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CUP-LENGTH ESTIMATE FOR PERIODIC SOLUTIONS OF CONTINUOUS HAMILTONIAN SYSTEMS ON THE SYMPLECTIC MANIFOLDS

BY

GUANGCUN LU (盧廣存)

Abstract. In this note we give the cup-length estimate of numbers of periodic solutions of the Hamiltonian system defined by a time-dependent C^1-function on the closed symplectic manifolds (M,ω) with $\omega|_{\pi_2(M)}=0$.

Let (M,ω) be a $2n$-dimensional symplectic manifold and $H:\mathbf{S}^1\times M\to\mathbf{R}$ a C^k-function($k\ge 1$). Here $\mathbf{S}^1=\mathbf{R}/\mathbf{Z}$. It gives rise to a C^{k-1} Hamiltonian vector field $X_H:\mathbf{R}\times M\to TM$ via $i_{X_{H_t}}\omega=dH_t$, where $X_{H_t}(\cdot)=X_H(t,\cdot)$ and $H_t(x)=H(t,x)$. Consider the Hamiltonian differential equation

$$\dot{x}(t)=X_H(t,x(t)) \tag{1}$$

on M. It is well-known that when $k\ge 2$ the solutions $x(t)$ of (1) determine a 1-parameter family of C^{k-1}-diffeomorphisms $\phi_H^t\in \mathrm{Diff}^{k-1}(M)$ satisfying $\phi_H^t(x(0))=x(t)$. These ϕ_H^t are also C^{k-1}-symplectic in the sense of $\omega_x(u,v)=\omega_{\phi_H^t(x)}(d\phi_H^t(x)(u),d\phi_H^t(x)(v))$ for any $x\in M$ and $u,v\in T_xM$. In particular $\phi_H:=\phi_H^1$ is called as the *Hamiltonian map* associated with H. In these cases there is a one-to-one corresponding between the set of fixed points of ϕ_H and that of contractible 1-periodic solutions of (1). The famous Arnold conjecture states that the number of fixed points of every ϕ_H is at least as many as the number of critical points of a smooth function on M if

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H is smooth([1]). Its homological version is

$$\sharp \mathrm{Fix}\phi_H \geq \mathrm{cuplength}(M) + 1. \tag{2}$$

So far for many symplectic manifolds the estimate (2) has been proved(see [3, 4, 6, 10]). For Arnold's original conjecture some progress had also been made in [8].

From the views of the dynamic system and application the case of $k = 1$ is very important and interesting. However, in this time the vector field X_H is merely continuous, and thus the uniquess of 1-periodic solutions of (1) through a given point on M can not be ensured in general. Furthermore, one cannot talk anything about ϕ_H yet. In fact, approximating it with the smooth Hamiltonians and directly applying the conclusion of the smooth Hamiltonian one can only prove that (1) has at least one solution. The main result of this note shows that the estimate (2) is still valid for C^1-Hamiltonian. In order to illustrate the precise sense of it we define a set-valued map $\psi_H : M \to 2^M$ as follows: for every $x \in M$

$$\psi_H(x) \tag{3}$$

is defined to be the subset of M consisting of all points y for which there exist a sequence of smooth functions $H_k : \mathbf{S}^1 \times M \to \mathbf{R}$ such that $\lim_{k\to\infty} \|H_k - H\|_{C^1} = 0$ and $\phi_{H_k}(x) \to y$. It is easy to check that this set-valued map is always nonempty and becomes the usual Hamiltonian map ϕ_H when H is C^2-smooth. A point $x \in M$ is called as the fixed point of ψ_H if $x \in \psi_H(x)$. Denote by $\mathcal{P}(H)$ the set of all contractible 1-periodic solutions of (1). It should be noted that in present case there is no obvious bijection between $\mathrm{Fix}(\psi_H)$ and $\mathcal{P}(H)$. But if we replace $\mathcal{P}(H)$ by its subset $\mathcal{P}_0(H)$ of all 1-periodic solutions y of (1) for which there exist a sequence of smooth functions $H_k : \mathbf{S}^1 \times M \to \mathbf{R}$ with $\lim_{k\to\infty} \|H_k - H\|_{C^1} = 0$ and $y_k \in \mathcal{P}(H_k)$ such that $y_k \xrightarrow{C^1} y$, then it is easy to prove that every element in $\mathcal{P}_0(H)$ may determine a fixed point of ψ_H and different elements does its distinct fixed points.

When $k \geq 2$ it is not hard to verify that $\mathcal{P}_0(H) = \mathcal{P}(H)$, and it easily follows from the arguments in [3, 6] that (2) holds. In this note we actually prove the following theorem.

Main Theorem. *For a closed symplectic manifold* (M, ω) *with* $\omega|_{\pi_2(M)} = 0$, *if the equation* (1) *defined by a* C^1*-function* $H : \mathbf{S}^1 \times M \to \mathbf{R}$ *has only finite many contractible* 1*-periodic solutions it holds that*

$$\sharp\mathcal{P}_0(H) \geq \mathrm{cuplength}(M) + 1. \tag{4}$$

Consequently, in this case $\sharp\mathrm{Fix}(\psi_H) \geq \mathrm{cuplength}(M) + 1$ *for* ψ_H *defined by* (3). *Furthermore, every Hamiltonian system associated with a* C^1*-function* $H : \mathbf{S}^1 \times M \to \mathbf{R}$ *has at least* $\mathrm{cuplength}(M) + 1$ *contractible* 1*-periodic solutions.*

Proof. The arguments will proceed along line of [3, 6]. In particular we adopt notations in [6] without special statements. First of all, we point out that under our assumption there are no nonconstant J-holomorphic spheres for any $J \in \mathcal{J}(M, \omega)$. Therefore, it follows from Corollary 4.2 in [6] that the associativity of the cap action holds. For a given C^1-smooth $H : \mathbf{S}^1 \times M \to \mathbf{R}$ we first take a sequence of smooth functions $\tilde{H}_k : \mathbf{S}^1 \times M \to \mathbf{R}$ such that $\|\tilde{H}_k - H\|_{C^1} < \frac{1}{k}$ for every $k \geq 1$, then for each fixed k pick a F-generic pair([6, p.277]) (H_k, J_k) such that $\|\tilde{H}_k - H_k\|_\varepsilon < \frac{1}{k}$ and $J_k \in \mathcal{U}_{\frac{1}{k}}(J_0)$. Here $J_0 \in \mathcal{J}(M, \omega)$ is fixed and $\mathcal{U}_{\frac{1}{k}}(J_0)$ as in [5,§2]. By the definition of the norm $\|\cdot\|_\varepsilon$ in [2] it must hold that

$$\lim_{k\to\infty} \|H_k - H\|_{C^1} = 0. \tag{5}$$

From Gromov's compactness theorem and Lemma 3.5 in [7] it follows that there exists a positive constant $\hbar_0 > 0$ such that (i) $E(v) \geq \hbar_0$ for any nonconstant J_k-holomorphic sphere $v : \mathbf{S}^2 \to M$ and $k = 0, 1, \cdots$, (ii) $E(u) \geq \hbar_0$ for any nontrivial s-dependent solution $u : \mathbf{R} \times \mathbf{S}^1 \to M$ of

$$\frac{\partial u}{\partial s} + J_k(u)\frac{\partial u}{\partial t} + \nabla H_k(t, u) = 0 \quad \text{or} \quad \frac{\partial u}{\partial s} + J_0(u)\frac{\partial u}{\partial t} + \nabla H(t, u) = 0. \tag{6}$$

Here the energy

$$E(v)=\int_{\mathbf{S}^2} v^*\omega \quad \text{and} \quad E(u)=\int_{-\infty}^{\infty}\int_0^1\Big|\frac{\partial u}{\partial s}\Big|^2 dtds.$$

Note that the above connecting orbits determined by H is only C^1 because of our H being C^1. Hence we make use of Lemma 3.5 in [7] instead of Theorem 3.3 in [5]. From now on we assume that M is embedded in $\mathbf{R}^q$ for a sufficiently large q. By means of the Ascoli-Arzela's theorem it follows from $\lim_{k\to\infty}\|H_k-H\|_{C^1}=0$ that for any $\epsilon>0$ there exists k_0 such that every $y\in\mathcal{P}(H_k)(k\ge k_0)$ satisfies $\|y-y_0\|_{C^1}<\epsilon$ for some $y_0\in\mathcal{P}_0(H)$.

By the assumption $\mathcal{P}(H)$ is a finite set. Let

$$\mathcal{P}(H):=\{y_1,\dots,y_p\}.$$

Here it should be emphasized that unlike the case of C^k-smooth$(k\ge 2)$ the elements of $\mathcal{P}(H)$ are not necessiarly isolated. That is, even if $y_i\ne y_j$ for $i\ne j$ we cannot guarantee $y_i(\mathbf{S}^1)\cap y_j(\mathbf{S}^1)=\emptyset$. But there must be a $t_0\in\mathbf{S}^1$ such that $y_i(t_0)\ne y_j(t_0)$. In fact, $\max_{t\in\mathbf{S}^1}|y_i(t)-y_j(t)|>0$. Therefore, we need to modify the definition of ϵ_0 in Lemma 5.1 of [6] as follows. Firstly, we define

$$\begin{aligned}\epsilon(y_i) &:= \min\{d(y_i,y_j):\ y_i(\mathbf{S}^1)\cap y_j(\mathbf{S}^1)=\emptyset,\ j\ne i\}\\ \hat\epsilon(y_i) &:= \min\{\max_{t\in\mathbf{S}^1}|y_i(t)-y_j(t)|:\ y_i(\mathbf{S}^1)\cap y_j(\mathbf{S}^1)\ne\emptyset,\ j\ne i\}\end{aligned}$$

for every $y_i\in\mathcal{P}(H)$. Here $\epsilon(y_i)$ and $\hat\epsilon(y_i)$ are understood as $+\infty$ as the minimum is taken in an empty set. Then $\min\{\epsilon(y_i),\ \hat\epsilon(y_i)\}$ is always a finite positive number as $p>1$. Denote by

$$\epsilon_0:=\min_{y_i\in\mathcal{P}(H)}\min\{\epsilon(y_i),\ \hat\epsilon(y_i),1\}.$$

Lemma 1.([6, lemma 5.1]) *For any $\epsilon\in(0,\epsilon_0)$ there exists an integer $k_1\ge k_0$ such that if $k\ge k_1$ and a connecting orbit u between 1-periodic* *solutions z and z' in $\mathcal{P}(H_k)$ satisfy:*

$$E(u) \leq \frac{\hbar_0}{2} \quad \text{and } \|z - z_0\|_{C^1} \leq \frac{\epsilon}{2} \quad \text{for some } z_0 \in \mathcal{P}_0(H),$$

then $u(\mathbf{R} \times \mathbf{S}^1)$ must be contained in 2ϵ-neighborhood of z_0.

For convenience we sketch its proof though it is almost a repeat of that in [6].

Proof of Lemma 1. Assume that there exists a $0 < \epsilon < \epsilon_0$, and a sequence of connecting trajectories u_{k_j} between $z_{k_j} \in \mathcal{P}(H_{k_j})$ and $z'_{k_j} \in \mathcal{P}(H_{k_j})$ such that

$$E(u_{k_j}) \leq \frac{\hbar_0}{2} \quad \text{and } \|z_{k_j} - z_0\|_{C^1} \leq \frac{\epsilon}{2} \quad \text{for some } z_0 \in \mathcal{P}_0(H),$$

and $u_{k_j}(\mathbf{R} \times \mathbf{S}^1)$ cannot be contained in 2ϵ-neighborhood of z_0. Then for every fixed j and sufficiently large $-s \in \mathbf{R}$ it holds that

$$\begin{aligned} &\|u_{k_j}(s,\cdot) - z_0\|_{C^1} \leq \|u_{k_j}(s,\cdot) - z_{k_j}\|_{C^1} + \|z_{k_j} - z_0\|_{C^1} < \epsilon, \\ &s_j := \sup\{s \in \mathbf{R} \,|\, \sup_{\tau \leq s, t \in \mathbf{S}^1} |u_{k_j}(\tau,t) - z_0(t)| < \epsilon\} \in \mathbf{R}. \end{aligned}$$

Let $u_{k_j} \cdot s_j(\tau,t) = u_{k_j}(\tau + s_j, t)$. Replaceing u_{k_j} with $u_{k_j} \cdot s_j$ we may assume that u_{k_j} satisfies

$$|u_{k_j}(\tau,t) - z_0(t)| < \epsilon, \ \forall \tau < 0, \ t \in \mathbf{S}^1. \tag{7}$$

But that $u_{k_j}(\mathbf{R} \times \mathbf{S}^1)$ cannot be contained in 2ϵ-neighborhood of z_0 implies that

$$|u_{k_j}(0,t_j) - z_0(t_j)| = \epsilon \tag{8}$$

for a sequence $t_j \in \mathbf{S}^1$ and $j = 1, 2, \cdots$. Without loss of generality we may assume $t_j \to t^* \in \mathbf{S}^1$. Since $\omega|_{\pi_2(M)} = 0$ and $E(u_{k_j}) \leq \hbar_0/2$, the weak compactness arguments shows that after passing to a subsequence u_{k_j} converges to u_∞ in C^1_{loc}-topolgy, where u_∞ is a solution of the second equation in (6) and $E(u_\infty) \leq \hbar_0/2$. By (7) it holds that $|u_\infty(\tau,t) - z_0(t)| < \epsilon$ for all $\tau < 0$ and $t \in \mathbf{S}^1$. This leads to $|u_\infty(-\infty, t) - z_0(t)| \le \epsilon$ for all $t \in \mathbf{S}^1$. Now our definition of ϵ_0 above shows that $u_\infty(-\infty, \cdot) = z_0(\cdot)$. Moreover (8) implies that $|u_\infty(0, t^*) - z_0(t^*)| = \epsilon$. Hence u_∞ is nontrivial. It follows from the definition of $\hbar_0$ above that $E(u_\infty) \ge \hbar_0$. But Fatou lemma leads to $E(u_\infty) < \hbar_0/2$. This contradiction completes the proof.

Continuing Proof of Main Theorem. As in [6] it suffice to prove that if there exist m elements of positive degree $\alpha_i \in H^{l_i}(M; \mathbf{Z}_2)$, $i = 1, \cdots, m$, such that $(\alpha_1 \cap) \circ \cdots \circ (\alpha_m \cap)$ acts on the Floer homology $FH_*(M; \mathbf{Z}_2)$ nontrivially, then $\sharp \mathcal{P}_0(H) \ge m + 1$. By the definition of $\mathcal{P}_0(H)$ any sequence $x_k \in \mathcal{P}(H_k)$ approximating the elements of $\mathcal{P}_0(H)$ in C^1-topology. Under the above assumptions the definition of the cap action shows that there exist the sequences of elements $\tilde{z}_k^i \in \tilde{\mathcal{P}}(H_k)$, $i = 0, \cdots, m$, such that

$$\mu(\tilde{z}_k^i) - \mu(\tilde{z}_k^{i+1}) = l_i \quad \text{and} \quad m^{\alpha_j}(\tilde{z}_k^i, \tilde{z}_k^{i+1}) = 1.$$

Let $\tilde{z}_k^i = [z_k^i, v_k^i]$. Passing to a subsequence we may assume that $z_k^i \to z_0^i \in \mathcal{P}_0(H)$ in C^1-topology for $i = 0, \dots, m$. *We wish to prove that* $z_0^i \ne z_0^j$ for $i < j$. Otherwise, assume $z_0^i = z_0^j$ for some $i < j$. Since $\mathcal{M}(\tilde{z}_k^l, \tilde{z}_k^{l+1}; H_k, J_k) \ne \emptyset$ for $l = 0, \cdots, m$, it holds that $a_{H_k}(\tilde{z}_k^i) > a_{H_k}(\tilde{z}_k^{i+1}) \ge \cdots \ge a_{H_k}(\tilde{z}_k^j)$. Here

$$a_{H_k}(\tilde{z}) = -\int_{D^2} v^*\omega + \int_0^1 H(t, z(t))dt \quad \text{for } \tilde{z} = [z, v] \in \tilde{\mathcal{L}}(M). \tag{9}$$

For any $u_k^i \in \mathcal{M}(\tilde{z}_k^i, \tilde{z}_k^{i+1}; H_k, J_k)$ the direct computation shows

$$\begin{aligned} E(u_k^i) &= a_{H_k}(\tilde{z}_k^i) - a_{H_k}(\tilde{z}_k^{i+1}) \\ &\le a_{H_k}(\tilde{z}_k^i) - a_{H_k}(\tilde{z}_k^j) \\ &= -\int_{D^2}(v_k^i)^*\omega + \int_{D^2}(v_k^j)^*\omega + \int_0^1 H_k(t, z_k^i(t))dt - \int_0^1 H_k(t, z_k^j(t))dt. \end{aligned} \tag{10}$$

We say that the case (b) in the proof of lemma 5.2 of [6] cannot occur in the case of $\omega|_{\pi_2(M)} = 0$. That is, $\limsup_{k\to\infty}(a_{H_k}(\tilde{z}_k^i) - a_{H_k}(\tilde{z}_k^{i+1}))$ is not more than $\hbar_0/2$. In fact, for sufficiently large k both z_k^i and z_k^j are sufficiently close to $z_0^i = z_0^j$. Thus we can choose a homotopy $W_k^{ij} : \mathbf{S}^1 \times [0, 1] \to M$

between z_k^i and z_k^j whose image set is contained in a small neighborhood of $z_0^i(\mathbf{S}^1) = z_0^j(\mathbf{S}^1)$. Gluing it with v_k^i and v_k^j together one get a map $g_k^{ij} : \mathbf{S}^2 = D^2\sharp\mathbf{S}^1 \times [0,1]\sharp(-D^2) \to M$ and easily check its homotopy class $[g_k^{ij}]$ being independent of the choice of W_k^{ij}. Therefore (10) becomes

$$\int_0^1 H_k(t, z_k^i(t))dt - \int_0^1 H_k(t, z_k^j(t))dt - \int_{\mathbf{S}^1\times[0,1]} (W_k^{ij})^*\omega + \langle[\omega], [g_k^{ij}]\rangle.$$

But $\langle[\omega], [g_k^{ij}]\rangle = 0$. The third term and the difference of the first and second terms trend all to zero as $k \to \infty$ we arrive at the conclusion. Hence

$$\limsup_{k\to\infty}(a_{H_k}(\tilde{z}_k^i) - a_{H_k}(\tilde{z}_k^{i+1})) \le \limsup_{k\to\infty}(a_{H_k}(\tilde{z}_k^i) - a_{H_k}(\tilde{z}_k^j)) \le \hbar_0/2,$$

that is, only case (a) may holds. As in [6] one can still get a contradiction in this case. The main theorem is proved.

Remark 2. We here cannot generalize Main Theorem in [6] to the case of H being only C^1-smooth. The difficulty is that we now cannot ensure Claim 5.4 there holds yet. In fact, the Conley-Zehnder index of solutions in $\mathcal{P}(H)$ is not defined whether we take the methods in [9] or [6] since we cannot linearize the equation $\dot{x} = X_H(x)$ along its solutions under our case. Thus the proof of Claim 5.4 in [6] does not work for our case. On the other hand if there are two distinct $z_k, z_k' \in \mathcal{P}(H_k)$ such that both $\{z_k\}_{k=1}^\infty$ and $\{z_k'\}_{k=1}^\infty$ converge to some $z \in \mathcal{P}(H)$ in C^1-topology, then for bounding disks v_k and v_k' obtained by the procedure described in the proof of Lemma 5.2 of [6] we cannot ensure that the paths of symplectic matrices $\{\Phi_{z_k}(t)\}_{0\le t\le 1}$ and $\{\Phi_{z_k'}(t)\}_{0\le t\le 1}$ defined by (5.1) in [9] are C^0-close each other as $k \to \infty$. The reason is that the sequence $\{H_k\}_{k=1}^\infty$ is only C^1-convergent, and the definitions of $\Phi_{z_k}(t)$ and $\Phi_{z_k'}(t)$ involved the second deriviates of H_k which must not converge. Therefore, it is very interesting question whether one can find a C^1-smooth function on some closed symplectic manifold (M,ω) satisfying Main Theorem in [6] such that $\sharp\mathcal{P}(H) < \text{cuplength}(M) + 1$.

Finally, carefully checking the proofs of Theorem A.2.1 and Proposition A.2.5 in [6] it is not hard to find that the case (b) in the proof of Lemma

5.2 there cannot happen. Thus we may make use of our above methods to prove:

Proposition 3. *Let ω_i be the standard symplectic form on $\mathbf{CP}^{n_i}$ with $\langle[\omega_i],[\mathbf{CP}^1]\rangle = 1$ and $M = \mathbf{CP}^{n_1}\times\cdots\times\mathbf{CP}^{n_k}$. Denote by*

$$\Omega := (n_1+1)\omega_1 \oplus \cdots (n_k+1)\omega_k \quad \text{and} \quad \Omega' := m_1\omega_1 \oplus \cdots \oplus m_k\omega_k.$$

Here $m_1,\cdots,m_k$ are positive integers. Then for every C^1-smooth time-dependent Hamiltion $H : \mathbf{S}^1 \times M \to \mathbf{R}$ the number of contractible 1-periodic solutions of $\dot{x} = X_{H_t}(t,x)$ is greater than or equal to the greatest common divisor of $\{n_i+1\}$, but that of $\dot{x} = X'_{H_t}(t,x)$ is at least $\max\{(n_i+1)/m_i\}$. Here X_{H_t} and X'_{H_t} are given by $i_{X_{H_t}}\Omega = dH_t$ and $i_{X'_{H_t}}\Omega' = dH_t$ respectively.

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Department of Mathematics, Beijing Normal University, Beijing 100875, P. R. China.

E-mail: gclu@bnu.edu.cn.