$$\begin{aligned}\alpha_1(t)&=A_1\cos(lt)+B_1\sin(lt),\\ \alpha_2(t)&=A_2\cos(lt)+B_2\sin(lt),\\ \alpha_3(t)&=A_3\cos(lt)+B_3\sin(lt).\end{aligned}$$

Let $A=(A_1,A_2,A_3)$, $B=(B_1,B_2,B_3)$. Then equation of α reads $\alpha(t)=A\cos(lt)+B\sin(lt)$, so α lies in the plane spanned by $\{A,B\}$. After some coordinate transformations, it follows that α is a circle, centered at the origin. Knowing that $g_1(\alpha,\alpha)=\pm 1$ and using the linear independence of functions $\sin(x)$ and $\cos(x)$, we find $g_1(A,A)=g_1(B,B)=1, g_1(A,B)=0$. Moreover from $g_1(\dot{\alpha},\dot{\alpha})=1$ we get $l^2=1$. Next, substituting $k=l^2=1$ in (24) gives equation (15). Finally, subcases (II.1.1.) and (II.1.2.) imply that β is either the circle or an orthogonal hyperbola, which proves statement (iv).

Case II.2 If $g_2(\beta,\dot{\beta})=0$, then $-\beta_1\dot{\beta}_1+\beta_2\dot{\beta}_2=0$ and therefore β is the orthogonal hyperbola, centered at the origin, given by $\beta(s)=(a\sinh(s),a\cosh(s))$, or else by $\beta(s)=(a\cosh(s),a\sinh(s)),\quad a\in\mathbb{R}^+$. We may assume that β has the equation of the form $\beta(s)=(a\cosh(s),a\sinh(s))$, $a\in\mathbb{R}^+$. Then we have

$$\begin{aligned}&g_{11}=-a^2g_1(\dot{\alpha},\dot{\alpha}),\quad g_{22}=a^2g_1(\alpha,\alpha),\\ &\beta_1\dot{\beta}_2-\dot{\beta}_1\beta_2=\ddot{\beta}_1\dot{\beta}_2-\dot{\beta}_1\ddot{\beta}_2=a^2.\end{aligned}\tag{28}$$

Substituting (28) in (13), we obtain

$$\begin{aligned} &g_1(\alpha,\alpha)(\ddot{\alpha}_1\dot{\alpha}_2-\dot{\alpha}_1\ddot{\alpha}_2)-g_1(\dot{\alpha},\dot{\alpha})(\alpha_1\dot{\alpha}_2-\dot{\alpha}_1\alpha_2)=0,\\ &g_1(\alpha,\alpha)(\ddot{\alpha}_2\dot{\alpha}_3-\dot{\alpha}_2\ddot{\alpha}_3)-g_1(\dot{\alpha},\dot{\alpha})(\alpha_2\dot{\alpha}_3-\dot{\alpha}_2\alpha_3)=0. \end{aligned} \tag{29}$$

If α has any constant component,we distinguish the following two cases:

Case II.2.1. Assume that $\alpha_1(t)=$ constant. Then relation (29) implies $\alpha_1(t)=0$. Thus α can be parameterized by $\alpha(t)=(0,\rho(t)\sin(t),\rho(t)\cos(t))$, so the second equation in (29) becomes

$$\rho\ddot{\rho}-3\dot{\rho}^2-2\rho^2=0.$$

Putting $w=\frac{\dot{\rho}}{\rho}$, it follows that $\dot{w}=\frac{\rho\ddot{\rho}-\dot{\rho}^2}{\rho^2}$, and the previous differential equation becomes

$$\dot{w}=2(1+w^2).$$

The solution of this differential equation is

$$w=\tan(2t+c_1),\ \ c_1\in\mathbb{R}.$$

Moreover, after one more integration we find that

$$\rho(s)=\frac{c_2}{\sqrt{|\cos(2t+c_1)|}},\quad c_1\in\mathbb{R},\ c_2\in\mathbb{R}^+.$$

Taking $c_1=0$, we obtain that α is an orthogonal hyperbola given by

$$\alpha(t)=\frac{c_2}{\sqrt{|\cos(2t)|}}(0,\sin(t),\cos(t)),\ \ c_2\in\mathbb{R}^+.$$

Case II.2.2. Assume that $\alpha_3(t)=$ constant. Then relation (29) implies $\alpha_3(t)=0$. Thus α can be parameterized by $\alpha(t)=(\rho(t)\cosh(t),\rho(t)\sinh(t),0)$. The first equation in (29) becomes

$$\rho\ddot{\rho} - 3\dot{\rho}^2 + 2\rho^2 = 0.$$

We obtain the same differential equation as in (II.1.1.) and (II.1.2.). It follows that α is the circle with the equation

$$\alpha(t) = \frac{c_2}{\sqrt{\cosh(2t)}}(\cosh(t), \sinh(t), 0), \ \ c_2 \in \mathbb{R}^+.$$

or else an orthogonal hyperbola with the equation

$$\alpha(t) = \frac{d_2}{\sqrt{|\sinh(2t)|}}(\cosh(t), \sinh(t), 0), \ \ d_2 \in \mathbb{R}^+.$$

By cases (II.2.1.) and (II.2.2.), we proved statement (v).

If α has no constant components, we distingish the following two cases:

Case II.2.3. In this case, we may choose parameter t, such that $g_1(\alpha, \alpha) = g_1(\dot{\alpha}, \dot{\alpha})$. Then relation (29) turn out to be

$$\begin{aligned} &\dot{\alpha}_2(\ddot{\alpha}_1 - \alpha_1) - \dot{\alpha}_1(\ddot{\alpha}_2 - \alpha_2) = 0, \\ &\dot{\alpha}_3(\ddot{\alpha}_1 - \alpha_1) - \dot{\alpha}_1(\ddot{\alpha}_3 - \alpha_3) = 0, \end{aligned}$$

and therefore

$$\frac{\ddot{\alpha}_1 - \alpha_1}{\dot{\alpha}_1} = \frac{\ddot{\alpha}_2 - \alpha_2}{\dot{\alpha}_2} = \frac{\ddot{\alpha}_3 - \alpha_3}{\dot{\alpha}_3}. \tag{30}$$

Differentiating the equation $g_1(\alpha, \alpha) = g_1(\dot{\alpha}, \dot{\alpha})$ with respect to t, gives

$$\dot{\alpha}_1(\ddot{\alpha}_1 - \alpha_1) + \dot{\alpha}_2(\ddot{\alpha}_2 - \alpha_2) + \dot{\alpha}_3(\ddot{\alpha}_3 - \alpha_3) = 0. \tag{31}$$

Then (30) and (31) imply that

$$\begin{aligned} \ddot{\alpha}_1 &= \alpha_1, \\ \ddot{\alpha}_2 &= \alpha_2, \\ \ddot{\alpha}_3 &= \alpha_3. \end{aligned} \tag{32}$$

The solutions of differential equations (32) are

$$\alpha_1(t) = A_1 e^t + B_1 e^{-t},$$
$$\alpha_2(t) = A_2 e^t + B_2 e^{-t},$$
$$\alpha_3(t) = A_3 e^t + B_3 e^{-t},$$

where $A_i, B_i \in \mathbb{R}$, $i = 1, 2, 3$, are integration constants or equivalently

$$\alpha_1(t) = C_1 \cosh(t) + D_1 \sinh(t),$$
$$\alpha_2(t) = C_2 \cosh(t) + D_2 \sinh(t),$$
$$\alpha_3(t) = C_3 \cosh(t) + D_3 \sinh(t),$$

where $C_i = A_i + B_i$, $D_i = A_i - B_i$, $i = 1, 2, 3$. Let us put $C = (C_1, C_3, C_3)$ and $D = (D_1, D_3, D_3)$. Hence the equation of α reads

$$\alpha(t) = C \cosh(t) + D \sinh(t),$$

so α is an orthogonal hyperbola, centered at the origin and lying in the plane spanned by $\{C, D\}$. Finally, using the equation $g_1(\alpha, \alpha) = g_1(\dot{\alpha}, \dot{\alpha})$, we get $||C|| = ||D||$.

Case II.2.4. In this case, we may choose parameter t, such that $g_1(\alpha, \alpha) = -g_1(\dot{\alpha}, \dot{\alpha})$. Then relation (29) turn out to be

$$\dot{\alpha}_2(\ddot{\alpha}_1 + \alpha_1) - \dot{\alpha}_1(\ddot{\alpha}_2 + \alpha_2) = 0,$$
$$\dot{\alpha}_3(\ddot{\alpha}_1 + \alpha_1) - \dot{\alpha}_1(\ddot{\alpha}_3 + \alpha_3) = 0,$$

and therefore

$$\frac{\ddot{\alpha}_1 + \alpha_1}{\dot{\alpha}_1} = \frac{\ddot{\alpha}_2 + \alpha_2}{\dot{\alpha}_2} = \frac{\ddot{\alpha}_3 + \alpha_3}{\dot{\alpha}_3}. \tag{33}$$

Differentiating the equation $g_1(\alpha, \alpha) = -g_1(\dot{\alpha}, \dot{\alpha})$ with respect to t, gives

$$\dot{\alpha}_1(-\ddot{\alpha}_1 - \alpha_1) + \dot{\alpha}_2(\ddot{\alpha}_2 + \alpha_2) + \dot{\alpha}_3(\ddot{\alpha}_3 + \alpha_3) = 0. \tag{34}$$

Then (33) and (34) imply that

$$\begin{aligned} \ddot{\alpha}_1 &= -\alpha_1, \\ \ddot{\alpha}_2 &= -\alpha_2, \\ \ddot{\alpha}_3 &= -\alpha_3. \end{aligned} \tag{35}$$

The solutions of differential equations (35) are

$$\begin{aligned} \alpha_1(t) &= A_1\cos(t) + B_1\sin(t), \\ \alpha_2(t) &= A_2\cos(t) + B_2\sin(t), \\ \alpha_3(t) &= A_3\cos(t) + B_3\sin(t), \end{aligned}$$

where $A_i, B_i \in \mathbb{R}$, $i = 1, 2, 3.$ are integration constants. Let us put $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$. Hence the equation of α reads

$$\alpha(t) = A\cos(t) + B\sin(t),$$

so α is an circle centered at the origin and lying in the plane spanned by $\{A, B\}$. Finally, using the equation $g_1(\alpha, \alpha) = -g_1(\dot{\alpha}, \dot{\alpha})$, we get $||A|| = ||B||$. Therefore, cases (II.2.3.) and (II.2.4.) imply statement (vi).

Remark 1. If $\mathrm{Im}(\alpha \otimes \beta)$ is a minimal surface in $\mathbb{E}^6_3$, then both curves α and β are plane curves.

In the next two theorems, we classify totally real and complex tensor product surfaces in the semi-Euclidean space $\mathbb{E}^6_3$.

Theorem 2. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian space curve $\alpha : \mathbb{R} \to \mathbb{E}^3_1$ and a Lorentzian plane curve $\beta : \mathbb{R} \to \mathbb{E}^2_1$, is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure J_0, $J_0(p, q, u, v, z, w) = (-q, p, -v, u, -w, z)$, on $\mathbb{E}^6_3$ if and only if*

(i) *α lies in a pseudosphere $\mathbb{S}^2_1$ or in a pseudohyperbolical space $\mathbb{H}^2_0$;*

(ii) *β is an orthogonal hyperbola centered at the origin.*

Proof. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a Lorentzian space curve and $\beta(s) = (\beta_1(s), \beta_2(s))$ be a Lorentzian plane curve. Assume that the tensor product $f = \alpha \otimes \beta$ defined by (1) is totally real. Then the following two conditions hold.

$$
\begin{aligned}
&g(J_0(\tfrac{\partial f}{\partial t}), \tfrac{\partial f}{\partial s}) = 0,\\
&g(J_0(\tfrac{\partial f}{\partial s}), \tfrac{\partial f}{\partial t}) = 0.
\end{aligned} \tag{36}
$$

From (1) we get

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= (\dot\alpha_1\beta_1, \dot\alpha_1\beta_2, \dot\alpha_2\beta_1, \dot\alpha_2\beta_2, \dot\alpha_3\beta_1, \dot\alpha_3\beta_2),\\
\frac{\partial f}{\partial s} &= (\alpha_1\dot\beta_1, \alpha_1\dot\beta_2, \alpha_2\dot\beta_1, \alpha_2\dot\beta_2, \alpha_3\dot\beta_1, \alpha_3\dot\beta_2),
\end{aligned}
$$

Hence the condition (36) turn out to be

$$
\begin{aligned}
&g((-\dot\alpha_1\beta_2, \dot\alpha_1\beta_1, -\dot\alpha_2\beta_2, \dot\alpha_2\beta_1, -\dot\alpha_3\beta_2, \dot\alpha_3\beta_1),\\
&\qquad (\alpha_1\dot\beta_1, \alpha_1\dot\beta_2, \alpha_2\dot\beta_1, \alpha_2\dot\beta_2, \alpha_3\dot\beta_1, \alpha_3\dot\beta_2)) = 0,\\
&g((-\alpha_1\dot\beta_2, \alpha_1\dot\beta_1, -\alpha_2\dot\beta_2, \alpha_2\dot\beta_1, -\alpha_3\dot\beta_2, \alpha_3\dot\beta_1),\\
&\qquad (\dot\alpha_1\beta_1, \dot\alpha_1\beta_2, \dot\alpha_2\beta_1, \alpha_2\dot\beta_2, \dot\alpha_3\beta_1, \dot\alpha_3\beta_2)) = 0.
\end{aligned} \tag{37}
$$

Further, both equation in (37) are equivalent with the equation

$$
(\dot\beta_1\beta_2 + \beta_1\dot\beta_2)(-\alpha_1\dot\alpha_1 + \alpha_2\dot\alpha_2 + \alpha_3\dot\alpha_3) = 0, \tag{38}
$$

so we have distinguish two possibilities. If $-\alpha_1\dot\alpha_1 + \alpha_2\dot\alpha_2 + \alpha_3\dot\alpha_3 = 0$, then $-\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \text{constant} = c, \quad c \in \mathbb{R}$, which means that α lies in a pseudosphere if constant $= c > 0$ or α lies in a pseudohyperbolical space if constant $= c < 0$. If $\dot\beta_1\beta_2 + \beta_1\dot\beta_2 = 0$, then $\beta_1\beta_2 = \text{constant}$. Consequently, $\beta(s)$ is an orthogonal hyperbola centered at the origin with the equation

$\beta(s) = \frac{a}{\sqrt{|\sinh(2s)|}}(\cosh(s), \sinh(s))$, or $\beta(s) = \frac{a}{\sqrt{|\sinh(2s)|}}(\sinh(s), \cosh(s))$, $a \in \mathbb{R}^+$. This completes the proof of theorem.

Theorem 3. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian space curve $\alpha : \mathbb{R} \to \mathbb{E}^3_1$ and a Lorentzian plane curve $\beta : \mathbb{R} \to \mathbb{E}^2_1$ is a complex Lorentzian immersion with respect to the pseudo-Hermitian structure J_0 given by $J_0(p, q, u, v, z, w) = (-q, p, -v, u, -w, z)$, on $\mathbb{E}^6_3$ if and only if α is a straight line through the origin.*

Proof. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a Lorentzian space curve and $\beta(s) = (\beta_1(s), \beta_2(s))$ be a Lorentzian plane curve. Assume that the tensor product $f = \alpha \otimes \beta$ defined by (1) is a complex immersion. Then we have

$$g(J_0(\frac{\partial f}{\partial t}), n_i) = g(J_0(\frac{\partial f}{\partial s}), n_i) = 0, \quad i = 1, 2, 3, 4. \tag{39}$$

Where $\{n_1, n_2, n_3, n_4\}$ given by (2) is a basis of the normal space. Therefore by straightforward computations the equations (39) can be reduced to

$$(\beta_1^2 + \beta_2^2)(\dot{\alpha}_2\alpha_1 - \dot{\alpha}_1\alpha_2) = 0$$
$$(\beta_1^2 + \beta_2^2)(\alpha_2\dot{\alpha}_3 - \alpha_3\dot{\alpha}_2) = 0.$$

It follows that $\dot{\alpha}_2\alpha_1 - \dot{\alpha}_1\alpha_2 = 0$ and $\dot{\alpha}_3\alpha_2 - \alpha_3\dot{\alpha}_2 = 0$, which means that α is a straight line through the origin.

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