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SOLVABILITY OF MULTI-POINT BOUNDARY VALUE PROBLEMS FOR 2n-ORDER ORDINARY DIFFERENTIAL EQUATIONS AT RESONANCE(II)

BY

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Abstract. In this paper, we prove existence results for solutions of multi-point boundary value problems at resonance (Theorems 2.1-2.2) and for positive solutions at non-resonance (Theorems 2.3) for a 2n-th order differential equation. Our method is based upon the coincidence degree theory of Mawhin. The interest is that the degree of some variables among $x_0, x_1, \ldots, x_{2n-1}$ in the function $f(t, x_0, x_1, \ldots, x_{2n-1})$ is allowable to be greater than 1. The results obtained are new.

1. Introduction. In this paper, we investigate the existence of solutions and positive solutions of the multi-point boundary value problem for 2n-th order differential equations

$$(-1)^{n-1}x^{(2n)} = f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)), \quad t \in (0,1), \tag{1}$$

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subject to the boundary value conditions

$$(2)\qquad \begin{cases} x^{(2i-1)}(0)=0,\ i=1,\dots,n,\\ x^{(2i-1)}(1)=\sum_{k=1}^{p_i}\alpha_{i,k}x^{(2i-1)}(\xi_{i,k}),\ i=1,\dots,n-1,\\ x(1)=\sum_{i=1}^{m}\beta_i x(\xi_i), \end{cases}$$

where $f:[0,1]\times R^{2n}\to R$ is a continuous function, $n\ge 1$, $p_i\ge 1$ for $i=1,\dots,n-1$, and $m\ge 1$ are integers, $0<\xi_{i,1}<\cdots<\xi_{i,p_i}<1$ for $i=1,\dots,n-1$, $0<\xi_1<\cdots<\xi_m<1$ and $\alpha_{i,k}\in R$ for $i=1,\dots,n-1$ and $k=1,\dots,p_i$, $\beta_i\in R$ for $i=1,\dots,m$. Our purpose here is to provide sufficient conditions for the existence of solutions of boundary value problem (1) and (2) at resonance and positive solutions at non-resonance. These will be done by applying the well known coincidence degree theory and Schauder fixed point theorem.

The motivation for this paper is as follows. First, there were many papers concerned with the solvability of the second-order differential equations

$$(3)\qquad x''(t)+f(t,x(t),x'(t))=0,\ t\in(0,1),$$

subject to two-point boundary conditions

$$\alpha x(0)-\beta x'(0)=\delta x(1)+\gamma x'(1)=0$$

or the different multi-point boundary conditions at resonance or at non-resonance, we refer the readers to [1-8] and the references therein. For example, in [6], Liu and Yu studied the solvability of the boundary value problems for second order differential equation

$$(4)\qquad \begin{aligned} x''(t) &= f(t,x(t),x'(t))+e(t),\ t\in(0,1),\\ x'(0) &= 0,\ x(1)=\sum_{i=1}^{m}\alpha_i x(\xi_i), \end{aligned}$$

where $\sum_{i=1}^{m}\alpha_i = 1$, which shows that such a problem is a resonance problem. They proved that under some assumptions it has at least one solution. One of the main assumptions is as follows:

$$(*) \qquad |f(t,x,y)| \le a(t)|x| + b(t)|y| + p(t)|x|^{\delta} + q(t)|y|^{\theta} + r(t),$$

where a, b, p, q are non-negative continuous functions and r is a continuous function. To the best of our knowledge, the existence of solutions of *multi-point boundary value problems at resonance for higher order differential equations* were not investigated till now. The question is that under what conditions above problems have solutions if $(*)$ is not valid and under what conditions BVP(4) has positive solutions?

Second, the solvability of fourth-order differential equations

$$(5) \qquad x^{(4)}(t) = f(t, x(t), -x''(t)),\ t \in (0,1),$$

or

$$(**) \qquad x^{(4)}(t) = f(t, x(t)),\ t \in (0,1),$$

subject to different boundary conditions have been studied by many authors, please see [16-21]. However, the solvability problems of equations (5) or (6) subject to following boundary value conditions

$$x(1) = x'(0) = x'(1) = x'''(0) = 0,$$

has not been studied.

Third, very recently, Chyan and Henderson, in [14], studied the following 2mth-order differential equation

$$(6) \qquad x^{(2m)}(t) = f(t, x(t), x''(t), \ldots, x^{(2m-2)}(t)),\ \ 0 < t < 1,$$

with either the Lidstone boundary value condition

$$x^{(2i)}(0) = x^{(2i)}(1) = 0 \;\; for \; i = 0, 1, \ldots, m-1, \tag{7}$$

or the focal boundary value condition

$$x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \;\; for \; i = 0, 1, \ldots, m-1. \tag{8}$$

They proved the existence of at least one positive solution in the case either f is super-linear or f is sub-linear.

The similar problems were also investigated in [15] by Palamides by using an analysis of the corresponding field on the face-plane and the well known Sperner's Lemma. The method there is different from that in [10-14]. In the papers mentioned above, the nonlinearity f depends on $x, x'', \ldots, x^{(2(m-1))}$.

For BVP(1) and (2), the corresponding linear differential equation is

$$(-1)^{n-1} x^{(2n)} = 0, \quad t \in (0, 1). \tag{9}$$

It is easy to know that equation (10) subject to boundary conditions (2) has nontrivial solutions $x(t) = c$ if $\sum_{i=1}^{m} \beta_i = 1$, where $c \in R$. As usual, we say that BVP(1) and (2) is a resonance problem. The problem appears naturally considering this boundary value problem:

(P). Under what conditions problem (1) and (2) has at least one positive solution?

In this paper, we will solve above problems, please see Theorems 2.1-2.3. The results obtained are new.

2. Main results In this section, we establish sufficient conditions for the existence of at least one solution of BVP(1)-(2) and one positive solution of BVP(1) and (2). respectively. For convenience, we first introduce some notations and an abstract existence theorem by Gaines and Mawhin [9].

Let X and Y be Banach spaces, L: dom $L \subset X \to Y$ be a Fredholm operator of index zero, $P: X \to X$, $Q: Y \to Y$ be projectors such that

$$\text{Im } P = \text{Ker } L, \ \text{Ker } Q = \text{Im } L, \ \ X = \text{Ker } L \oplus \text{Ker } P, \ \ Y = \text{Im } L \oplus \text{Im } Q.$$

It follows that

$$L|_{\text{dom } L \cap \text{Ker } P}: \ \text{dom } L \cap \text{Ker } P \to \text{Im } L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, dom $L \cap \overline{\Omega} \neq \varnothing$, the map $N: X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \overline{\Omega} \to X$ is compact.

Theorem GM[9]. *Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(domL/\text{Ker}L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin \text{Im}L$ *for every* $x \in \text{Ker}L \cap \partial\Omega$*;*

(iii) *deg*$(\Lambda QN|_{\text{Ker}L}, \ \Omega \cap \text{Ker}L, 0) \neq 0$*, where* $\Lambda: \ Y/\text{Im}L \to \text{Ker}L$ *is the isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $domL \cap \overline{\Omega}$*.*

We use the classical Banach space $C^k[0,1]$, let $X = C^{2n-1}[0,1]$ and $Y = C^0[0,1]$. Y is endowed with the norm $||y||_\infty = \max_{t\in[0,1]}|y(t)|$, X is endowed with the norm $||x|| = \max\{||x||_\infty, ||x'||_\infty, \ldots, ||x^{(2n-1)}||_\infty\}$. Define the linear operator L and the nonlinear operator N by

$$L: \ X \cap \text{dom}L \to Y, \qquad Lx(t) = (-1)^{n-1}x^{(2n)}(t) \ for \ x \in X \cap \text{dom}L,$$

$$N: \ X \to Y, \qquad Nx(t) = f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)), \ for \ x \in X,$$

respectively, where

$$\begin{aligned}\mathrm{dom}L &= \Big\{x\in C^{n-1}[0,1],\ x^{(2i-1)}(0)=0 \text{ for } i=1,\ldots,n\\ &\quad x^{(2i-1)}(1)=\sum_{k=1}^{p_i}\alpha_{i,k}x^{(2i-1)}(\xi_{i,k}), \text{for } i=1,\ldots,n-1,\\ &\quad x(1)=\sum_{i=1}^{m}\beta_i x(\xi_i)\Big\}.\end{aligned}$$

Suppose $\sum_{k=1}^{p_i}\alpha_{i,k}\xi_{i,k}\neq 1$ for $i=1,\ldots,n-1$. Let, for $i=1,\ldots,n-1$, $G_{i-1}(t,s)$ be the Green's function of problem

$$-u''(t)=\alpha(t),\ \ u(0)=u(1)-\sum_{k=1}^{p_i}\alpha_{i,k}u(\xi_{i,k})=0,$$

for some α. Let

$$G(t,s)=\int_0^1\cdots\int_1^1 G_1(t,\tau_1)\cdots G_{n-1}(\tau_{n-2},s)d\tau_1\cdots d\tau_{n-2}.$$

Lemma 2.1. *For problem* (1) *and* (2), *let* $\sum_{i=1}^m\beta_i=1$. *Assume*

(i) *There is* k_i *so that* $\alpha_{i,k}\geq 0$ *for* $k=1,\ldots,k_i$ *and* $\alpha_{i,k}\leq 0$ *for* $k=k_i+1,\ldots,p_i$ *with* $\sum_{k=1}^{p_i}\alpha_{i,k}<1$;

(ii) *There is nonnegative integer* l *such that*

$$\Delta=\int_0^1\int_0^1 G(s,\tau)\int_0^\tau u^l du d\tau ds-\sum_{i=1}^m\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau u^l dud\tau ds\neq 0.$$

Then the following results hold.

(i) $\mathrm{Ker}L=\{x(t)\equiv c,\ t\in[0,1],\ c\in R\}$;

(ii) $\mathrm{Im}L=\left\{\begin{array}{ll} y\in Y, & \int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds\\ & \quad =\sum_{i=1}^m\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds\end{array}\right\}$;

(iii) *L is a Fredholm operator of index zero;*

(iv) *There are projectors* $P:\ X\to X$ *and* $Q:\ Y\to Y$ *such that*

$\text{Ker}L = \text{Im}P$ *and* $\text{Ker}Q = \text{Im}L$. *Furthermore, let* $\Omega \subset X$ *be an open bounded subset with* $\overline{\Omega} \cap \text{dom}L \neq \emptyset$, *then* N *is* $L-$*compact on* $\overline{\Omega}$*;*

(v) $x(t)$ *is a solution of* $BVP(1)$ *and* (2) *if and only if* x *is a solution of the operator equation* $Lx = Nx$ *in* $\text{dom}L$.

Proof. (i) The proof is easy and is omitted.

(ii) If $y \in \text{Im}L$, then

$$(10)\qquad \begin{aligned} (-1)^{n-1}x^{(2n)} &= y(t),\ t \in (0,1), \\ x^{(2i-1)}(0) &= x^{(2i-1)}(1) - \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}(\xi_{i,k}) = 0,\ i=1,\dots,n-1, \\ x^{(2n-1)}(0) &= 0,\ x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i). \end{aligned}$$

This implies $x^{(2n-1)}(t) = (-1)^{n-1}\int_0^t y(u)du$ since $x^{(2n-1)}(0) = 0$. We get

$$x^{(2n-3)}(t) = (-1)^{n-2}\int_0^1 G_{n-1}(t,\tau)\int_0^\tau y(u)dud\tau,$$

Similarly, we get

$$x'(t) = \int_0^1 G(t,\tau)\int_0^\tau y(u)dud\tau.$$

So

$$(11)\qquad x(t) = c + \int_0^t \int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds.$$

It follows from $x(1) = \sum_{i=1}^m \beta_i x(\xi_i)$ that

$$(12)\quad \int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds = \sum_{i=1}^m \beta_i \int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds.$$

On the other hand, assume (13) holds. Let

$$x(t) = c + \int_0^t \int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds.$$

Then $x(t)$ satisfies (11). Hence (ii) is complete.

(iii) From (i), $\dim\mathrm{Ker}L = 1$. On the other hand, for $y \in Y$, let

$$y_0 = y - \frac{t^k}{\Delta}\Big(\int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds\Big).$$

It is easy to check that $y_0 \in \mathrm{Im}L$. Let

$$\overline{R} = \{ct^k : \ t \in [0,1], \ c \in R \}.$$

We get $Y = \overline{R} + \mathrm{Im}L$. It follows from $\overline{R} \cap \mathrm{Im}L = \{0\}$ that $Y = \overline{R} \oplus \mathrm{Im}L$. Hence $\dim Y/\mathrm{Im}L = 1$. On the other hand, f is continuous and $\mathrm{Im}L$ is closed. So L is a Fredholm operator of index zero.

(iv) Define the projectors $P : X \to X$ and $Q : Y \to Y$ by

$$\begin{aligned} Px(t) &= x(0) \text{ for } x \in X, \\ Qy(t) &= \frac{t^k}{\Delta}\Big(\int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds \\ &\quad - \sum_{i=1}^m \beta_i \int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds\Big) \quad \text{for } y \in Y. \end{aligned}$$

It is easy to check that $\mathrm{Ker}L = \mathrm{Im}P$ and $\mathrm{Im}L = \mathrm{Ker}Q$. The generalized inverse $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ of L can be written by

$$K_P y(t) = \int_0^t \int_0^1 G(s,\tau)\int_0^\tau y(u)dud\tau ds.$$

(v) The proof is easy and is omitted.

Theorem 2.1. *Suppose following conditions hold.*

(A_1) *There are a continuous function* $e(t)$ *and nonnegative functions*

$g_i(t,x)$ $(i=0,1,\ldots,2n-1)$ *such that* f *satisfies*

$$|f(t,x_0,x_1,\ldots,x_{2n-1})|\leq e(t)+\sum_{i=0}^{2n-1}g_i(t,x_i),$$

for all $t\in[0,1]$ *and* $(x_0,x_1,\ldots,x_{2n-1})\in R^{2n}$ *and*

$$\lim_{|x|\to\infty}\sup_{t\in[0,1]}\frac{|g_i(t,x)|}{|x|}=r_i,\ \textit{for } i=0,1,\ldots,2n-1$$

with $r_i\geq 0$ *for* $i=0,1,\ldots,2n-1$*;*

(A_2) *There is a constant* $M>0$ *so that for all* $x\in \mathrm{dom}L$*, if* $|x(t)|>M$ *for all* $t\in[0,1]$*, we have*

$$\int_0^1\int_0^1 G(s,\tau)\int_0^\tau f_x(u)dud\tau ds\neq\sum_{i=1}^m\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau f_x(u)dud\tau ds;$$

(A_3) *There is a constant* $M^*>0$ *such that*

$$f(t,c,0,\ldots,0)>0$$

for $t\in[0,1]$ *and* $c>M^*$ *or*

$$f(t,c,0,\ldots,,0)<0$$

for $t\in[0,1]$ *and* $c<-M^*$*;*

(A_4) *There is* k_i *so that* $\alpha_{i,k}\geq 0$ *for* $k=1,\ldots,k_i$ *and* $\alpha_{i,k}\leq 0$ *for* $k=k_i+1,\ldots,p_i$ *with* $\sum_{k=1}^{p_i}\alpha_{i,k}<1$*;*

(A_5) *There is* i_0 *so that* $\beta_i\geq 0$ *for all* $i=1,\ldots,i_0$ *and* $\beta_i<0$ *for all* $i=i_0+1,\ldots,m$ *with* $\sum_{i=1}^m\beta_i=1$.

Then $BVP(1)$ and (2) has at least one solution provided

$$\sum_{i=0}^{2n-1} r_i < \frac{1}{2}. \tag{13}$$

Proof. To apply Theorem GM, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \text{dom}L/\text{Ker}L,\ Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. For $x \in \Omega_1$, we claim that there is $\xi_i \in [0,1]$ such that $x^{(2i)}(\xi_i) = 0$ for $i = 1, 2, \ldots, n-1$. Assume to the contrary that $x^{(2i)}(t) \neq 0$ for all $t \in [0,1]$. Then either $x^{(2i)}(t) > 0$ for all $t \in [0,1]$ or $x^{(2i)}(t) < 0$ for all $t \in [0,1]$. If the first case holds, from $x^{(2i-1)}(\xi_{i,k}) = 0$, we get $x^{(2i-1)}$ is positive and increasing on $[0,1]$. From (A_4), we get

$$\begin{aligned} x^{(2i-1)}(1) &= \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}(\xi_{i,k}) \\ &\leq \sum_{k=1}^{k_i} \alpha_{i,k} x^{(2i-1)}(\xi_{i,k_i}) + \sum_{k_i+1}^{p_i} \alpha_{i,k} x^{(2i-1)}(\xi_{i,k_i}) \\ &= \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}(\xi_{i,k_i}) \\ &< x^{(2i-1)}(\xi_{i,k_i}) < x^{2i-1)}(1), \end{aligned}$$

which is a contradiction. If the second case holds, we can deduce the same contradiction. Hence, we have

$$|x^{(2n-2)}(t)| = \left| x^{(2n-2)}(\xi_{n-1}) + \int_t^{\xi_{n-1}} x^{(2n-1)}(s)ds \right|$$

$$\begin{aligned}
&\leq \int_0^1 |x^{(2n-1)}(s)|ds, \\
|x^{(2n-3)}(t)| &= \left| x^{(2n-3)}(0) + \int_0^t x^{(2n-2)}(s)ds \right| \\
&\leq \int_0^1 |x^{(2n-2)}(s)|ds \leq \int_0^1 |x^{(2n-1)}(s)|ds, \\
&\vdots \qquad \vdots \\
|x'(t)| &\leq \int_0^1 |x^{(2n-1)}(s)|ds.
\end{aligned} \tag{14}$$

It follows from $x \in \Omega_1$ that $\lambda Nx \in \text{Im}L$, so

$$\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u))dud\tau ds = 0.$$

Since (A_2), there is $\xi \in [0,1]$ so that $x(\xi) \leq M$. Hence

$$\begin{aligned}
|x(t)| &\leq |x(\xi)| + \left| \int_\xi^t x'(s)ds \right| \\
&\leq M + \int_0^1 |x'(s)|ds \leq M + \int_0^1 |x^{(2n-1)}(s)|ds.
\end{aligned} \tag{15}$$

It suffices to prove there is a constant $B > 0$ such that

$$||x|| = \max\{||x||_\infty, ||x'||_\infty, \ldots, ||x^{(2n-1)}||_\infty\} \leq B.$$

We divide this step into two Sub-steps.

Sub-step 1.1. We prove that there is a constant $\overline{M} > 0$ such that

$$\int_0^T |x^{(2n-1)}(s)|^2 ds \leq \overline{M}.$$

For $x \in \Omega_1$, we have

$$(-1)^{n-1} x^{(2n)}(t) = \lambda f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)). \tag{16}$$

It is easy to know that there is $\eta \in [0,1]$ such that

$$\int_0^1 |x^{(2n-1)}(s)|^2 ds = |x^{(2n-1)}(\eta)|^2.$$

Multiplying two sides of (16) by $x^{(2n-1)}(t)$ and integrating it from 0 to η, using (A_1), we get

$$\begin{aligned}
& \frac{1}{2}\int_0^1 |x^{(2n-1)}(s)|^2 ds = \frac{1}{2}|x^{(2n-1)}(\eta)|^2 \\
= & \frac{1}{2}|x^{(2n-1)}(\eta)|^2 - \frac{1}{2}|x^{(2n-1)}(0)|^2 \\
= & \lambda \int_0^\eta (-1)^{n-1} f(s, x(s), x'(s), \dots, x^{(2n-1)}(s)) x^{(2n-1)}(s) ds \\
\le & \int_0^\eta |f(s, x(s), x'(s), \dots, x^{(2n-1)}(s))||x^{(2n-1)}(s)| ds \\
\le & \sum_{i=0}^{2n-1} \int_0^1 g_i(s, x^{(i)}(s)) x^{(2n-1)}(s) ds + \int_0^1 |e(s)||x^{(2n-1)}(s)| ds \\
\le & \sum_{i=0}^{2n-1} \int_0^1 |g_i(s, x^{(i)}(s))||x^{(2n-1)}(s)| ds + ||e||_\infty \int_0^1 |x^{(2n-1)}(s)| ds \\
\le & \sum_{i=0}^{2n-1} \int_0^1 |g_i(s, x^{(i)}(s))||x^{(2n-1)}(s)| ds + ||e||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.
\end{aligned}$$

Let $\epsilon > 0$ satisfy

$$\frac{1}{2} > \sum_{i=0}^{2n-1} (r_i + \epsilon).$$

For such a $\epsilon > 0$, we find from (A_1) that there is a constant $\delta > M$ such that for every $i = 0, 1, \dots, 2n-1$,

$$|g_i(t,x)| < (r_i + \epsilon)|x| \text{ uniformly for } t \in [0,1] \text{ and } |x| > \delta.$$

Let, for $i = 0, 1, \dots, 2n-1$,

$$\begin{aligned}
\Delta_{1,i} &= \{t : \ t \in [0,1], \ |x^{(i)}(t)| \le \delta\}, \\
\Delta_{2,i} &= \{t : \ t \in [0,1], \ |x^{(i)}(t)| > \delta\},
\end{aligned}$$

$$g_{\delta,i} = \max_{t\in[0,1],|x|\le\delta} |g_i(t,x)|.$$

Then

$$\begin{aligned}
& \frac{1}{2}\int_0^1 \left|x^{(2n-1)}(s)\right|^2 ds \\
\le\; & \sum_{i=0}^{2n-1}\int_{\Delta_{1,i}} |g_i(s,x^{(i)}(s))||x^{(2n-1)}(s)|ds \\
& + \sum_{i=0}^{2n-1}\int_{\Delta_{2,i}} |g_i(s,x^{(i)}(s)||x^{(2n-1)}(s)|ds + ||e||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\
\le\; & \sum_{i=0}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1}(r_i+\epsilon)\int_0^1 |x^{(i)}(s)||x^{(2n-1)}(s)|ds \\
& + ||e||_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.
\end{aligned}$$

Again using (14) and (15), we get

$$\begin{aligned}
& \int_0^1 |x(s)|ds \le M + \int_0^1 |x^{(2n-1)}(s)|ds, \\
& \int_0^1 |x^{(i)}(s)|ds \le \int_0^1 |x^{(2n-1)}(s)|ds, \\
& \int_0^1 |x(s)||x^{(2n-1)}(s)|ds \le \left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right)\int_0^1 |x^{(2n-1)}(s)|ds, \\
& \int_0^1 |x^{(i)}(s)||x^{(2n-1)}(s)|ds \le \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^2,\ i = 1,\ldots,2n-2.
\end{aligned} \tag{17}$$

So, from

$$\int_0^1 |x^{(2n-1)}(s)|ds \le \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2},$$

we get

$$\begin{aligned}
& \frac{1}{2}\int_0^1 |x^{(2n-1)}(s)|^2 ds \\
\le\; & g_{\delta,0}\left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right) + \sum_{i=1}^{2n-1} g_{\delta,i}\int_0^1 |x^{(2n-1)}(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&+(r_0+\epsilon)\left(M+\int_0^1|x^{(2n-1)}(s)|ds\right)\int_0^1|x^{(2n-1)}(s)|ds\\
&+\sum_{i=1}^{2n-2}(r_i+\epsilon)\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^2+||e||_\infty\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
&+(r_{2n-1}+\epsilon)\int_0^1|x^{(2n-1)}(s)|^2ds\\
\leq\ & g_{\delta,0}\left[M+\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\right]+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
&+(r_0+\epsilon)\left[M\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}+\int_0^1|x^{(2n-1)}(s)|^2ds\right]\\
&+\sum_{i=1}^{2n-1}(r_i+\epsilon)\int_0^1|x^{(2n-1)}(s)|^2ds+||e||_\infty\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}.
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\left(\frac{1}{2}-(r_0+\epsilon)-\sum_{i=1}^{2n-1}(r_i+\epsilon)\right)\int_0^1|x^{(2n-1)}(s)|^2ds\\
\leq\ & g_{\delta,0}\left[M+\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\right]+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
&+(r_0+\epsilon)M\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}+||e||_\infty\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}.
\end{aligned}
$$

From the definition of ϵ, there is a constant $\overline{M}>0$ such that

$$
\int_0^1|x^{(2n-1)}(s)|^2ds\leq\overline{M}.
$$

Sub-step 1.2. Prove there is $B>0$ such that $||x||\leq B$.

From sub-step 1.1, we have

$$
\begin{aligned}
||x||_\infty &\leq M+\int_0^1|x^{(2n-1)}(s)|ds\\
&\leq M+\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
&\leq M+\overline{M}^{1/2}.
\end{aligned}
$$

$$\begin{aligned}
\|x^{(i)}\|_\infty &\le \int_0^1 |x^{(2n-1)}(s)|ds \\
&\le \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\
&\le \overline{M}^{1/2},\ i=1,\dots,n-2.
\end{aligned}$$

Multiplying two sides of (16) by $x^{(2n-1)}(t)$, integrating them from 0 to t, using (A_1), we get

$$\begin{aligned}
&\frac{1}{2}|x^{(2n-1)}(t)|^2 \\
= &\ \lambda \int_0^t (-1)^{n-1} f(s,x(s),x'(s),\dots,x^{(2n-1)}(s))x^{(2n-1)}(s)ds \\
= &\int_0^1 |f(s,x(s),x'(s),\dots,x^{(2n-1)}(s))x^{(2n-1)}(s)||x^{(2n-1)}(s)|ds \\
\le &\sum_{i=0}^{2n-1}\int_0^1 g_i(s,x^{(i)}(s))|x^{(2n-1)}(s)|ds + \int_0^1 |e(s)||x^{(2n-1)}(s)|ds \\
\le &\sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))||x^{(2n-1)}(s)|ds + \|e\|_\infty \int_0^1 |x^{(2n-1)}(s)|ds \\
\le &\sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))||x^{(2n-1)}(s)|ds + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.
\end{aligned}$$

Similarly to Sub-step 1.1, we can get

$$\begin{aligned}
&\frac{1}{2}|x^{(2n-1)}(t)|^2 \\
\le &\sum_{i=0}^{2n-1}\int_{\Delta_{1,i}} |g_i(s,x^{(i)}(s))||x^{(2n-1)}(s)|ds \\
&+\sum_{i=0}^{2n-1}\int_{\Delta_{2,i}} |g_i(s,x^{(i)}(s)||x^{(2n-1)}(s)|ds + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\
\le &\sum_{i=0}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1}(r_i+\epsilon)\int_0^1 |x^{(i)}(s)||x^{(2n-1)}(s)|ds \\
&+\|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.
\end{aligned}$$

Using (17), we get

$$\begin{aligned}
&\frac{1}{2}|x^{(2n-1)}(t)|^2\\
\leq\ & g_{\delta,0}\left(M+\int_0^1|x^{(2n-1)}(s)|ds\right)+\sum_{i=1}^{2n-1}g_{\delta,i}\int_0^1|x^{(2n-1)}(s)|ds\\
&+(r_0+\epsilon)\left(M+\int_0^1|x^{(2n-1)}(s)|ds\right)\int_0^1|x^{(2n-1)}(s)|ds\\
&+\sum_{i=1}^{2n-2}(r_i+\epsilon)\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^2+||e||_\infty\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
&+(r_{2n-1}+\epsilon)\int_0^1|x^{(2n-1)}(s)|^2ds\\
\leq\ & g_{\delta,0}\left[M+\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\right]\\
&+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
&+(r_0+\epsilon)\left[M\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}+\int_0^1|x^{(2n-1)}(s)|^2ds\right]\\
&+\sum_{i=1}^{2n-1}(r_i+\epsilon)\int_0^1|x^{(2n-1)}(s)|^2ds+||e||_\infty\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\\
\leq\ & g_{\delta,0}\left(M+\overline{M}^{1/2}\right)+\sum_{i=1}^{2n-1}g_{\delta,i}\overline{M}^{1/2}\\
&+(r_0+\epsilon)\left[M\overline{M}^{1/2}+\overline{M}\right]+\sum_{i=1}^{2n-1}(r_i+\epsilon)\overline{M}+||e||_\infty\overline{M}^{1/2}.
\end{aligned}$$

So there is $\overline{M}'>0$ such that $|x^{(2n-1)}(t)|\leq\overline{M}'$. Hence $||x^{(2n-1)}||_\infty\leq\overline{M}'$. It follows from above discussion that there is $B>0$ such that

$$||x||\leq B.$$

Hence Ω_1 is bounded. This completes the step 1.

Step 2. Let

$$\Omega_2=\{x\in \mathrm{Ker}L,\ Nx\in \mathrm{Im}L\}.$$

We prove Ω_2 is bounded. Suppose $x \in \Omega_2$, then $x(t) = c \in R$, we prove $|c| \leq M^*$. In fact, if $c > M^*$, then (A_3) implies $f(t, c, 0, \dots, 0) > 0$, then, using (A_5),

$$\begin{aligned}
& \int_0^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds \\
& -\sum_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds \\
= & \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds \\
> & \sum_{i=1}^{i_0} \beta_i \int_{\xi_{i_0}}^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds \\
& + \sum_{i=i_0+1}^m \beta_i \int_{\xi_{i_0}}^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds \\
= & \sum_{i=1}^m \beta_i \int_{\xi_{i_0}}^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds \\
= & \int_{\xi_{i_0}}^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds \\
> & 0.
\end{aligned}$$

Similarly, if $c < -M^*$, then (A_3) implies $f(t, c, 0, \dots, 0) < 0$, we have

$$\begin{aligned}
& \int_0^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds \\
& -\sum_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds \\
= & \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds \\
< & 0.
\end{aligned}$$

On the other hand, if $x \in \mathrm{Ker}L$ and $Nx \in \mathrm{Im}L$, we have $QNx = 0$, i.e.

$$\int_0^1 \int_0^1 G(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds$$

$$
\begin{aligned}
& -\sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^{\tau} f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))dud\tau ds \\
= & \sum_{i=1}^{m}\beta_i\int_{\xi_i}^{1}\int_0^1 G(s,\tau)\int_0^{\tau} f(u,c,0,\dots,0)dud\tau ds \\
= & \ 0.
\end{aligned}
$$

This is a contradiction. So $|c| \le M^*$. It follows that Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \{x \in \mathrm{Ker}L,\ \mathrm{sgn}(\Delta)\lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge : \mathrm{Ker}L \to \mathrm{Im}Q$ is the linear isomorphism given by $\wedge(c) = ct^k$ for all $c \in R$. Now we show that Ω_3 is bounded. Suppose $x_n(t) = c_n \in \Omega_3$ and $|c_n| \to +\infty$ as n tends to infinity. Then there exist $\lambda_n \in [0,1]$ such that

$$
\begin{aligned}
& \mathrm{sgn}(\Delta)\lambda_n c_n \\
& +\frac{1-\lambda_n}{\Delta}\Big(\int_0^1\int_0^1 G(s,\tau)\int_0^{\tau} f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))dud\tau ds \\
& -\sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^{\tau} f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))dud\tau ds\Big) \\
= & \ \mathrm{sgn}(\Delta)\lambda_n c_n + \frac{1-\lambda_n}{\Delta}\sum_{i=1}^{m}\beta_i\int_{\xi_i}^{1}\int_0^1 G(s,\tau)\int_0^{\tau} f(u,c,0,\dots,0)dud\tau ds \\
= & \ 0.
\end{aligned}
$$

So

$$\mathrm{sgn}(\Delta)\Delta\lambda_n c_n = -(1-\lambda_n)\sum_{i=1}^{m}\beta_i\int_{\xi_i}^{1}\int_0^1 G(s,\tau)\int_0^{\tau} f(u,c,0,\dots,0)dud\tau ds.$$

It is easy to see that λ_n has a convergent subsequence, without loss of generality, suppose $\lambda_n \to \lambda_0$, Again, since $|c_n| \to +\infty$, there two cases to be considered, i.e. there is subsequence of c_n that tends to $+\infty$(without loss of generality suppose $c_n \to +\infty$) or there is subsequence of c_n that tends to

$-\infty$(without loss of generality suppose $c_n \to -\infty$). If $c_n \to +\infty$ as n tends to infinity, then for sufficiently large n, we have $c_n > M^*$. Hence, using (A_3), similar to Step 2, we see

$$\begin{aligned}\operatorname{sgn}(\Delta)\Delta\lambda_n c_n^2 &= -(1-\lambda_n)c_n\sum_{i=1}^m \beta_i\int_{\xi_i}^1\int_0^1 G(s,\tau)\int_0^\tau f_c(u)dud\tau ds\\ &< 0,\end{aligned}$$

a contradiction, where $f_c(u) = f(u, c, 0, \ldots, 0)$. If $c_n \to -\infty$, then for sufficiently large n, $c_n < -M^*$. Hence using (A_3), we see

$$\begin{aligned}\operatorname{sgn}(\Delta)\Delta\lambda_n c_n^2 &= -(1-\lambda_n)c_n\sum_{i=1}^m \beta_i\int_{\xi_i}^1\int_0^1 G(s,\tau)\int_0^\tau f_c(u)dud\tau ds\\ &< 0,\end{aligned}$$

a contradiction. So Ω_3 is bounded.

In the following, we shall show that all conditions of Theorem GM are satisfied. Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^3\overline{\Omega_i}$. By Lemma 2.1, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have

(a). $Lx \neq \lambda Nx$ for $x \in (\mathrm{dom}L/\mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b). $Nx \notin \mathrm{Im}L$ for $x \in \mathrm{Ker}L \cap \partial\Omega$.

Step 4. We prove $\deg(QN|_{\mathrm{Ker}L},\ \Omega \cap \mathrm{Ker}L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \operatorname{sgn}(\Delta)\lambda\wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned}&\deg(QN|\mathrm{Ker}L, \Omega\cap\mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega\cap\mathrm{Ker}L, 0)\\ = \ &\deg(H(\cdot,1), \Omega\cap\mathrm{Ker}L, 0) = \deg(I, \Omega\cap\mathrm{Ker}L, 0) \neq 0.\end{aligned}$$

Thus by Theorem GM, $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$, which is a solution of BVP(1)-(2). The proof is complete.

Theorem 2.2. *Suppose following conditions hold.*

(A_1'). *There are continuous functions $h(t, x_0, x_1, \ldots, x_{2n-1})$, $e(t)$ and nonnegative functions $g_i(t, x)(i = 0, 1, \ldots, 2n-1)$ and positive numbers β and m such that f satisfies*

$$(-1)^{n-1} f(t, x_0, x_1, \ldots, x_{2n-1}) = e(t) + h(t, x_0, x_1, \ldots, x_{2n-1}) + \sum_{i=0}^{2n-1} g_i(t, x_i),$$

and also that h satisfies

$$x_{2n-1} h(t, x_0, x_1, \ldots, x_{2n-1}) \leq -\beta |x_{2n-1}|^{m+1}$$

for all $t \in [0,1]$ and $(x_0, x_1, \ldots, x_{2n-1}) \in R^{2n}$ and

$$\lim_{|x|\to\infty} \sup_{t\in[0,1]} \frac{|g_i(t,x)|}{|x|^m} = r_i, \text{ for } i = 0, 1, \ldots, 2n-1$$

with $r_i \geq 0$ for $i = 0, 1, \ldots, 2n-1$;

(A_2'). *There exist constants $L \geq 0$, $\alpha > 0$ and $\alpha_i \geq 0 (i = 1, \ldots, 2n-2)$ such that*

$$|f(t, x_0, x-1, \ldots, x_{2n-1})| \geq \alpha |x_0| - \sum_{i=1}^{2n-2} \alpha_i |x_i| - L$$

for all $t \in [0,1]$ and $(x_0, x_1, \ldots, x_{2n-1}) \in R^{2n}$.

Furthermore, (A_3), (A_4) and (A_5) of Theorem 2.1 hold. Then $BVP(1)$ and (2) has at least one solution provided

$$\left(1 + \frac{\sum_{i=1}^{2n-2} \alpha_i}{\alpha}\right)^m r_0 + \sum_{i=1}^{2n-1} r_i < \beta. \tag{18}$$

Proof. To apply Theorem GM, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \mathrm{dom}L/\mathrm{Ker}L,\ Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. Similar to Step 1 of Theorem 2.1, we have (14). We claim that there is $\xi \in (0,1)$ such that

$$f(\xi, x(\xi), x'(\xi), \ldots, x^{(2n-1)}(\xi)) = 0.$$

In fact, if $f(t, x(t), \ldots, x^{(n-1)}(t)) \neq 0$ for all $t \in [0,1]$, then either

$$\begin{aligned} &f(t, x(t), \ldots, x^{(n-1)}(t)) > 0 \qquad \text{for all } t \in [0,1] \text{ or}\\ &f(t, x(t), \ldots, x^{(n-1)}(t)) < 0 \qquad \text{for all } t \in [0,1]. \end{aligned}$$

So we get

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \beta_i \int_{\xi_i}^{1} \int_0^1 G(s,\tau) \int_0^{\tau} f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \\ &\geq \sum_{i=1}^{i_0} \beta_i \int_{\xi_{i_0}}^{1} \int_0^1 G(s,\tau) \int_0^{\tau} f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \\ &\quad + \sum_{i=i_0+1}^{m} \beta_i \int_{\xi_{i_0}}^{1} \int_0^1 G(s,\tau) \int_0^{\tau} f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \\ &= \sum_{i=1}^{m} \beta_i \int_{\xi_{i_0}}^{1} \int_0^1 G(s,\tau) \int_0^{\tau} f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \\ &= \int_{\xi_{i_0}}^{1} \int_0^1 G(s,\tau) \int_0^{\tau} f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \\ &> 0, \end{aligned}$$

which is a contradiction. If the second case holds, the same contradiction can be deduced. By (A_2'), we see that

$$\begin{aligned}|x(\xi)| &\le \frac{L}{\alpha}+\frac{1}{\alpha}\sum_{i=1}^{2n-2}\alpha_i|x^{(i)}(\xi)|\\ &\le \frac{L}{\alpha}+\frac{1}{\alpha}\sum_{i=1}^{2n-2}\alpha_i\int_0^1|x^{(2n-1)}(s)|ds.\end{aligned}$$

Hence

$$\begin{aligned}|x(t)| &\le |x(\xi)|+\left|\int_\xi^t x'(s)ds\right|\\ &\le \frac{L}{\alpha}+\frac{1}{\alpha}\sum_{i=1}^{2n-2}\alpha_i\int_0^1|x^{(2n-1)}(s)|ds+\int_0^1|x^{(2n-1)}(s)|ds \qquad (19)\\ &= A\left(M+\int_0^1|x^{(2n-1)}(s)|\right)ds,\end{aligned}$$

where $A=1+\frac{\sum_{i=1}^{2n-2}\alpha_i}{\alpha}$ and $M=\frac{L}{\alpha A}$. It suffices to prove there is a constant $B>0$ such that

$$||x||=\max\{||x||_\infty,||x'||_\infty,\dots,||x^{(2n-1)}||_\infty\}\le B.$$

We divide this step into two sub-steps.

Sub-step 1.1. We prove that there is a constant $\overline{M}>0$ such that

$$\int_0^T|x^{(2n-1)}(s)|^{m+1}ds\le\overline{M}.$$

For $x\in\Omega_1$, we have

$$(-1)^{n-1}x^{(n)}(t)=\lambda f(t,x(t),x'(t),\dots,x^{(2n-1)}(t)). \qquad (20)$$

Multiplying two sides of (20) by $x^{(2n-1)}(t)$ and integrating it from 0 to 1, using (A_1'), we get

$$\begin{aligned}
0 &\leq \frac{1}{2}|x^{(2n-1)}(1)|^2 \\
&= \lambda\int_0^1 (-1)^{n-1} f(s, x(s), x'(s), \ldots, x^{(2n-1)}(s))x^{(2n-1)}(s)ds \\
&= \lambda\Big(\int_0^1 h(s, x(s), x'(s), \ldots, x^{(2n-1)}(s))x^{(2n-1)}(s)ds \\
&\quad + \sum_{i=0}^{2n-1}\int_0^1 g_i(s, x^{(i)}(s))x^{(2n-1)}(s)ds + \int_0^1 e(s)x^{(2n-1)}(s)ds\Big).
\end{aligned}$$

Thus, from the second part of (A_1'),

$$\begin{aligned}
&\lambda\beta\int_0^1 \left|x^{(2n-1)}(s)\right|^{m+1} ds \\
\leq\ & -\lambda\int_0^1 h(s, x(s), x'(s), \ldots, x^{(2n-1)}(s))x^{(2n-1)}(s)ds \\
=\ & \lambda\sum_{i=0}^{2n-1}\int_0^1 g_i(s, x^{(i)}(s))x^{(2n-1)}(s)ds + \lambda\int_0^1 e(s)x^{(2n-1)}(s)ds \\
\leq\ & \lambda\sum_{i=0}^{2n-1}\int_0^1 |g_i(s, x^{(i)}(s))||x^{(2n-1)}(s)|ds + \lambda\int_0^1 |e(s)||x^{(2n-1)}(s)|ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\beta\int_0^1 \left|x^{(2n-1)}(s)\right|^{m+1} ds &\leq \sum_{i=0}^{2n-1}\int_0^1 |g_i(s, x^{(i)}(s))||x^{(2n-1)}(s)|ds \\
&\quad + \int_0^1 |e(s)||x^{(2n-1)}(s)|ds.
\end{aligned}$$

Let $\epsilon > 0$ satisfy

$$\beta > \left(1 + \frac{\sum_{i=1}^{2n-2}\alpha_i}{\alpha}\right)^m (r_0 + \epsilon) + \sum_{i=1}^{2n-1}(r_i + \epsilon).$$

By the conditions of theorem, we see $\epsilon > 0$. For such a $\epsilon > 0$, we find from

(A_1') that there is a constant $\delta > M$ such that for every $i = 0, 1, \ldots, 2n-1$,

$$|g_i(t,x)| < (r_i + \epsilon)|x|^m \text{ uniformly for } t \in [0,1] \text{ and } |x| > \delta.$$

Let, for $i = 0, 1, \ldots, 2n-1$,

$$\begin{aligned}
\Delta_{1,i} &= \{t : \ t \in [0,1], \ |x^{(i)}(t)| \leq \delta\}, \\
\Delta_{2,i} &= \{t : \ t \in [0,1], \ |x^{(i)}(t)| > \delta\}, \\
g_{\delta,i} &= \max_{t \in [0,1], |x| \leq \delta} |g_i(t,x)|.
\end{aligned}$$

Then

$$\begin{aligned}
& \beta \int_0^1 \left| x^{(2n-1)}(s) \right|^{m+1} ds \\
\leq & \sum_{i=0}^{2n-1} \int_{\Delta_{1,i}} |g_i(s, x^{(i)}(s))||x^{(2n-1)}(s)|ds \\
& + \sum_{i=0}^{2n-1} \int_{\Delta_{2,i}} |g_i(s, x^{(i)}(s)||x^{(2n-1)}(s)|ds + \int_0^1 |e(s)||x^{(2n-1)}(s)|ds \\
\leq & \sum_{i=0}^{2n-1} g_{\delta,i} \int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1} (r_i + \epsilon) \int_{\Delta_{2,i}} |x^{(i)}(s)|^m |x^{(2n-1)}(s)|ds \\
& + \int_0^1 |e(s)||x^{(2n-1)}(s)|ds \\
\leq & \sum_{i=0}^{2n-1} g_{\delta,i} \int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1} (r_i + \epsilon) \int_0^1 |x^{(i)}(s)|^m |x^{(2n-1)}(s)|ds \\
& + \int_0^1 |e(s)||x^{(2n-1)}(s)|ds.
\end{aligned}$$

Again

$$\begin{aligned}
& \int_0^1 |x(s)|^m |x^{(2n-1)}(s)|ds \\
\leq & \ A^m \left(M + \int_0^1 |x^{(2n-1)}(s)|ds \right)^m \int_0^1 |x^{(2n-1)}(s)|ds,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 |x^{(i)}(s)|^m |x^{(2n-1)}(s)|ds \\
\leq\ & \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^1 |x^{(2n-1)}(s)|ds, \quad i=1,\ldots,2n-2.
\end{aligned}$$

So

$$\begin{aligned}
&\beta \int_0^1 |x^{(2n-1)}(s)|^{m+1} ds \\
\leq\ & A^m \left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right)^m (r_0+\epsilon) \int_0^1 |x^{(2n-1)}(s)|ds \\
& + \sum_{i=1}^{2n-1} (r_i+\epsilon) \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^T |x^{(2n-1)}(s)|ds \\
& + \int_0^1 |e(s)||x^{(2n-1)}(s)|ds + \sum_{i=0}^{2n-1} g_{\delta,i} \int_0^1 |x^{(i)}(s)|ds \\
\leq\ & A^m \left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right)^m (r_0+\epsilon) \int_0^1 |x^{(2n-1)}(s)|ds \\
& + \sum_{i=1}^{2n-2} (r_i+\epsilon) \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^1 |x^{(2n-1)}(s)|ds \\
& +(r_{2n-1}+\epsilon) \int_0^1 |x^{(2n-1)}(s)|^{m+1} ds + ||e||_\infty \int_0^1 |x^{(2n-1)}(s)|ds \\
& + \sum_{i=1}^{2n-1} g_{\delta,i} \int_0^1 |x^{(i)}(s)|ds + g_{\delta,0} \int_0^1 |x(s)|ds \\
=\ & A^m \left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right)^m (r_0+\epsilon) \int_0^1 |x^{(2n-1)}(s)|ds \\
& + \sum_{i=1}^{2n-2} (r_i+\epsilon) \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1} \\
& +(r_{2n-1}+\epsilon) \int_0^1 |x^{(2n-1)}(s)|^{m+1} ds + ||e||_\infty \int_0^1 |x^{(2n-1)}(s)|ds \\
& + \sum_{i=1}^{2n-1} g_{\delta,i} \int_0^1 |x^{(2n-1)}(s)|ds + g_{\delta,0} A \left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right).
\end{aligned}$$

We claim that there is a constant $\sigma \in (0,1)$, independent of λ, such that $(1+x)^m \leq 1+(m+1)x$ for all $x \in (0,\sigma)$. In fact, let $q(x) = (1+x)^m -$

$(1+(m+1)x)$, we see $q(0)=0$, and $q'(0)=-1<0$, so the claim is valid. To obtain $\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \le \overline{M}$, we consider two cases.

Case 1. $\int_0^1 |x^{(2n-1)}(s)|ds \le \frac{M}{\sigma}$.

So

$$\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m_0} \le M^m\left(1+\frac{1}{\sigma}\right)^{m_0}.$$

Since

$$\int_0^1 |x^{(2n-1)}(s)|ds \le \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)},$$

we get

$$\begin{aligned}
&\beta\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\\
\le\ & A^m M^m\left(1+\frac{1}{\sigma}\right)^m (r_0+\epsilon)\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+\sum_{i=1}^{2n-1}(r_i+\epsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\\
&+g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right]\\
&+||e||_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}.
\end{aligned}$$

i.e.

$$\begin{aligned}
&\left(\beta-\sum_{i=1}^{2n-1}(r_i+\epsilon)\right)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\\
\le\ &\left[A^m M^m\left(1+\frac{1}{\sigma}\right)^m (r_0+\epsilon)+||e||_\infty\right]\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}
\end{aligned}$$

$$+g_{\delta,0}A\left[M+\left(\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right].$$

From the definition of ϵ, we find that $\beta-\sum_{i=1}^{2n-1}(r_i+\epsilon)>0$ and that there is a constant $M_1>0$ such that

$$\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\le M_1.$$

Case 2. $\int_0^1|x^{(2n-1)}(s)|ds>\frac{M}{\sigma}$.

In this case, $0<\frac{M}{\int_0^1|x^{(2n-1)}(s)|ds}<\sigma$. Using $(1+x)^m\le 1+(m+1)x$ for $x\in(0,\sigma)$, we have

$$\begin{aligned}
&\left(M+\int_0^1|x^{(2n-1)}(s)|ds\right)^m\\
=&\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m\left(1+\frac{M}{\int_0^1|x^{(2n-1)}(s)|ds}\right)^m\\
\le&\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m\left(1+\frac{(m+1)M}{\int_0^1|x^{(2n-1)}(s)|ds}\right)\\
=&\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m+(m+1)M\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^{m-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\beta\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\\
\le\ &A^m(r_0+\epsilon)\int_0^1|x^{(2n-1)}(s)|ds\left[\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m\right.\\
&\left.+(m+1)M\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^{m-1}\right]+\\
&\sum_{i=1}^{2n-2}(r_i+\epsilon)\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^{m+1}\\
&+(r_{2n-1}+\epsilon)\int_0^1|x^{(2n-1)}(s)|^{m+1}ds+||e||_\infty\int_0^1|x^{(2n-1)}(s)|ds
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right] \\
= \;& A^m(r_0+\epsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1} \\
& +\sum_{i=1}^{2n-2}(r_i+\epsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1} \\
& +A^m(r_0+\epsilon)(m+1)M\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m} \\
& +(r_{2n-1}+\epsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds+||e||_\infty\int_0^1 |x^{(2n-1)}(s)|ds \\
& +\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right] \\
\leq \;& \left(A^m(r_0+\epsilon)+\sum_{i=1}^{2n-2}(r_i+\epsilon)\right)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \\
& +(r_{2n-1}+\epsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \\
& +A^m(r_0+\epsilon)(m+1)M\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{m/(m+1)} \\
& +||e||_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right].
\end{aligned}
$$

Hence

$$
\left(\beta-A^m(r_0+\epsilon)-\sum_{i=1}^{2n-1}(r_i+\epsilon)\right)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds
$$

$$\begin{aligned}
&\leq A^m(1+m)(r_0+\epsilon)M\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{m/(m+1)} \\
&\quad +||e||_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&\quad +\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(i)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&\quad +g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right].
\end{aligned}$$

From the definition of ϵ, we find that there is $M_2 > 0$ such that

$$\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \leq M_2.$$

Thus we obtain from Case 1 and 2 that

$$\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \leq \max\{M_1, M_2\} =: \overline{M}.$$

Sub-step 1.2. Prove there is $B > 0$ such that $||x|| \leq B$.

From Sub-step 1.1, we have

$$\begin{aligned}
||x||_\infty &\leq A\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right) \\
&\leq A\left(M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right) \\
&\leq A\left(M+\overline{M}^{1/(m+1)}\right). \\
||x^{(i)}||_\infty &\leq \int_0^1 |x^{(2n-1)}(s)|ds \\
&\leq \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
&\leq \overline{M}^{1/(m+1)},\ i=1,\ldots,n-2.
\end{aligned}$$

Multiplying two sides of (20) by $x^{(2n-1)}(t)$, integrating it from 0 to t, using

(A_1'), we get

$$\begin{aligned}
&\frac{1}{2}|x^{(2n-1)}(t)|^2\\
=\ & \lambda\int_0^t (-1)^{n-1} f(s,x(s),x'(s),\ldots,x^{(2n-1)}(s))x^{(2n-1)}(s)ds\\
=\ & \lambda\int_0^t h(s,x(s),x'(s),\ldots,x^{(2n-1)}(s))x^{(2n-1)}(s)ds\\
& +\lambda\sum_{i=0}^{2n-1}\int_0^t g_i(s,x^{(i)}(s))x^{(2n-1)}(s)ds+\lambda\int_0^t e(s)x^{(2n-1)}(s)ds\\
\leq\ & -\lambda\beta\int_0^t |x^{(2n-1)}(s)|^{m+1}ds\\
& +\lambda\sum_{i=0}^{2n-1}\int_0^t g_i(s,x^{(i)}(s))x^{(2n-1)}(s)ds+\lambda\int_0^t e(s)x^{(2n-1)}(s)ds\\
\leq\ & \lambda\sum_{i=0}^{2n-1}\int_0^t g_i(s,x^{(i)}(s))x^{(2n-1)}(s)ds+\lambda\int_0^t e(s)x^{(2n-1)}(s)ds\\
\leq\ & \sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))||x^{(2n-1)}(s)|ds+\int_0^1 |e(s)||x^{(2n-1)}(s)|ds\\
\leq\ & \sum_{i=0}^{2n-1}\int_{\Delta_{1,i}} |g_i(s,x^{(i)}(s))||x^{(2n-1)}(s)|ds\\
& +\sum_{i=0}^{2n-1}\int_{\Delta_{2,i}} |g_i(s,x^{(i)}(s)||x^{(2n-1)}(s)|ds+\int_0^1 |e(s)||x^{(2n-1)}(s)|ds\\
\leq\ & \sum_{i=1}^{2n-1} g_{\delta,i}\int_0^1 |x^{(2n-1)}(s)|ds+\sum_{i=0}^{2n-1}(r_i+\epsilon)\int_{\Delta_{2,i}} |x^{(i)}(s)|^m|x^{(2n-1)}(s)|ds\\
& +\int_0^1 |e(s)||x^{(2n-1)}(s)|ds+g_{\delta,0}\int_0^1 |x(s)|ds.
\end{aligned}$$

Similarly to Sub-step 1.1, we can get

$$\begin{aligned}
&\frac{1}{2}|x^{(2n-1)}(t)|^2\\
\leq\ & \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
& +\sum_{i=0}^{2n-1}(r_i+\epsilon)\int_0^1 |x^{(i)}(s)|^m|x^{(2n-1)}(s)|ds
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^1 |e(s)||x^{(2n-1)}(s)|ds + g_{\delta,0}A\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right) \\
\leq\ & \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +\sum_{i=1}^{2n-1}(r_i+\epsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1} \\
& +||e||_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right] \\
& +(r_0+\epsilon)A^m\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^m\int_0^1 |x^{(2n-1)}(s)|ds \\
\leq\ & \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +\sum_{i=1}^{2n-1}(r_i+\epsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \\
& +||e||_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
& +g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right] \\
& +(r_0+\epsilon)A^m\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right]^m\times \\
& \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} \\
\leq\ & \sum_{i=1}^{2n-1} g_{\delta,i}\overline{M}^{1/(m+1)}+\sum_{i=1}^{2n-1}(r_i+\epsilon)\overline{M}+||e||_\infty\overline{M}^{1/(m+1)} \\
& +(r_0+\epsilon)A^m\left[M+\overline{M}^{1/(m+1)}\right]^m\overline{M}^{1/(m+1)}+g_{\delta,0}A\left(M+\overline{M}^{1/(m+1)}\right).
\end{aligned}
$$

So there is $M_3>0$ such that $|x^{(2n-1)}(t)|\leq M_3$. Hence $||x^{(2n-1)}||_\infty\leq M_3$.

It follows from above discussion that there is $B>0$ such that

$$||x||\leq B.$$

Hence Ω_1 is bounded. This completes the step 1.

Step 2. Let

$$\Omega_2 = \{x \in \text{Ker}L,\ Nx \in \text{Im}L\}.$$

It is similar to that of Step 2 of the proof of Theorem 2.1 to prove that Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \{x \in \text{Ker}L,\ \text{sgn}(\Delta)\lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge : \text{Ker}L \to \text{Im}Q$ is the linear isomorphism given by $\wedge(c) = ct^k$ for all $c \in R$. It is similar to that of proof of Theorem 2.1 to show that Ω_3 is bounded.

In the following, we shall show that all conditions of Theorem GM are satisfied. Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$. By Lemma 2.1, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have (a) and (b) if the proof of Theorem 2.1.

Step 4. We prove $\deg(QN|_{\text{Ker}L},\ \Omega \cap \text{Ker}L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \text{sgn}(\Delta)\lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \text{Ker}L$. The remainder of the proof is similar to that of Theorem 2.1 and is omitted.

Thus by Theorem GM, $Lx = Nx$ has at least one solution in $\text{dom}L \cap \overline{\Omega}$, which is a solution of BVP(1)-(2). The proof is complete.

Remark 1. In Theorems 2.1, the degree of the variables $x_0, x_1, \ldots,$ x_{2n-1} in function f may be different from each other, in Theorem 2.2, the degree greater than 1.

Now, we assume:

(A_6). $\alpha_{i,k} \geq 0$ for all $i = 1, \ldots, n-1, k = 1, \ldots, p_i$ with $\sum_{k=1}^{p_i} \alpha_{i,k} < 1$. $\beta_i \geq 0$ for $i = 1, \ldots, m$ with $\sum_{i=1}^{m} \beta_i < 1$.

(A_7). $f(t, x_0, \ldots, x_{2n-1}) \leq 0$ for all $t \in [0,1]$ and $(x_0, \ldots, x_{2n-1}) \in R^{2n}$ and $f(t, 0, \ldots, 0) \not\equiv 0$ on any interval $[\alpha, \beta]$, where $0 \leq \alpha < \beta \leq 1$.

In this case, problem (1) and (2) is a non-resonance boundary value problem. We have the following results.

Theorem 2.3. *Suppose* $(A_1) - (A_5)$ *of Theorem* 2.1 *and* (A_6) *and* (A_7) *hold, Then* $BVP(1)$ *and* (2) *has at least one positive solution provided* (13) *holds.*

Proof. For problem (1) and (2), from (11) together with $x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i)$, we get c in (12) by

$$c = \frac{1}{\sum_{i=1}^{m} \beta_i - 1} \Big(\int_0^1 \int_0^1 G(s,\tau) \int_0^\tau y(u) du d\tau ds - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau y(u) du d\tau ds \Big).$$

Let $f_x(t) = f(t, x(t), \ldots, x^{(2n-1)}(t))$. Define an operator T by

$$\begin{aligned} &Tx(t) \\ = \ & \frac{1}{\sum_{i=1}^{m} \beta_i - 1} \Big(\int_0^1 \int_0^1 G(s,\tau) \int_0^\tau f_x(u) du d\tau ds - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau f_x(u) du d\tau ds \Big) \\ &+ \int_0^t \int_0^1 G(s,\tau) \int_0^\tau f_x(u) du d\tau ds \end{aligned}$$

for every $x \in C^{n-1}[0,1]$.

Consider the set $\Omega = \{x \in C^{n-1}[0,1], x = \lambda Tx,\ \lambda \in [0,1]\}$. For $x \in \Omega$, we have

$$(-1)^{n-1} x^{(2n)}(t) = f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)).$$

Similar to the proof of Theorem 2.1, we can prove that there is a constant

$B > 0$ such that $||x|| \leq B$ for every $x \in \Omega$. Then by Schauder fixed point theorem, T has at least one fixed point, which is a solution of BVP(1) and (2). Since $G(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$ and (A_6), we see

$$\begin{aligned} x(t) &= \frac{1}{\sum_{i=1}^m \beta_i - 1}\Big(\int_0^1 \int_0^1 G(s,\tau)\int_0^\tau f(u,x(u),\ldots,x^{(2n-1)}(u))dud\tau ds \\ &\quad -\sum_{i=1}^m \beta_i \int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau f(u,x(u),\ldots,x^{(2n-1)}(u))dud\tau ds\Big) \\ &\quad +\int_0^t \int_0^1 G(s,\tau)\int_0^\tau f(u,x(u),x'(u),\ldots,x^{(2n-1)}(u))dud\tau ds \\ &= \frac{1}{\sum_{i=1}^m \beta_i - 1}\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 \times \\ &\quad G(s,\tau)\int_0^\tau f(u,x(u),x'(u),\ldots,x^{(2n-1)}(u))dud\tau ds \\ &\quad -\int_0^t \int_0^1 G(s,\tau)\int_0^\tau f(u,x(u),x'(u),\ldots,x^{(2n-1)}(u))dud\tau ds \\ &\geq 0. \end{aligned}$$

On the other hand, (A_7) implies that $x(t) > 0$ for all $t \in (0,1)$.

References

1. V. Il'in and E. Moiseev, *Non-local boundary value problems of first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differential Eqns., **23**(1987), 803-810.

2. C. P. Gupta, S. K. Ntougas and P. C. Tsamatos, *Solvability of an m-point boundary value problem for second order ordinary differential equations*, J. Math. Anal. Appl., **189**(1995), 575-584.

3. W. Feng and J. R. L. Webb, *Solvability of three-point boundary value problems at resonance*, Nonlinear Anal., **30**(1997), 3227-3238.

4. W. Feng and J. R. L. Webb, *Solvability of m-point boundary value problems with nonlinear growth*, J. Math. Anal. Appl., **212**(1997), 467-489.

5. B. Liu and J. Yu, *Solvability of multi-point boundary value problems at resonance* (I), India J. Pure Appl. Math., **33**(4)(2002), 475-494.

6. B. Liu and J. Yu, *Solvability of multi-point boundary value problems at resonance* (II), Appl. Math. Comput., **136**(2003), 353-377.

7. B. Liu, *Solvability of multi-point boundary value problems at resonance* (III), Appl. Math. Comput., **129**(2002), 119-143.

8. B. Liu, *Solvability of multi-point boundary value problems at resonance* (IV), Appl. Math. Comput., **143**(2003), 275-299.

9. R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Math., 568, Springer, Berlin, 1977.

10. R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

11. J. R. Graef and B. Yang, *Existence and non-existence of positive solutions of fourth-order nonlinear boundary value problems*, Appl. Anal., **74**(2000), 201-204.

12. R. Ma, H. Wang, *On the existence of positive solutions of fourth-order differential equations*, Appl. Anal., **59**(1995), 225-231.

13. J. R. Graef and B. Yang, *On a nonlinear boundary value problem for fourth-order differential equations*, Appl. Anal., **72**(1999), 439-448.

14. C. J. Chyan and J. Henderson, *Positive solutions of $2m^{th}$-order boundary value problems*, Appl. Math. Letters, **15**(2002), 767-774.

15. P. K. Palamides, *Positive solutions for higher-order Lidstone boundary value problems: A new approach Via Sperner's Lemma*, Comput. Math. Appl., **42**(2001), 75-89.

16. J. R. Graef and B.Yang, *On a nonlinear boundary-value problem for fourth-order equations*, Appl. Anal., **72**(1999), 439-448.

17. J. R. Graef and B. Yang, *Existence and non-existence of positive solutions of fourth-order nonlinear boundary-value problems*, Appl. Anal., **74**(2000), 201-214.

18. B. G. Zhang and L. Z. Kong, *Multiple positive solutions of a class of p-Laplacian equations*(in Chinese), Annals Math., **6**(2001), 1-10.

19. R. Ma, J. Zhang and S. Fu, *The method of lower and upper solutions for fourth order two-point boundary value problems*, J. Math. Anal. Appl., **215**(1997), 415-422.

20. Y. Liu and W. Ge, *Double positive solutions of fourth-order nonlinear boundary value problems*, Appl. Anal., **82**(4)(2002), 369-380.

21. R. I. Avery, J. M. Davis and J. Henderson, *Three symmetric positive solutions for Lidstone-Williams theorem*, Electronic J. Diff. Equ., **40**(2000), 1-15.

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