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SOME MULTIVALENT STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS

BY

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Abstract. Let $A_p(n)$ $(p, n \in N)$ be the class of functions $f(z) = z^p + a_{p+n}z^{p+n} + \cdots$ which are analytic in the unit disk. By using the method of differential subordination, we investigate the multivalent starlikeness of certain analytic functions in $A_p(n)$ and their convolutions. Several interesting consequences are also given.

1. Introduction. Let $A_p(n)$ $(p, n \in N = \{1, 2, 3, \ldots\})$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A_p(n)$ is called p-valently starlike in E if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \quad (z \in E).$$

We denote by $S_p(n)$ the subclass of $A_p(n)$ consisting of functions $f(z)$ which are p-valently starlike in E. Also, we write $A_1(1) = A$ and $S_1(1) = S$.

Let $f(z)$ and $g(z)$ be analytic in E. Then we say that $f(z)$ is subordinate

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to $g(z)$, denoted by $f(z) \prec g(z)$, if there exists an analytic function $\omega(z)$ in E such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for $z \in E$. If $g(z)$ is univalent in E, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

A function $f(z) \in A_p(n)$ is said to be in the class $B_p(n, \alpha, \lambda)$ $(\alpha, \lambda > 0)$ if it satisfies

$$\left| \frac{zf'(z)}{pf(z)} \left(\frac{f(z)}{z^p} \right)^{\alpha} - 1 \right| < \lambda \quad (z \in E). \tag{1}$$

Note that $B_1(1, \alpha, \lambda)$ $(0 < \lambda \leq 1)$ is a subclass of Bazilevic functions in E and each function in $B_1(1, \alpha, \lambda)$ is univalent in E.

The celebrated Polya-Schoenberg conjecture proved by Ruscheweyh and Sheil-Small [4] states that if $f(z)$, $g(z) \in S$, then

$$\int_0^z \frac{(f * g)(t)}{t} dt \in S,$$

where $*$ stands for the Hadamard product or convolution. It is also known that $(f * g)(z)$ need not even be univalent in E for $f(z)$, $g(z) \in S$.

There are many papers in which various sufficient conditions for multivalent starlikeness were obtained. The object of the present paper is to investigate the multivalent starlikeness for certain functions in $A_p(n)$ and their convolutions.

2. Preliminaries.

We shall use the following lemmas to prove our results.

Lemma 1.([2]) *Let $h(z)$ be a convex function in E (i.e. $h(z)$ is analytic and univalent in E and $h(E)$ is a convex domain), $h(0) = 1$, and let $g(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ be analytic in E. If $g(z) + \frac{1}{c} z g'(z) \prec h(z)$, then*

$$g(z) \prec \frac{c}{n} z^{-c/n} \int_0^z t^{c/n-1} h(t) dt,$$

where $c \neq 0$ *and* $Rec \geq 0$.

Lemma 2. *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=1}^{\infty} b_n z^n$ *be analytic in* E. *If* $f(z) \prec g(z)$, *then*

$$\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2 \quad (n \in N).$$

This lemma is due to Rogosinski (see [1, p.192]).

3. Main results.

Theorem 1. *If* $f(z) \in B_p(n, \alpha, \lambda)$ *with*

$$\lambda = \frac{p\alpha + n}{[(p\alpha)^2 + (p\alpha + n)^2]^{1/2}}, \tag{2}$$

then $f(z) \in S_p(n)$.

Proof. We use a technique in [5]. Since $f(z) \in A_p(n)$ satisfies (1), we can write

$$\frac{zf'(z)}{pf(z)} \left(\frac{f(z)}{z^p} \right)^{\alpha} \prec 1 + \lambda z, \tag{3}$$

where λ is given by (2).

Let $g(z) = (f(z)/z^p)^{\alpha}$. Then $g(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic in E and

$$\frac{zf'(z)}{pf(z)} \left(\frac{f(z)}{z^p} \right)^{\alpha} = g(z) + \frac{1}{p\alpha} zg'(z).$$

Therefore, it follows from (3) that

$$g(z) + \frac{1}{p\alpha} zg'(z) \prec 1 + \lambda z,$$

and an application of Lemma 1 with $h(z) = 1 + \lambda z$ yields

$$g(z) \prec 1 + \frac{p\alpha\lambda}{p\alpha + n} z$$

or, equivalently,

$$\left| \left(\frac{f(z)}{z^p} \right)^\alpha - 1 \right| < \frac{p\alpha\lambda}{p\alpha + n} \quad (z \in E). \tag{4}$$

From (1) and (4), we easily have

$$\left| \arg \left\{ \frac{zf'(z)}{pf(z)} \left(\frac{f(z)}{z^p} \right)^\alpha \right\} \right| < \arcsin \lambda,$$

$$\left| \arg \left(\frac{f(z)}{z^p} \right)^\alpha \right| < \arcsin \frac{p\alpha\lambda}{p\alpha + n},$$

and hence

$$\begin{aligned} \left| \arg \frac{zf'(z)}{pf(z)} \right| &\leq \left| \arg \left\{ \frac{zf'(z)}{pf(z)} \left(\frac{f(z)}{z^p} \right)^\alpha \right\} \right| + \left| \arg \left(\frac{f(z)}{z^p} \right)^\alpha \right| \\ &< \arcsin \lambda + \arcsin \frac{p\alpha\lambda}{p\alpha + n} \\ &= \frac{\pi}{2} \quad (z \in E), \end{aligned} \tag{5}$$

which implies that $f(z) \in S_p(n)$. This completes the proof of the theorem.

For $\alpha = 1$, our theorem gives

Corollary 1. *If* $f(z) \in B_p(n, 1, \lambda)$ *with*

$$\lambda = \frac{p + n}{[p^2 + (p + n)^2]^{1/2}},$$

then $f(z) \in S_p(n)$.

Remark 1. For $p = n = 1$, Mocanu [3] proved the corollary by another method.

Putting $\alpha = 2$ in Theorem 1, we have

Corollary 2. *If* $f(z) \in B_p(n, 2, \lambda)$ *with*

$$\lambda = \frac{2p+n}{[4p^2 + (2p+n)^2]^{1/2}},$$

then $f(z) \in S_p(n)$.

Theorem 2. *Let* $Rec > -p\alpha$. *If* $f(z) \in B_p(n, \alpha, \lambda)$ *with*

$$\lambda = \frac{|c + p\alpha + n|(p\alpha + n)}{|c + p\alpha|\,[(p\alpha)^2 + (p\alpha + n)^2]^{1/2}}, \tag{6}$$

then the function $F(z)$ *defined by*

$$F(z) = \Big[\frac{c + p\alpha}{z^c} \int_0^z t^{c-1}(f(t))^\alpha dt\Big]^{1/\alpha}$$

belongs to $S_p(n)$.

Proof. It is clear that the function $F(z)$ is in $A_p(n)$. Differentiating both sides of the equality

$$z^c(F(z))^\alpha = (c + p\alpha) \int_0^z t^{c-1}(f(t))^\alpha dt,$$

we have

$$c(F(z))^{\alpha-1}F'(z) + (z(F(z))^{\alpha-1}F'(z))' = (c + p\alpha)(f(z))^{\alpha-1}f'(z). \tag{7}$$

Letting

$$G(z) = \frac{zF'(z)}{pF(z)} \left(\frac{F(z)}{z^p}\right)^\alpha = 1 + b_n z^n + \cdots,$$

then (7) becomes

$$G(z) + \frac{zG'(z)}{c + p\alpha} = \frac{zf'(z)}{pf(z)} \left(\frac{f(z)}{z^p}\right)^\alpha. \tag{8}$$

It follows from (1) and (8) that

$$G(z)+\frac{zG'(z)}{c+p\alpha}\prec 1+\lambda z,$$

where λ is given by (6).

Now, an application of Lemma 1 yields

$$G(z)\prec 1+\frac{\lambda(c+p\alpha)}{c+p\alpha+n}z,$$

and hence

$$\left|\frac{zF'(z)}{pF(z)}\left(\frac{F(z)}{z^p}\right)^{\alpha}-1\right|<\frac{p\alpha+n}{[(p\alpha)^2+(p\alpha+n)^2]^{1/2}} \tag{9}$$

for $z\in E$. By replacing $f(z)$ by $F(z)$ in Theorem 1, it follows from (9) that $F(z)\in S_p(n)$.

Taking $\alpha=1$ in Theorem 2, we obtain

Corollary 3. *If $Rec>-p$ and $f(z)\in B_p(n,1,\lambda)$ with*

$$\lambda=\frac{|c+p+n|(p+n)}{|c+p|\,[p^2+(p+n)^2]^{1/2}},$$

then

$$\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt\in S_p(n).$$

Remark 2. For $p=n=1$ and $c=Rec>-1$, Corollary 3 was proved by Mocanu [3].

Theorem 3. *Let $Re\beta\geq 0$, $\beta\neq 0$, and $0<\lambda<|p+n\beta|/p$. If $f(z)$ in $A_p(n)$ satisfies*

$$\left|(1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}-1\right|<\lambda\quad(z\in E), \tag{10}$$

then

$$\left|\frac{zf'(z)}{pf(z)}-1\right|<\frac{\lambda(p+|p+n\beta|)}{|\beta|(|p+n\beta|-p\lambda)}\quad (z\in E). \tag{11}$$

Proof. The function $g(z)=f(z)/z^p=1+b_nz^n+\cdots$ is analytic in E and it follows from (10) that

$$(1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}=g(z)+\frac{\beta}{p}zg'(z)\prec 1+\lambda z.$$

By Lemma 1, we have

$$\frac{f(z)}{z^p}\prec 1+\frac{p\lambda}{p+n\beta}z, \tag{12}$$

which implies that

$$\left|\frac{f(z)}{z^p}-1\right|<\frac{p\lambda}{|p+n\beta|},\quad \left|\frac{f(z)}{z^p}\right|>1-\frac{p\lambda}{|p+n\beta|}>0\quad \text{for}\quad z\in E. \tag{13}$$

Making use of (10) and (13), we deduce that

$$\begin{aligned}|\beta|\left|\frac{f'(z)}{pz^{p-1}}-\frac{f(z)}{z^p}\right| &\le \left|\frac{f(z)}{z^p}-1+\beta\left(\frac{f'(z)}{pz^{p-1}}-\frac{f(z)}{z^p}\right)\right|+\left|\frac{f(z)}{z^p}-1\right|\\ &< \frac{\lambda(p+|p+n\beta|)}{|p+n\beta|-p\lambda}\left(1-\frac{p\lambda}{|p+n\beta|}\right)\\ &< \frac{\lambda(p+|p+n\beta|)}{|p+n\beta|-p\lambda}\left|\frac{f(z)}{z^p}\right|\quad (z\in E),\end{aligned}$$

which yields (11) and the proof is complete.

From Theorem 3, we easily have

Corollary 4. *If* $Re\beta\ge 0$, $\beta\ne 0$, *and* $f(z)$ *in* $A_p(n)$ *satisfies*

$$\left|(1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}-1\right|<\frac{|\beta(p+n\beta)|}{p(1+|\beta|)+|p+n\beta|}\quad (z\in E),$$

then $f(z)\in S_p(n)$ *and* $|zf'(z)/(pf(z))-1|<1$ $(z\in E)$.

For $p=n=\beta=1$, Theorem 3 reduces to

Corollary 5. *If* $0<\lambda<2$ *and* $f(z)$ *in* A *satisfies* $|f'(z)-1|<\lambda$ $(z\in E)$, *then*

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{3\lambda}{2-\lambda}\quad (z\in E).$$

Theorem 4. *Let* $Re\beta_1\geq 0$, $Re\beta_2\geq 0$, $\beta_1\beta_2\neq 0$, *and let* $f(z)$, $g(z)\in A_p(n)$ *satisfy*

$$\left|(1-\beta_1)\frac{f(z)}{z^p}+\beta_1\frac{f'(z)}{pz^{p-1}}-1\right| < \lambda_1, \tag{14}$$

$$\left|(1-\beta_2)\frac{g(z)}{z^p}+\beta_2\frac{g'(z)}{pz^{p-1}}-1\right| < \lambda_2 \tag{15}$$

for $z\in E$, *where* $\lambda_1>0$, $\lambda_2>0$, *and*

$$\lambda_1\lambda_2\leq\left|\left(1+\frac{n\beta_1}{p}\right)\left(1+\frac{n\beta_2}{p}\right)\right|. \tag{16}$$

Then

$$\varphi(z)=p\int_0^z\frac{(f*g)(t)}{t}dt\in S_p(n) \tag{17}$$

and

$$\left|\frac{z\varphi'(z)}{p\varphi(z)}-1\right|<1\quad (z\in E). \tag{18}$$

The bound of $\lambda_1\lambda_2$ *in* (16) *is the best possible.*

Proof. Let $f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k}\in A_p(n)$ satisfy (14). Then, by (12) in the proof of Theorem 3, we have

$$\frac{f(z)}{z^p}=1+\sum_{k=n}^{\infty}a_{p+k}z^k\prec 1+\frac{p\lambda_1}{p+n\beta_1}z.$$

Thus an application of Lemma 2 yields

$$\sum_{k=n}^{\infty}|a_{p+k}|^2 \leq \left(\frac{p\lambda_1}{|p+n\beta_1|}\right)^2.$$

Similarly, if $g(z)=z^p+\sum_{k=n}^{\infty}b_{p+k}z^{p+k}\in A_p(n)$ satisfies (15), then

$$\sum_{k=n}^{\infty}|b_{p+k}|^2 \leq \left(\frac{p\lambda_2}{|p+n\beta_2|}\right)^2.$$

Therefore, by the Cauchy-Schwarz inequality and (16), we obtain

$$\begin{aligned}\sum_{k=n}^{\infty}|a_{p+k}b_{p+k}| &\leq \left(\sum_{k=n}^{\infty}|a_{p+k}|^2\right)^{1/2}\left(\sum_{k=n}^{\infty}|b_{p+k}|^2\right)^{1/2}\\ &\leq \frac{p^2\lambda_1\lambda_2}{|(p+n\beta_1)(p+n\beta_2)|}\leq 1.\end{aligned}\tag{19}$$

Since the function $\varphi(z)$ in (17) can be expressed by

$$\varphi(z)=z^p+\sum_{k=n}^{\infty}\frac{p}{p+k}a_{p+k}b_{p+k}z^{p+k},$$

it follows from (19) that

$$\begin{aligned}\left|\frac{z\varphi'(z)}{p\varphi(z)}-1\right| &= \left|\frac{\sum_{k=n}^{\infty}\frac{k}{p+k}a_{p+k}b_{p+k}z^k}{1+\sum_{k=n}^{\infty}\frac{p}{p+k}a_{p+k}b_{p+k}z^k}\right|\\ &< \frac{\sum_{k=n}^{\infty}\frac{k}{p+k}|a_{p+k}b_{p+k}|}{1-\sum_{k=n}^{\infty}\frac{p}{p+k}|a_{p+k}b_{p+k}|}\\ &\leq 1 \quad (z\in E).\end{aligned}$$

This proves (18).

By considering the functions

$$f(z) = z^p + \frac{p\lambda_1}{p+n\beta_1} z^{p+n}, \quad g(z) = z^p + \frac{p\lambda_2}{p+n\beta_2} z^{p+n},$$

it is easy to verify that the bound of $\lambda_1\lambda_2$ in (16) is sharp. The proof is now complete.

For $p = n = \beta_1 = \beta_2 = 1$, Theorem 4 gives the following

Corollary 6. *Let $f(z)$, $g(z) \in A$ satisfy $|f'(z)-1| < \lambda_1$, $|g'(z)-1| < \lambda_2$ $(z \in E)$, where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1\lambda_2 \leq 4$, then*

$$\varphi(z) = \int_0^z \frac{(f*g)(t)}{t} dt \in S$$

and $|z\varphi'(z)/\varphi(z) - 1| < 1$ for $z \in E$. The bound 4 of $\lambda_1\lambda_2$ is sharp.

Theorem 5. *Let $f(z)$, $g(z) \in A_p(n)$ satisfy*

$$\left|(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}} - 1\right| < \lambda_1, \tag{20}$$

$$\left|(1-\beta)\frac{g(z)}{z^p} + \beta\frac{g'(z)}{pz^{p-1}} - 1\right| < \lambda_2 \tag{21}$$

for $z \in E$, where $\beta > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and

$$\lambda_1\lambda_2 \leq 1 + n\beta/p. \tag{22}$$

If $\psi(z) = (f*g)(z)$, then

$$\left|\frac{z\psi'(z)}{p\psi(z)} - 1\right| < \frac{1}{\beta} \quad (z \in E). \tag{23}$$

The bound of $\lambda_1\lambda_2$ in (22) is the best possible.

Proof. Let $f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k}$ and $g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}$

be in $A_p(n)$. Then (20) is equivalent to

$$\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)a_{p+k}z^k \prec \lambda_1 z.$$

For $\beta > 0$, an application of Lemma 2 gives

$$\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)^2 |a_{p+k}|^2 \le \lambda_1^2,$$

and so

$$\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)|a_{p+k}|^2 \le \frac{p\lambda_1^2}{p+n\beta}.$$

Similarly, it follows from (21) and Lemma 2 that

$$\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)|b_{p+k}|^2 \le \frac{p\lambda_2^2}{p+n\beta}.$$

Hence, by the Cauchy-Schwarz inequality and (22), we have

$$\begin{aligned}
&\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)|a_{p+k}b_{p+k}| \\
\le\ &\left(\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)|a_{p+k}|^2\right)^{1/2}\left(\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)|b_{p+k}|^2\right)^{1/2} \qquad (24)\\
\le\ &\frac{p\lambda_1\lambda_2}{p+n\beta} \le 1.
\end{aligned}$$

Now, in view of (24), we deduce that

$$\left|\frac{z\psi'(z)}{p\psi(z)}-1\right| = \left|\frac{\sum_{k=n}^{\infty} k a_{p+k}b_{p+k}z^k}{p(1+\sum_{k=n}^{\infty} a_{p+k}b_{p+k}z^k)}\right| < \frac{\sum_{k=n}^{\infty} k|a_{p+k}b_{p+k}|}{p(1-\sum_{k=n}^{\infty}|a_{p+k}b_{p+k}|)} \le \frac{1}{\beta}$$

for $z \in E$, which proves (23).

Next, the functions

$$f(z) = z^p + \frac{p\lambda_1}{p+n\beta} z^{p+n}, \quad g(z) = z^p + \frac{p\lambda_2}{p+n\beta} z^{p+n} \tag{25}$$

show that the bound of $\lambda_1\lambda_2$ in (22) is sharp. This completes the proof.

Taking $\beta = 1$ in Theorem 5, we have

Corollary 7. *Let $f(z)$, $g(z) \in A_p(n)$ satisfy*

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < \lambda_1, \quad \left|\frac{g'(z)}{pz^{p-1}} - 1\right| < \lambda_2, \quad (z \in E),$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1\lambda_2 \le 1 + n/p$. Then $\psi(z) = (f * g)(z) \in S_p(n)$ and

$$\left|\frac{z\psi'(z)}{p\psi(z)} - 1\right| < 1 \quad (z \in E).$$

The bound $1 + n/p$ of $\lambda_1\lambda_2$ is sharp.

Theorem 6. *Let $Re\beta \ge 0$, $\beta \ne 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and let $f(z)$, $g(z) \in A_p(n)$ satisfy*

$$\begin{aligned}\left|(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}} - 1\right| &< \lambda_1, \\ \left|(1-\beta)\frac{g(z)}{z^p} + \beta\frac{g'(z)}{pz^{p-1}} - 1\right| &< \lambda_2\end{aligned}$$

for $z \in E$. Then

$$\left|(1-\beta)\frac{\psi(z)}{z^p} + \beta\frac{\psi'(z)}{pz^{p-1}} - 1\right| < \frac{p\lambda_1\lambda_2}{|p+n\beta|} \quad (z \in E), \tag{26}$$

where $\psi(z) = (f * g)(z)$. The bound in (26) is the best possible.

Proof. Proceeding as in the proof of Theorems 4 and 5, we have

$$\sum_{k=n}^{\infty}\left|1+\frac{k\beta}{p}\right|^2|a_{p+k}|^2\le\lambda_1^2,\quad \sum_{k=n}^{\infty}|b_{p+k}|^2\le\left(\frac{p\lambda_2}{|p+n\beta|}\right)^2,$$

and hence

$$\begin{aligned}\left|(1-\beta)\frac{\psi(z)}{z^p}+\beta\frac{\psi'(z)}{pz^{p-1}}-1\right| &= \left|\sum_{k=n}^{\infty}\left(1+\frac{k\beta}{p}\right)a_{p+k}b_{p+k}z^k\right|\\ &< \left(\sum_{k=n}^{\infty}\left|1+\frac{k\beta}{p}\right|^2|a_{p+k}|^2\right)^{1/2}\left(\sum_{k=n}^{\infty}|b_{p+k}|^2\right)^{1/2}\\ &\le \frac{p\lambda_1\lambda_2}{|p+n\beta|}\quad (z\in E).\end{aligned}$$

This proves (26).

The bound in (26) is sharp for the functions $f(z)$ and $g(z)$ given by (25) (with $Re\beta\ge 0$ and $\beta\ne 0$).

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