BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 33, Number 1, March 2005

HOMOTOPY THEORY IN GROUPOID ENRICHED CATEGORIES

BY

AKRUR BEHERA

Abstract. The concepts of h-limits, strong h-limits (and their duals) and partial proofs of homotopy limit reduction theorems relating to h-limits and strong h-limits are already known for a groupoid enriched category (g.e. category). In this paper the concepts of weak h-limits, quasi-limits (and their duals) are introduced in a g.e. category and the fuller version of the homotopy limit reduction theorems concerning the four types of limits, i.e., weak h-limits, h-limits, strong h-limits and quasi-limits are proved. The previously called Brown Complement Theorem is proved under the restricted assumption that the g.e. category admits only weak h-limits instead of h-limits and the generalized version of the Brown Complement Theorem is also proved which is relevant to the problem of showing under suitable smallness conditions that if a g.e. category admits all h-limits then it also admits all h-colomits.

1. Introduction. In [3] Fantham and Moore have discussed the technique and language of category theory for doing homotopy theory. They have presented a reasonable approach for the category-theoretic aspects of homotopy theory via an enriched category. They have also proposed some concepts that arise from spaces, maps, and homotopy classes of homotopies of maps. As it stands, they comprise a special type of 2-category [4] in

Received by the editors January 13, 2003.

AMS 2000 Subject Classification: 18D20.

Key words and phrases: Groupoid enriched category, homotopy limits, Brown Complement Theorem, smallness conditions.

which the morphism sets form a groupoid, and accordingly a groupoid enriched category (g.e. category) [3]. The one important difference between g.e. categories and ordinary categories is that the role of the category of sets in ordinary category theory is replaced by the category of groupoids. In ordinary category, the morphisms from X to Y form a set $\hom(X, Y)$, whereas in the g.e. category the morphisms from X to Y and the homotopies form a groupoid $\mathrm{Hom}(X, Y)$. The definitions of g.e. category, pseudo functor (p-functor), pseudo natural (p-natural) transformatiom, pseudo equivalence (p-equivalence) and their examples can be found in [3]. Also in [3] Fantham and Moore have extensively studied (strong) h-limits, h-pullbacks, h-equalizers, h-products and their dual notions namely, (strong) h-colomits, h-pushouts, h-coequalizers, h-coproducts in a g.e. category. We point out that there is a certain confusion in the statement of h-limits ([3], p.395). We introduce the concepts of weak h-limits and quasi-limits. Our main purpose of introducing the concept of weak h-limits is to obtain a generalized version of the Brown Complement Theorem. We use quasi-limits to prove that if a g.e. category admits all h-limits, then under appropriate smallness conditions, it also admits all h-colimits.

If we concentrate on the g.e. category $\mathcal{CW}$ of CW-complexes with continuous maps and homotopy classes of homotopies of maps then although a model for the h-product exists in $\mathcal{CW}$ ([3], p.409), the h-projection map for this model has scarcely any topological significance (although at this point one might have expected the h-projection map to be a fibration). In view of this observation we investigate a stronger version of the definitions of h-limits and h-colimits, which we call 2-limits and 2-colimits respectively. In order to define 2-limits (2-colimits) we introduce the concepts of pre 2-limits (pre 2-colimits) (which have already appeared with various names in general works on 2-categories [5]) and then we define 2-limit as pre 2-limit+h-limit and 2-colimit as pre 2-colimit+h-colimit, so that 2-limit always implies h-limit (whereas the converse is not true!) and dually 2-colimit implies h-colimit.

The definitions and narrative can be recalled from [3]. For a groupoid G, $\pi_0(G)$ denotes the isomorphism classes of points, i.e., the set of path components and $G(x,x)$ with identity e_x is written as $\pi_1(x)$ [3].

2. Homotopy limits in g.e. categories. Consider the diagram $T : \mathcal{I} \to \mathcal{C}$ of the graph $\mathcal{I}$ in the g.e. category $\mathcal{C}$ and the constant diagram $k_L : \mathcal{I} \to \mathcal{C}$ at the object L of $\mathcal{C}$ and a p-map $\theta : k_L \to T$ [3]. For each X in $\mathcal{C}$ we have a groupoid homomorphism θ^*_X: $\mathrm{Hom}(X, L) \to \mathrm{Hom}(k_X, T)$.

2.1 Definition. (a) The p-map θ and L are called the *weak h-limit* of T and *weak h-limit object* of T respectively if the groupoid homomorphism θ^*_X is a π_0-equivalence, i.e., the induced map $\pi_0(\theta^*_X) : \pi_0(\mathrm{Hom}(X, L)) \to \pi_0(\mathrm{Hom}(k_X, T))$ is a bijection. **(b)** θ is called an *h-limit* and L an *h-limit object* of T if θ^*_X is π_0-equivalence and π_1-surjective. Such a homomorphism of groupoids will be henceforth called an *h-equivalence.* **(c)** θ is called a strong *h-limit* and L a *strong h-limit object* [3] of T if θ^*_X is an equivalence (π_0-equivalence+π_1-equivalence) of groupoids. In [3] this limit is called a *pseudo limit*, but since the concept is stronger than h-limit for linguistic reasons we rename it: *strong h-limit.* **(d)** θ is called a *quasi-limit* and L a *quasi-limit object* of T if θ^*_X is π_0-surjective. **(e)** θ is called a *pre 2-limit* and L a *pre 2-limit object* of T if θ^*_X is surjective on objects. **(f)** θ is called a *2-limit* and L a *2-limit object* of T if θ^*_X is surjective on objects as well as an h-equivalence. **(g)** θ is called a *strong 2-limit* and L a *strong 2-limit object* of T if θ^*_X is an isomorphism of groupoids.

2.2 Note. In [3], 2.1 (a) is given as the definition of h-limit and there is a remark (Remark 2, p. 395) to the effect that 2.1 (b) follows. However the authors admit that their original explanation of this is lost and possibly incorrect and that we should regard what appears in [3] as an incorrect definition of h-limit and 2.1 (b) should take its place. However in cases 1 - 4 and examples ([3], p. 396-400), Limit Reduction Theorem ([3], p. 400) and

Brown Complement Theorem ([3], p. 406) 2.1 (b) is used for the definition of h-limit.

Now we define the contracted or substitute concepts of various limits. In fact by composing homotopies we can get alternative forms of various limits corresponding to pullbacks and equalizers in an ordinary category.

2.3 Definition. For the various forms of pullbacks of $f : A \to C$, $g : B \to C$ in $\mathcal{C}$ we consider the quintuplet $\{(u, f, \alpha, v, g) \mid \alpha : fu \simeq gv, u : P \to A, v : P \to B\}$ such that any other quintuplet $\{(r, f, \beta, s, g)|\beta : fr \simeq gs, r : X \to A, s : X \to B\}$ factorizes as $\varphi : r \simeq uk$, $\psi : vk \simeq s$, $k : X \to P$ with $\beta = (g*\psi)\cdot(\alpha*k)\cdot(f*\varphi)$ where **(1)** for a weak *h-pullback*, although there may be different factorizations, k is unique to within homotopy, **(2)** for an *h-pullback* if there exists another factorization $\varphi' : r \simeq uk'$, $\psi' : vk' \simeq s$, $k' : X \to P$ satisfying $\beta = (g * \psi') \cdot (\alpha * k') \cdot (f * \varphi')$, then there exists a homotopy $\mu : k \simeq k'$ such that $\varphi' = (u * \mu) \cdot \varphi$ and $\psi = \psi' \cdot (u * \mu)$, **(3)** for a *strong h-pullback* the homotopy μ in (2) is unique and **(4)** quasi-pullback in $\mathcal{C}$ means a quintuplet $\{(u, f, \alpha, v, g) \mid \alpha : fu \simeq gv, u : P \to A, v : P \to B\}$ such that any quintuplet $\{(r, f, \beta, s, g) \mid \beta : fr \simeq gs, r : X \to A, s : X \to B\}$ factorizes through it; **(5)** in the case of *pre* 2-*pulllback* there exists factorization with $r = uk$, $vk = s$, $k : X \to P$ and $\beta = \alpha * k$, **(6)** a pre 2-pullback becomes a 2-*pullback* if it is also an h-pullback and **(7)** finally a 2-pullback is a *strong* 2-*pullback* provided it is a strong h-pullback.

2.4 Definition. Given an indexing set $J + \{0\}$ with base point 0 and a family of maps f_0, $(f_i) : A \to B$, $i \in J$ we consider the quintuplet $\{(e, (f_i), (\alpha_i), e, f_0) \mid (\alpha_i) : (f_i)e \simeq f_0e, e : E \to A\}$ such that any other quintuplet $\{(e', (f_i), (\beta_i), e', f_0) \mid (\beta_i) : f_ie' \simeq f_0e', e' : X \to A\}$ factorizes as $\varphi : e' \simeq ek$, $k : X \to E$ with $\beta_i = (f_0*\varphi^{-1})\cdot(\alpha_i*k)\cdot(f_i*\varphi)$ where **(i)** for a *weak h-equalizer*, although there may be different factorizations, k is unique to within homotopy, **(ii)** for an h-equalizer if there exists another factorization $\varphi' : e' \simeq ek'$, $k' : X \to E$ satisfying $\beta_i = (f_0 * \varphi'^{-1}) \cdot (\alpha_i * k') \cdot (f_i * \varphi')$ for each

$i \in J$, then there exists a homotopy $\mu : k \simeq k'$ such that $e' = (e*\mu)\cdot\varphi$, **(iii)** for a strong *h-equalizer* the homotopy μ in (ii) is unique and **(iv)** *quasi-equalizer* in $\mathcal{C}$ means a quintuplet $\{(e, (f_i), (\alpha_i), e, f_0) \mid (\alpha_i) : (f_i)e \simeq f_0 e, e : X \to A\}$ such that any quintuplet $\{(e', (f_i), (\beta_i), e', f_0) \mid (\beta_i) : (f_i)e' \simeq f_0 e', e' : X \to A\}$ factorizes through it. **(v)** For maps $f,\ g : A \to B$ we obtain *pre* 2-*equalizer* as the quintuplet $\{(l, f, \alpha, l, g) \mid \alpha : fl \simeq gl, l : E \to A\}$ such that any other quintuplet $\{(f, r, \beta, g, r) \mid \beta : fr \simeq gr, r : X \to A\}$ factorizes as $r = lk$, $k : X \to E$, with $\beta = \alpha * k$. **(vi)** A pre 2-equalizer is a 2-*equalizer* if it is also an h-equalizer. **(vii)** A 2-equalizer is called a *strong* 2-*equalizer* if it is also a strong h-equalizer.

2.5 Definition. If the graph $\mathcal{I}$ is considered to be discrete in cases 2.1 (a-g) then L becomes the *weak h-product*, *h-product*, *strong h-product*, *quasi-product*, *pre* 2-*product*, 2-*product* and *strong* 2-*product* of the family of objects $\{T(i) : i \in I\}$ respectively.

2.6 Note. We observe that in a g.e. category $\mathcal{C}$

$$\begin{array}{ccccccc}
\text{strong 2-limit} & \Rightarrow & \text{2-limit} & \Rightarrow & \text{pre 2-limit} & & \\
\Downarrow & & \Downarrow & & & & \\
\text{strong } h\text{-limit} & \Rightarrow & h\text{-limit} & \Rightarrow & \text{weak } h\text{-limit} & \Rightarrow & \text{quasi-limit} \\
 & & & & & & \Downarrow \\
 & & & & \text{limit in } \pi\mathcal{C} & \Rightarrow & \text{quasi-limit in } \pi\mathcal{C}
\end{array}$$

where $\pi\mathcal{C}$ denotes the homotopy class category of $\pi\mathcal{C}$.

2.7 Examples. We present some examples of the various limits and substitute limits and their duals, other than given in [3]. We present some more examples in Section 5 also.

(1). Let $\mathcal{T}^*$ denote the g.e. category of pointed topological spaces with continuous base-point preserving maps and base-point preserving homotopy classes of base-point preserving homotopies of maps. For any $X \in \mathcal{T}^*$ we have an h-pullback $\{(p, i, \theta, p, i) \mid \theta : ip \simeq ip, i : * \to X, p : \Omega X \to *\}$ where ΩX is the loop space of X and $\theta : \Omega X \times I \to X$ is defined by $\theta(\omega, t) = \omega(t)$. Clearly this is not a strong h-pullback.

(2). Similarly in $\mathcal{T}^*$ we have an h-pushout $\{(j, q, \theta, j, q) \mid \theta : qj \simeq qj, j : X \to *, q : * \to \sum X\}$ where $\sum X$ is the suspension of X and $\theta : X \times I \to \sum X$ is defined by $\theta(x, t) = [x, t]$. Clearly this is not a strong h-pushout.

(3). Let $\mathcal{CW}^*$ denote the g.e. category of the compactly generated spaces of the homotopy type of well pointed CW-complexes with continuous base-point preserving maps and base-point preserving homotopy classes of base-point preserving homotopies of maps. Let $T : \mathcal{T}^* \to \mathcal{CW}^*$ be the realized singular complex functor, i.e., $T(Y) = |\text{Sing}Y|$. If $i : \mathcal{CW}^* \to \mathcal{T}^*$ is the inclusion functor, the map induced by the singular projection $|\text{Sing}Y| \to Y$ produces a homeomorphism $\eta_{X,Y} : \text{Hom}(X, |\text{Sing}Y|) \to \text{Hom}(i(X), Y)$ which is natural in X and Y and by traditional argument, $\eta_{X,Y}$ is a π_0-equivalence. In [3] it is proved that $\mathcal{CW}^*$ admits all h-limits and through $\eta_{X,Y}$ a model of the product in $\mathcal{CW}^*$ is obtained by $(X_i)_{i \in I} \to \text{Sing}(\prod_{i \in I} X_i)$ which is an h-product but not a 2-product.

(4). Let $\mathcal{T}^k$ be the g.e. category of compactly generated spaces with continuous maps and homotopy classes of homotopies. There is a standard h-pullback $\{(u, f, \theta, v, g) \mid \theta : fu \simeq gv, f : A \to C, u : P \to U, g : B \to C, v : P \to B\}$ of f, g by spaces of paths, i.e., the subspace of $A \times C^I \times B$ comprising (a, γ, b) with $\gamma(0) = a$, $\gamma(1) = b$ ($I = [0, 1]$, the product is that of compactly generated spaces rather than the topological product). This is a 2-pullback but not a strong 2-pullback because $\{(u\pi_P, f, \theta, u\pi_P, g) \mid \theta * \pi_P : fu\pi_P \simeq gv\pi_P, f : A \to C, u\pi_P : P \times D \to U, g : B \to C, v\pi_P : P \times D \to B\}$ with D contractible, is also a 2-pullback. Actually this example suggests what is additionally required for 2-limit to become a strong 2-limit.

(5). Topological spaces with respect to continuous maps and homotopy classes of homotopies of maps form a g.e. category $\mathcal{T}$. In $\mathcal{T}$ the 2-pullback of $f : A \to C$, $g : B \to C$ is given by $\{(p, f, \alpha, q, g) \mid \alpha : fp \simeq gq, p : P_{f,g} \to$

$A, q : P_{f,g} \to B\}$ where $P_{f,g} = \{(a, \omega, b) \in A \times C^I \times B : f(a) = \omega(0), g(b) = \omega(1)\}$, $\alpha : P_{f,g} \times I \to C$ is defined by $\alpha((a, \omega, b), t) = \omega(t)$ and p, q are the respective projections. This is not a strong 2-pullback.

(6). Chain complexes over a ring with chain maps and chain homotopy classes of chain maps form a g.e. category $\mathcal{C}\mathfrak{h}$. In $\mathcal{C}\mathfrak{h}$ the 2-pullback of $f : A \to X$, $g : B \to X$ is given by $\{(p, f, T, q, g) \mid T : fp \simeq gq, p : P \to A, q : P \to B\}$ where $P^n = \{(a, b, x, \overline{x}) \in A^n \oplus B^n \oplus X^n \oplus X^{n+1} : f^n(a) - g^n(b) = x + \partial_X^{n+1}(\overline{x})\}$, $T^n : P^n \to X^{n+1}$ is defined by $T^n(a, b, x, \overline{x}) = \overline{x}$ and p, q are the respective projections of chain complexes. This is not a strong 2-pullback.

(7). In $\mathcal{T}$ the 2-equalizer of $f, g : A \to B$ is given by $\{(p, f, \alpha, p, g) \mid \alpha : fp \simeq gp, p : E_{f,g} \to A\}$ where $E_{f,g} = \{(a, \omega) \in A \times B^I : f(a) = \omega(0), g(a) = \omega(1)\}$, $\alpha : E_{f,g} \times I \to B$ is defined by $\alpha((a, \omega), t) = \omega(t)$ and p is the projection map. This is not a strong 2-equalizer.

(8). In $\mathcal{C}\mathfrak{h}$ the 2-equalizer of chain maps $f, g : X \to Y$ is given by $\{(e, f, T, e, g) \mid T : fe \simeq ge, e : E \to X\}$ where $E^n = \{(x, y, \overline{y}) \in X^n \oplus Y^n \oplus Y^{n+1} : f^n(x) - g^n(x) = y + \partial_X^{n+1}(\overline{y})\}$, $T : E \to Y$ is defined by $T^n(x, y, \overline{y}) = \overline{y}$ and e is the usual projection. This is not a strong 2-equalizer.

(9). In $\mathcal{T}$ the 2-pushout of $f : A \to B$, $g : A \to C$ is given by $\{(f, i, \alpha, g, j) \mid \alpha : if \simeq jg, i : B \to C_{f,g}, j : C \to C_{f,g}\}$ where $C_{f,g} = (B \cup (A \times I) \cup C)/\{(a, 0) \sim f(a), (a, 1) \sim g(a)\}$, $\alpha : A \times I \to C_{f,g}$ is defined by $\alpha((a, t)) = [a, t]$ and i, j are the respective quotient maps. This is not a strong 2-pushout.

(10). In $\mathcal{C}\mathfrak{h}$ the 2-pushout of chain maps $f : X \to A$, $g : X \to B$ is given by $\{(f, i, T, g, j) \mid T : if \simeq jg, i : A \to Q, j : B \to Q\}$ where $Q^n = (A^n \oplus B^n \oplus X^n \oplus X^{n+1})/F^n$ with $F^n = \{(f^n(x), -g^n(x), -x, -\partial_X^n(X)) : x \in X^n\}$, $T : X \to Q$ is defined by $T^n(x) = [0, 0, 0, x]$ and i, j are the respective inclusions of the chain complexes. This is not a strong 2-pushout.

(11). In $\mathcal{T}$ the 2-coequalizer of $f, g : X \to Y$ is given by $\{(f, i, \alpha, g, i) \mid \alpha : if \simeq ig, i : Y \to M_{f,g}\}$ where $M_{f,g} = ((X \times I) \cup Y)/\{(x, 0) \sim$

$f(x),(x,1) \sim g(x)\}$, $\alpha : X \times I \to M_{f,g}$ is defined by $\alpha(x,t) = [x,t]$ and $i(y) = [y]$. This is not a strong 2-coequalizer.

(12). In $\mathcal{C}\mathfrak{h}$ the 2-coequalizer of $f, g : X \to A$ is given by $\{(f, q, T, g, q) \mid T : qf \simeq qg, q : A \to Q\}$ where $Q^n = (A^n \oplus X^{n-1} \oplus X^n)/L^n$ with $L^n = \{(f^n(x) - g^n(x), \partial^n_X(x), -x) : x \in X^n\}$, $T : X \to Q$ is defined by $T^n(x) = [0, x, 0]$ and $q^n(a) = [a, 0, 0]$. This is not a strong 2-coequalizer.

(13). In $\mathcal{T}^*$ the pushout of the inclusion maps $i : X \cap Y \hookrightarrow X$, $j : X \cap Y \hookrightarrow Y$ is $X \cup Y$ which is a quasi-pushout but not a weak h-pushout.

(14). In $\mathcal{T}^*$ for any $Y \in \mathcal{T}^*$ the pushout of $f : S^{n-1} \to Y$ and $i : S^{n-1} \to D^n$ is the space $Y \cup D^n$ which is a h-pushout but not a 2-pushout, D^n and S^{n-1} being the unit n-disk and unit n-sphere in $\mathbb{R}^n$ respectively.

3. The limit reduction theorems. In ([3], p. 400) the Limit Reduction Theorem is proved for h-limits only with the omission of a few steps. Since we are concerned with seven types of limits we give a fuller version of the same as follows. First we state and prove the Limit Reduction Theorem for the first four types of limits (2.1, (a) to (d))

3.1 Theorem. (The Limit Reduction Theorem) *Let the g.e. category $\mathcal{C}$ admit strong h-products. Then the following hold:*

(1) *$\mathcal{C}$ admits weak h-limits $\Leftrightarrow$ $\mathcal{C}$ admits weak h-equalizers $\Leftrightarrow$ $\mathcal{C}$ admits weak h-pullbacks.*

(2) *$\mathcal{C}$ admits h-limits $\Leftrightarrow$ $\mathcal{C}$ admits h-equalizers $\Leftrightarrow$ $\mathcal{C}$ admits h-pullbacks.*

(3) *$\mathcal{C}$ admits strong h-limits $\Leftrightarrow$ $\mathcal{C}$ admits strong h-equalizers $\Leftrightarrow$ $\mathcal{C}$ admits strong h-pullbacks.*

(4) *$\mathcal{C}$ admits quasi-limits $\Leftrightarrow$ $\mathcal{C}$ admits quasi-equalizers $\Leftrightarrow$ $\mathcal{C}$ admits quasi-pullbacks.*

Proof. We prove: (A) $\mathcal{C}$ admits pullbacks (1-4) $\Rightarrow \mathcal{C}$ admits equalizers (i-iv). (B) $\mathcal{C}$ admits equalizers (i-iv) $\Rightarrow \mathcal{C}$ admits pullbacks (1-4). (C) $\mathcal{C}$ admits equalizers (i-iv) $\Leftrightarrow \mathcal{C}$ admits limits (a-d).

(A): To construct the equalizers (i-iv) of the maps $f_1, f_2 : X \to Y$ consider the diagram $F : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a graph of the type $a, b : 1 \to 2$ and $F(1) = X$, $F(2) = Y$, $F(a) = f_1$, $F(b) = f_2$. We consider p-maps $k_Z \to F$ of the form $(l, m, (\alpha_i))$, $l : Z \to X$, $m : Z \to Y$, $\alpha_i : m \simeq f_i l$, $i = 1, 2$. Let $Y \times Y$ be the strong h-product of 2-copies of Y with projection maps p_i, $i = 1, 2$. We construct $\Delta : Y \to Y \times Y$ together with homotopies $\delta_i : 1_Y \simeq p_i \Delta$ and $h : X \to Y \times Y$ together with homotopies $\gamma_i : f_i \simeq p_i h$, $i = 1, 2$. Since $Y \times Y$ is a strong h-product, there is a unique homotopy $\alpha' : \Delta m \simeq hl$ with $\alpha_i = (\gamma_i^{-1} * l) \cdot (p_i * \alpha') \cdot (\delta_i * m)$, $i = 1, 2$. In this way we obtain a bijective correspondence $(l, m, (\alpha_i)) \mapsto (\Delta, m, \alpha', h, l)$ from maps $k_Z \to F$ to quintuplets $(m, \Delta, \alpha', l, h)$ extending the span of $h : X \to Y \times Y$ and $\Delta : Y \to Y \times Y$. Clearly this correspondence commutes with homotopies and with composition by a map $k : Z' \to Z$. Thus we conclude that a limit object (a-d) Z of F is the lead object of a pullback (1-4) and vice-versa. Relating the limit of F to equalizers, we obtain the required equalizers (i-iv) from pullbacks (1-4).

(B): For reasons of brevity this part was not given in [3]. We show that pullbacks (1-4) for $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ correspond to the equalizers (i-iv) for $f_1 p_1 : X_1 \times X_2 \to X$ and $f_2 p_2 : X_1 \times X_2 \to X$ where $X_1 \times X_2$ denotes a strong h-product of X_1 and X_2 with projections p_i, $i = 1, 2$. Corresponding to the quintuplet $\{(u_1, f_1, \alpha, u_2, f_2) \mid \alpha : f_1 u_1 \simeq f_2 u_2, u_1 : P \to X_1, u_2 : P \to X_2\}$ we construct homotopies $\alpha' : f_1 u_1 l \simeq f_2 u_2 l$, $\beta_i : u \simeq p_i l$, $i = 1, 2$, $l : P \to X_1 \times X_2$ such that $\alpha = (f_2 * \beta_2^{-1}) \cdot \alpha' \cdot (f_1 * \beta_1)$. The correspondence $\alpha' \leftrightarrow \alpha$ is bijective and commutes with composition of maps and homotopies and hence gives pullbacks from equalizers in all four cases (i)-(iv).

(C): Next we show that a limit object (a-d) of a diagram $S : \mathcal{I} \to \mathcal{C}$ can be constructed as an equalizer (i-iv) of strong h-products. Let P denote the strong h-product of the family $S(i)$ labelled over the points of $\mathcal{I}$ and let Q denote the strong h-product of $A(f)$ labelled over the arrows of $\mathcal{I}$ and for $f : i \to j$, $S(i)$, $S(j)$ are denoted by $D(f)$, $A(f)$ respectively. We construct maps $q, g : P \to Q$ together with homotopies $\delta_f : p_j \simeq p_j q$, $\gamma_f : S(f)p_i \simeq p_f g$, $p_f : Q \to A(f)$, $p_j : P \to A(f)$, $p_i : P \to D(f)$, $S(f) : D(f) \to A(f)$. Consider a p-map $((l_i), (\alpha_f)) : k_Z \to S$, $l_i : Z \to S(i)$, $\alpha_f : l_j \simeq S(f)l_i$. Corresponding to (l_i) we choose $l : Z \to P$ with homotopies $\varphi_i : l_i \simeq p_i l$. For $f : i \to j$ in $\mathcal{I}$ we write: $l_f = n_f$, $\varphi_j = \beta_f$, $l_i = m_f$, $\varphi_i = \varepsilon_f$. Since Q is a strong h-product we can find a unique homotopy $\alpha' : ql \simeq ql$ such that

$$(1) \qquad (S(f) * \varepsilon_f^{-1}) \cdot (\gamma_f^{-1} * l) \cdot (p_f * \alpha') \cdot (\delta_f * l) \cdot \beta_f = \alpha_f$$

and hence a cone $C' = (l, (q, g), \alpha')$. We consider the correspondence $((l_i), (\alpha_f)) \leftrightarrow (l, (\varphi_i), \alpha')$. Put $(l, (\varphi_i)) \sim (\bar{l}, (\overline{\varphi}_i))$ if there is a homotopy $\omega : l \simeq \bar{l}$ such that $\overline{\varphi}_i \cdot \omega = \varphi_i$ for all $i \in \mathcal{I}$. This is an equivalence relation and for given (l_i), (α_f) we extend this to an equivalence relation on triples by $(l, (\varphi_i), \alpha') \sim (\bar{l}, (\overline{\varphi}_i), \overline{\alpha}')$; in this case (relative to given (l_i), (α_f)), $\overline{\alpha}'$ is derived from (l_i), (φ_i) in the same way α' is derived from l, (φ_i). We denote the equivalence class by $< l, (\varphi_i), \alpha' >$ and we have a bijective correspondence

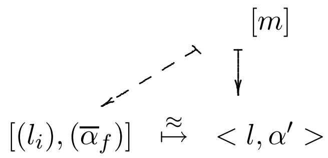
$$(2) \qquad ((l_i), (\alpha_f) \leftrightarrow < l, (\varphi_i), \alpha' > .$$

We also observe that by omitting the (φ_i) the resulting class of pairs $< l, \alpha' >$ (let us call it) is merely a homotopy class of equalizers $\{(l, q, \alpha', l, q) \mid \alpha' : ql \simeq ql\}$. We see easily that the correspondence given by (2) commutes with composition in the sense that $((l_i s), (\alpha_f * s)) \leftrightarrow < ls, (\varphi_i * s), a' * s >$ for appropriate s.

In order to examine the effect of a homotopy let $\theta_i : l_i \simeq \bar{l}_i$ determine a homotopy $\theta : ((l_i), (\alpha_f)) \simeq ((\bar{l}_i), (\overline{\alpha}_f))$, i.e., $\overline{\alpha}_f = (S(f) * \theta_i) \cdot \alpha_f \cdot \theta_j^{-1}$, $f : i \to j$. The correspondence (2) gives $((\bar{l}_i), (\overline{\alpha}_f)) \simeq < l, (\varphi_i \cdot \theta_i^{-1}), ? >$. We show that the undetermined homotopy $? = \alpha'$. Suppose that corresponding to $\overline{\alpha}_f$ we have a homotopy $? : ql \simeq gl$ such that $(S(f) * \overline{\varepsilon}_f^{-1}) \cdot (\gamma_f^{-1} * l) \cdot (p_f * ?) \cdot (\delta_f * l) \cdot \overline{\beta}_f = \overline{\alpha}_f$ where $\overline{\varepsilon}_f = \varepsilon_f \cdot \theta_i^{-1}$, $\overline{\beta}_f = \beta_f \cdot \theta_j^{-1}$. Thus $(S(f) * \varepsilon_f^{-1}) \cdot (\gamma_f^{-1} * l) \cdot (p_f * ?) \cdot (\delta_f * l) \cdot \beta_f = \alpha_f$ and hence by the uniqueness condition of the definition of strong h-product, $? = \alpha'$.

We note that if the equalizer (weak h-equalizer or quasi-equalizer) of q, g exists, then we have

(3)



with $m : Z \to E$ and $l \simeq um$ for an equalizer $\{(u, q, \beta, g, u) \mid \beta : qu \simeq gu, u : E \to P\}$. Hence through the reverse horizontal correspondence we have respectively a bijection (in the case of weak h-equalizer) and a surjection (in the case of quasi-equalizer) $[Z, E] \to [k_Z, S]$. Since the horizontal correspondence in the diagram (3) commutes with composition $W \to Z \to E$, it follows that E is displayed as the weak h-limit or quasi-limit of S respectively and vice-versa by some $\xi : k_E \to S$ corresponding to the identical maps $E \to E$ on the left hand side.

For the remainder (cases (2) and (3)) we consider the above bijection $[Z, E] \to [k_Z, S]$. Let $\alpha_f : l_j \simeq S(f) \cdot l_i$ and $\{(l, q, \alpha', l, g) \mid \alpha' : ql \simeq gl, l : Z \to P\}$ correspond through the original equivalence (depending on the choice (φ_i)). Let $\eta = (\eta_i)$, $\eta_i : l_i \simeq l_i$, define a homotopy of the cone $((l_i), (\alpha_f))$ to itself; this subjects the homotopies (η_i) to the condition $(S(f) * \eta_i) \cdot \alpha_f = \alpha_f \cdot \eta_j$. By the definition of strong h-product the homotopies η_i determine, through (φ_i), a unique homotopy $\eta' : l \simeq l$ such that $\varphi_i \cdot \eta_i = (p_i * \eta') \cdot \varphi_i$ for each i. This is a homotopy of the equalizer $\{(l, q, \alpha', l, g) \mid \alpha' : ql \simeq$

$gl, l : Z \to P\}$ with itself, i.e., $(g * \eta') \cdot \alpha' = \alpha' \cdot (q * \eta')$. Indeed, we have $(p_f * \alpha') \cdot ((p_f q) * \eta') \cdot (\delta_f * l) \cdot \beta_f = p_f(\alpha' \cdot (q * \eta')) \cdot (\delta_f * l) \cdot \beta_f = p_f((g * \eta') \cdot \alpha') \cdot (\delta_f * l) \cdot \beta_f = ((p_f g) * \eta') \cdot (p_f * \alpha') \cdot (\delta_f * l) \cdot \beta_f = ((p_f g) * \eta') \cdot (\gamma_f * l) \cdot (S(f) * \varepsilon_f) \cdot \alpha_f$ by (1). Conversely, by the reverse argument, each $\eta : l \simeq l$ gives a homotopy of the equalizer and determines unique homotopies $\eta_i : l_i \simeq l_i$ with $\varphi_i \cdot \eta_i = (p_i * \eta') \cdot \varphi_i$ for each i. This gives a homotopy of the given cone to itself. Thus the correspondence $\eta \leftrightarrow \eta'$ gives the bijection $\pi_1(k_Z, S; (l_i), (\varphi_i)) \to G(l, \alpha')$ where for $f : A \to B$, $\pi_1(A, B; f)$ denotes the group of homotopies $f \simeq f$ and $G(l, \alpha')$ is the group of homotopies in the groupoid $G(Z, q, g)$ of equalizers around the given equalizer (l, α'). It is evident that this bijection is a group isomorphism; indeed $(p_i * (\eta' \cdot \xi')) \cdot \varphi_i = (p_i * \eta') \cdot (p_i * \xi') \cdot \varphi_i = (p_i * \eta') \cdot \varphi_i \cdot \xi_i = (\varphi_i \cdot \eta_i) \cdot \xi_i = \varphi_i \cdot (\eta_i \cdot \xi_i)$.

Let $\{(u, q, \beta, u, g) \mid \beta : qu \simeq gu, u : E \to P\}$ be the weak h-equalizer of q, g and let it correspond to $m : Z \to E$ through $\lambda : um \simeq l$. We replace this equalizer by trivializing λ (i.e., $(um, \beta * m)$ which is a point in the same component (l, α')) of $G(Z, q, g)$. By composition the homotopy $\mu : m \simeq m$ determines a homotopy $(um, \beta * m) \simeq (um, \beta * m)$ of the replaced equalizer to itself and we get $c_Z : \mathrm{Hom}(Z, E) \to G(Z, q, g)$ which is an equivalence in case the equalizer is a strong h-equalizer and an h-equivalence (since it is a π_0-equivalence and by the proof for cases (1) and (4) it is π_1-surjective) in case the equalizer is merely an h-equalizer. We thus have

$$\begin{array}{ccc} & & \pi_1(Z, E; m) \\ & \swarrow & \downarrow \\ \pi_1(k_Z, S; (l_i), \alpha_f) & \xrightarrow{\approx} & G(l, \alpha') \end{array}$$

and $um = l$, $\beta * m = \alpha'$. The combined map $a_Z : \pi_1(Z, E; m) \to \pi_1(k_Z, S; (l_i), \alpha_f)$ commutes with the composition of maps because the horizontal and vertical maps do so. Thus from $\xi : k_E \to S$ defined in the first part we obtain by composition $b_Z : \mathrm{Hom}(Z, E) \to \mathrm{Hom}(k_Z, S)$ (which gives, on application of the path-class operator, the map $[Z, E] \to [k_Z, S]$

of the first part) and it is this map of groupoids that restricts to a_Z on appropriate path-groups. Thus the map b_Z is respectively an equivalence or h-equivalence when c_Z is so.

Next we use the above theorem to prove the Limit Reduction Theorem (Theorem 3.3, below) concerning pre 2-limits and 2-limits.

3.2 Note. It is to be noted that in [7] the concepts of pre 2-limits have already appeared before under various names in numerous works on 2-categories. Here we compare the lay-out the proof of Theorem 3.3 (theorem below) with that of Theorem 3.1 and see how greatly simplified the later would become if strong 2-products were available, namely through the omission of the homotopies (δ_i), (β_i) in parts (A), (B) and all of the homotopies γ, ε, δ, β in part (C) so that the homotopy α' is resulted by compounding the homotopies (α_f) into the product Q.

3.3 Theorem. *Let the g.e. category $\mathcal{C}$ admit strong 2-products. Then $\mathcal{C}$ admits 2-limits $\Leftrightarrow$ it admits 2-equalizers $\Leftrightarrow$ it admits 2-pullbacks.*

Proof. First we obtain 2-pullbacks from 2-equalizers. For the maps $f_i : A_i \to B$, $i = 1, 2$ we use ordinary product notation $A_1 \times A_2$ for strong 2-products, $p_i : A_1 \times A_2 \to A_i$ and if $\{(e, f_1p_1, \alpha, e, f_2p_2) \mid \alpha : f_1p_1e \simeq f_2p_2e, e : E \to A_1 \times A_2\}$ is a 2-equalizer then clearly $\{(p_1e, f_1, \alpha, p_2e, f_2) \mid \alpha : f_1p_1e \simeq f_2p_2e\}$ is a pre 2-pullback and hence a 2-pullback by Theorem 3.1.

Next we obtain 2-equalizers from 2-pullbacks. For f_1, $f_2 : A \to B$ if $\{e, (f_1, f_2), \alpha, e', \Delta) \mid \alpha : (f_1, f_2)e \simeq \Delta e', e : E \to A, e' : E \to B, \Delta : B \to B \times B\}$ is a 2-pullback, then clearly $\{(e, f_1, (q_2 * \alpha^{-1}) \cdot (q_1 * \alpha), e, f_2 \mid (q_2 * \alpha^{-1}) \cdot (q_1 * \alpha) : f_1e \simeq p_2e, q_i : B \times B \to B, i = 1, 2\}$ is a pre 2-equalizer and again the proof is completed by Theorem 3.1.

Next we show that a 2-limit object of a diagram $T : \mathcal{I} \to \mathcal{C}$ can be constructed as 2-equalizer of strong 2-products. For each arrow a in $\mathcal{I}$, we have $T(a) : T(d(a)) \to T(r(a))$, where $d(a)$ and $r(a)$ are the domain and range of a respectively. Let A be the strong 2-product of the family $\{T(i)\}$ labelled over the points of $\mathcal{I}$ and let B denote the strong 2-product of the family $\{T(r(a))\}$ labelled over the morphisms of $\mathcal{I}$ with projection maps $p_i : A \to T(i)$ and $p_{r(a)} : B \to T(r(a))$ respectively. By strong 2-product, for each arrow $a \in \mathcal{I}$, we obtain maps f, $g : A \to B$ such that $p_{r(a)} = p'_{r(a)}f$ and $T(a)p_{d(a)} = p'_{r(a)}g$ respectively. Let $\{(e, f, \varepsilon, e, g) \mid \varepsilon : fe \simeq ge, e : E \to A\}$ be the 2-equalizer of f, g. For each $a : i \to j$ in $\mathcal{I}$, let $i = d(a)$ and $j = r(a)$ and observe that we have a homotopy cone $p'_{r(a)} * \varepsilon^{-1} : T(a)p_{d(a)}e \simeq p_{r(a)}e$. For an arbitrary cone $q_i : X \to T(i)$, $i \in I$ with $\theta : T(a)q_{d(a)} \simeq q_{r(a)}$, by strong 2-product we obtain $h : X \to A$ such that $q_i = p_ih$ for each $i \in \mathcal{I}$ and hence a homotopy $\varphi : fh \simeq gh$ such that $\theta^{-1} = p'_{r(a)} * \varphi$. By 2-equalizer there is a factorization $l : X \to E$ with $el = h$ and $\varphi = \varepsilon * l$. Thus $q_i = p_ih = p_iel = (p_ie)l$. The rest follows from Theorem 3.1.

Conversely if $\mathcal{C}$ admits all 2-limits we consider the graph $\mathcal{I}$ with two objects 1, 2 and two arrows j, $j' : 1 \to 2$ only and the diagram $T : \mathcal{I} \to \mathcal{C}$ given by $T(1) = A$, $T(2) = B$, $T(j) = f$, $T(j') = g$. If $\theta : k_L \simeq T$ is the 2-limit of T we show that the 2-equalizer of f and g is $\{(\theta_1, f, \alpha, \theta_1, g) \mid \alpha : f\theta_1 \simeq g\theta_1\}$ where $\varphi : f\theta_1 \simeq \theta_2$, $\psi : g\theta_1 \simeq \theta_2$, $\alpha = \psi^{-1} \cdot \varphi$. Let $\beta : f\xi_1 \simeq g\xi_1$, $\xi_1 : X \to A$ and $f\xi_1 = \xi_2$; so $\beta^{-1} = g\xi_1 \simeq \xi_2$. By the presence of 2-limit there is a factorization $h : X \to L$ such that $\xi_1 = \theta_1 h$. By the choice of $f\xi_1 = \xi_2$, the homotopy $\beta^{-1} \cdot (\alpha * h)$ is trivial, i.e., $\beta = \alpha * h$. The rest follows from the analysis of h-equalizer in ([3], p.398).

3.4 Note. Theorems 3.1 and 3.3 can be dualized in the obvious way.

4. Fibrations and cofibrations in g.e. categories. We now investigate what is additionally needed for an h-limit to be 2-limit.

4.1 Definition. A map $f : X \to Y$ in a g.e. category C is called a *fibration* if for any map $g_0 : Z \to X$ and any homotopy $\alpha : fg_0 = h_0 \simeq h_1$, h_0, $h_1 : Z \to Y$ there exists a homotopy $\beta : g_0 \simeq g_1$, $g_1 : Z \to X$, such that $fg_1 = h_1$ and $f * \beta = \alpha$. The dual concept is that of *cofibration.*

4.2 Proposition. *If* $\{(u, f, \alpha, v, g) \mid \alpha : fu \simeq gv, u : P \to A, f : A \to C, v : P \to B, g : B \to C\}$ *is a* 2*-pullback, then* u, v *are joint fibrations in the sense that for every map* $k : X \to P$ *and homotopies* $\varphi : uk \simeq u'$, $u' : X \to A$, $\psi : vk \simeq v'$, $v' : X \to B$ *there is a homotopy* $\mu : k \simeq k'$, $k' : X \to P$ *such that* $u' = uk'$, $v' = vk$ *and* $\varphi = u * \mu$, $\psi = v * \mu$. *Conversely if* $\{(u, f, \alpha, v, g) \mid \alpha : fu \simeq gv, u : P \to A, f : A \to C, v : P \to B, g : B \to C\}$ *is an* h*-pullback and* u, v *are joint fibrations then the given* h*-pullback is a* 2*-pullback.*

Proof. Suppose we have a map $k : X \to P$ and homotopies $\varphi : uk \simeq u'$, $\psi : vk \simeq u'$. Let $\Phi = (g * \psi) \cdot (\alpha * k) \cdot (f * \varphi^{-1}) : fu' \simeq gv'$. By the definition of pre 2-pullback there exists a map $k' : X \to P$ such that $u' = uk'$, $v' = vk'$. By the definition of h-pullback there exists a homotopy $\mu : k \simeq k'$ such that $\varphi \cdot (u * \mu^{-1}) = e_{u'}$ (the trivial homotopy at u') i.e., $\varphi = u * \mu$ and $\psi = v * \mu$. Hence u, v are joint fibrations.

To prove the converse we consider $\{(u, f, \beta, v, g) \mid \beta : fu \simeq gv, u : X \to A, f : A \to C, v : X \to B, g : B \to C\}$. By the definition of h-pullback there exist a map $h : X \to P$ and homotopies $\varepsilon : uh \simeq p$, $\delta : vh \simeq q$ such that $\beta = (g * \delta) \cdot (a * h) \cdot (f * \varepsilon^{-1})$. Since u, v are joint fibrations there exists a homotopy $\eta : h \simeq h'$, $h' : X \to P$ such that $uh' = p$, $vh' = q$ and $\varepsilon = u * \eta$, $\delta = v * \eta$. Thus $\beta = ((gv) * \eta) \cdot (a * h) \cdot ((fu) * \eta^{-1})$. As composition is a functor of two variables, we get $(\alpha * h') \cdot ((fu) * \eta) = ((gv) * \eta) \cdot (a * h)$ and using this we get $\beta = \alpha * h'$. Thus we get the required 2-pullback.

4.3 Proposition. *If the g.e. categoery* C *is closed under* 2*-pullbacks, then any map* $f : X \to Y$ *can be factored as* $f = pt$ *where* p *is a fibration* *and t is a homotopy equivalence.*

Proof. Let $\{(p, 1_Y, \varphi, q, f) \mid \varphi : 1_Y p \simeq fq, f : X \to Y, 1_Y : Y \to Y\}$ be the 2-pullback of f, 1_Y. Let e_f be the trivial homotopy at f. By the definition of 2-pullback there is a factorization $pt = f$, $qt = 1_X$ and $\varphi * t = e_f$, $t : X \to P$. By Proposition 4.2, p is a fibration and P is homotopy equivalent to X since $\{(p, 1_Y, q, f) \mid \varphi : 1_Y p \simeq fq, f : X \to Y, 1_Y : Y \to Y\}$ is clearly an h-pullback and limit objects of h-pullbacks are unique to within homotopy equivalence.

The next result proves a similar fact for products.

4.4 Proposition. *If P is a 2-product of a family $\{X_i : i \in I\}$ then the family of maps $p_i : P \to X_i$ has joint fibration property in the sense that if $k : X \to P$ and homotopies $\alpha_i : p_i k \simeq h_i$ are given then there is a homotopy $\mu : k \simeq k'$ such that $p_i k' = h_i$ and $p_i * \mu = \alpha_i$ for each $i \in I$. Conversely if P is an h-product with projections $p_i : P \to X_i$ then P is a 2-product with respect to these projections if the family of maps p_i has the joint fibration property.*

Proof. Suppose that we are given a map $k : X \to P$ and homotopies $\alpha_i : p_i k \simeq h_i$. By the definition of pre 2-product there is a factorization $p_i k' = h_i$, $k' : X \to P$. By the definition of h-product there exists a homotopy $\mu : k \simeq k'$ such that $(p_i * \mu) \cdot \alpha_i^{-1} = e_{h_i}$ (trivial homotopy at h_i) i.e., $\alpha_i = p_i * \mu$. Thus the family of maps $p_i : P \to X_i$ has joint fibration property.

To prove the converse consider the family of maps $f_i : Z \to X_i$, $i \in I$. By the definition of h-product there exist a map $h : Z \to P$ and homotopies $\beta_i : f_i \to p_i h$. Since the family of maps $p_i : P \to X_i$ has joint fibration property, there exists a homotopy $v : h \simeq h'$, $h' : Z \to P$ such that $p_i h' = f_i$ and $p_i * v = \beta_i^{-1}$. Hence P is the pre 2-product of the family $\{X_i : i \in I\}$.

Propositions 4.2, 4.3 and 4.4 can be dualized in the obvious way.

4.5 Example. Each projection map p_i of a 2-product is a fibration (i.e., we augument a homotopy for a single $i \in I$ to a family of homotopies for the whole set I).

5. Extensions of the Brown Complement Theorem with applications. We will use the following forms of limits, pullbacks and equalizers in the g.e. category $\mathcal{G}$ of groupoids with respect to homomorphisms and homotopies.

Limits in $\mathcal{G}$: For a diagram $S : \Gamma \to \mathcal{G}$ the points of the limit object L are given by $L = \{(x_i, \alpha_f) : i \in \Gamma,\ f : i \to j \text{ in } \Gamma,\ x_i \in S(i),\ \alpha_f : S(f)(x_i) \simeq x_j\}$. A path $(\gamma_i) : (x_i, \alpha_f) \simeq (x'_i, \alpha'_f)$ in L is a family of paths $\gamma_i : x_i \simeq x'_i$ in $S(i)$ for each $i \in \Gamma$ with $\alpha'_f \cdot S(f)(\gamma_i) = \gamma_j \cdot \alpha_f$ (*limit path condition*). The cone $\theta : k_L \to S$ is given by projections $\theta_i : ((x_i, \alpha_f)) = x_i$, $\theta_f : ((x_i, \alpha_f)) = \alpha_f$. For a cone $\xi : k_X \to S$, we obtain the factorization through $X \to L$ given by $x \mapsto (\xi_i(x), \xi_f(x))$ and for $\alpha : x \to y$, $\alpha \mapsto \xi_i(\alpha))$. Thus $\theta : k_L \to S$ is a pre 2-limit. This is a strong 2-limit in the sense that for a homotopy of maps $\tau : \xi \simeq \xi'$ where ξ, $\xi' : k_X \to S$ and ξ, ξ' are obtained from the limit cone θ by $\xi = \theta \cdot k_h$, $\xi' = \theta \cdot k_h$ then there is a unique homotopy $\eta : h \simeq h'$ such that $\tau = \theta * \eta$. One checks that $\tau : x \mapsto (\tau_i(x))$. Note that for Γ discrete this becomes the concept of strong 2-product and also note that for 2-products a strong 2-limit as presently described is unique to within isomorphism.

Pullbacks in $\mathcal{G}$: For $f : A \to C$, $g : B \to C$ consider the quintuplet $\{(u, f, \alpha, v, g) \mid \alpha : fu \simeq gv, u : P \to A, v : P \to B\}$ where the points of P are given by $P = \{(a, \alpha, b) : a \in A_0, b \in B_0, \alpha : f(a) \simeq g(b)$ (A_0 means the point of A) and a path in P is given by $(\theta, \varphi) : (a, \alpha, b) \simeq (a', \alpha', b')$ with $\theta : a \simeq a'$, $\varphi : b \simeq b'$ and $g(\varphi) \cdot \alpha = \alpha' \cdot f(\theta)$ and u, v are given by $u : (a, \alpha, b)$, $(\theta, \varphi) \mapsto a$, θ and $v : (a, \alpha, b)$, $(\theta, \varphi) \mapsto b$, φ. For a general quintuplet $\{(l, f, \gamma, h, g) \mid \gamma : fl \simeq gh, l : X \to A, h : X \to B\}$ associated to f, g we factorise the first quintuplet via, $k : X \to P$ given by $x \mapsto (l(x), \gamma(x), h(x))$ and for a path $\beta : x \simeq y$, $\beta \mapsto (l(\beta), h(\beta))$. This shows that the first quintuplet is a pre 2-pullback. It is also a strong 2-pullback: For if $(\overline{\theta}, \overline{\varphi}) : (l, \gamma, h) \simeq (l', \gamma', h')$ is a homotopy of the quintuplets in the sense that $\overline{\theta} : l \simeq l'$, $\overline{\varphi} : h \simeq h'$ and $(g * \overline{\varphi}) \cdot \gamma \cdot (f * \overline{\theta}^{-1}) = \gamma'$, $l' : X \to A$, $h' : X \to B$ and the two quintuplets are obtained by factorization through k, $k' : X \to P$ then there is a unique homotopy $\mu : k \simeq k'$ such that $u * \mu = \overline{\theta}$, $v * \mu = \overline{\varphi}$; specifically, $\mu : x \mapsto (\overline{\theta}(x), \overline{\varphi}(x))$. It is easy to see that a strong 2-pullback in this sense is unique to within isomorphism.

Equalizers in $\mathcal{G}$: For a family of maps (f_i), $f_0 : X \to Y$, $i \in J$, J being an index set with base point 0, consider the multi-quintuplet $\{(u, (f_i), (\xi_i), u, f_0) \mid (\xi_i) : (f_i)u \simeq f_0 u, u : E \to X\}$ where the points of E are given by $E = \{(x, (\xi_i)) : x \in X_0, \xi_i : f_i(x) \to f_0(x), i \in (J - \{0\})\}$ and the paths in E are $\theta : (x, (\xi_i)) \to (x', (\xi_i'))$ with $\theta : x \simeq x'$ and $f_0(\theta) \cdot \xi_i = \xi_i' \cdot f_i(\theta)$ and u is given by $u : (x, \xi_i)$, $\theta \mapsto x$, θ. A general multi-quintuplet $\{(v, (f_i), (\beta_i), v, f_0) \mid (\beta_i) : (f_i)v \simeq f_0 v, v : Q \to X\}$ is factorized via, $k : Q \to E$ given by $z \mapsto (v(z), (\beta_i)(z))$ and for $\zeta : z \mapsto w$, $\zeta \mapsto v(\zeta)$. It is easy to check that E is a strong multiple 2-equalizer of the family of maps (f_i), f_0 in the sense that if two multi-quintuplets $\{(v, (f_i), (\beta_i), v, f_0) \mid (\beta_i) : (f_i)v \simeq f_0 v, v : Q \to X\}$ and $\{(v', (f_i), (\beta_i'), v', f_0) \mid (\beta_i') : (f_i)v' \simeq f_0 v', v' : Q \to X\}$ with $\varphi : v \simeq v'$ and $(f_0 * \varphi) \cdot \beta_i \cdot (f_0 * \varphi^{-1}) = \beta_i'$, are factored through k, $k' : Q \to E$, then there is a unique homotopy $\mu : k \simeq k'$ such that $u * \mu = \varphi$.

For what occurs later, we work out some induced limit maps.

5.1 Proposition. *In the g.e. category* $\mathcal{G}$ *let* $\{(u, (f_i), (\alpha_i), u, f_0) \mid \alpha_i : f_i u \simeq f_0 u, u : E \to X)\}$ *and* $\{(u', (f_i'), (\alpha_i'), u', f_0') \mid \alpha_i' : f_i' u \simeq f_0' u', u' : E' \to X)\}$ *be the multiple h-equalizers of the family of maps* f_i, $f_0 : X \to Y$ *and* f_i', $f_0' : X' \to Y'$ *respectively. Let* $p : X \to X'$ *be* π_0*-surjective,* $q : Y \to Y'$ *be* π_1*-surjective and* $\varphi_i : f_i' p \simeq q f_i$, $\varphi_0 : f_0' p \simeq q f_0$, $i \in J$, J *being an index set with base point* 0. *Then there exists a map* $k : E \to E'$ *with* $pu \simeq u'k$ *and* k *is* π_0*-surjective.*

Proof. Points of E are $(x,(\xi_i))$, $x \in E_0$ with $\xi_i : f_i(x) \to f_0(x)$ in Y and paths are $\theta : (x,(\xi_i)) \to (x',(\xi'_i))$ with $\theta : x \to x'$ and $\xi'_i = f_0(\theta)\cdot\xi_i\cdot f_i(\theta)^{-1}$; similarly for E'. We define groupoid homomorphism $k : E \to E'$ by setting $k(x,(\xi_i)) = (p(x), \varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x))$ whose right hand side is in E' and for a path $\theta : (x,(\xi_i)) \to (x',(\xi'_i))$ setting $k(\theta) = p(\theta)$ which also is in E' and this means we have to check in Y' that $f'_0 p(\theta)\cdot\varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x)) = \varphi_0^{-1}(x')\cdot q(\xi'_i)\cdot\varphi_i(x'))\cdot f'_i p(\theta)$. The homotopies $\varphi_i : f'_i p \simeq q f_i$, $\varphi_0 : f'_0 p \simeq q f_0$ and $\theta : x \to x'$ give $q f_i(\theta)\cdot\varphi_i(x) = \varphi_i(x')\cdot f'_i p(\theta)$, $q f_0(\theta)\cdot\varphi_0(x) = \varphi_0(x')\cdot f'_0 p(\theta)$. Thus $\varphi_0^{-1}(x')\cdot q(\xi'_i)\cdot\varphi_i(x')\cdot f'_i p(\theta) = \varphi_0^{-1}(x')\cdot q f_0(\theta)\cdot q(\xi_i)\cdot q f_i(\theta)^{-1}\cdot q f_i(\theta)\cdot \varphi_i(x) = \varphi_0^{-1}(x')\cdot q f_0(\theta)\cdot q(\xi_i)\cdot\varphi_i(x) = f'_0 p(\theta)\cdot\varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x))$, as desired.

To show that k is an induced map in the sense required, we observe that they are in fact equal, viz., $pu(x,(\xi_i)) = p(x)$, $u'k(x,(\xi_i)) = u'(p(x), \varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x)) = p(x)$.

In order to show that k is π_0-surjective for any $(y',(\eta'_i))$ in E', we have to find $(x,(\xi_i))$ in E such that $k(x,(\xi_i)) = (p(x), \varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x)) \simeq (y',(\eta'_i))$, i.e., we have to find ε: $p(x) \to y'$ and $\xi_i : f_i(x) \to f_0(x)$ such that $f'_0(\varepsilon)\cdot\varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x) = \eta'_i\cdot f'_i(\varepsilon)$. Since p is π_0-surjective we get such an $\varepsilon : p(x) \to y'$. To have $\eta'_i = f'_0(\varepsilon)\cdot\varphi_0^{-1}(x)\cdot q(\xi_i)\cdot\varphi_i(x)\cdot f'_i(\varepsilon)^{-1}$, it is enough to take ξ_i so that $q(\xi_i) = \varphi_0(x)\cdot f_0(\varepsilon)^{-1}\cdot\eta'_i\cdot f'_0(\varepsilon)\cdot\varphi_i(x)^{-1}$ and the right hand side here is a path $q f_i(x) \simeq q f_0(x)$; this is possible since q is π_1-surjective (viz., $\theta : G \to H$ is π_1-surjective means by definition that if $h \in H_0$ is in the image of θ so is any path at h and this easily implies that if h ,$h' \in H_0$ are both in the image of θ so also is any path $h \simeq h'$).

5.2 Corollary. *Let S, $T : \Gamma \to \mathcal{G}$ be diagrams and $\theta : S \to T$ a p-map such that $\theta(i) : S(i) \to T(i)$ is an h-equivalence for $i \in \Gamma$. Then an induced limit map $L_\theta : L_S \to L_T$ is π_0-surjective.*

Proof. The result follows from the above proposition since straight products of h-equivqlences of groupoids are obviously h-equivalences.

5.3 Proposition. *Let* $(u, f, \alpha, v, g) \mid \alpha : fu \simeq gv, u : P \to X, f : X \to Z, v : P \to Y, g : Y \to Z\}$ *and* $\{(u', f', \alpha', v', g') \mid \alpha' : f'u' \simeq g'v', u' : P' \to X', f' : X' \to Z', v' : P' \to Y', g' : Y' \to Z'\}$ *be two h-pullbacks in the g.e. category* $\mathcal{G}$*. Let* $r : X \to X'$*,* $s : Y \to Y'$ *be* π_0*-surjective and* $t : Z \to Z'$ *be* π_1*-surjective and* $\Phi : tf \simeq f'r$*,* $\Psi : tg \simeq g's$*. Then the induced map* $k : P \to P'$ *is* π_0*-surjective.*

Proof. Points of P are given by $P = \{(x, y, \xi), x \in X_0, y \in Y_0, \xi : f(x) \simeq g(y)\}$ and paths are $(\theta, \varphi) : (x, y, \xi) \to (\overline{x}, \overline{y}, \overline{\xi})$ with $\theta : x \to \overline{x}$, $\varphi : y \to \overline{y}$, $\overline{\xi} = g(\varphi) \cdot \xi \cdot f(\theta)^{-1}$ and u, v, α are defined by evaluating on (x, y, ξ) as x, y, ξ respectively; similarly for P'.

We define a groupoid homomorphism $k : P \to P'$ by setting $k(x, y, \xi) = (r(x), s(y), \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x))$ where $r(x) \in X_0'$, $s(y) \in Y_0'$ and for paths $(\theta, \varphi) : (x, y, \xi) \to (\overline{x}, \overline{y}, \overline{\xi})$ by setting $k(\theta, \varphi) = (r(\theta), s(\varphi))$, which again is in P' and this means we have to check that $g's(\varphi) \cdot \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x) = \Psi(\overline{y}) \cdot t(\overline{\xi}) \cdot \Phi^{-1}(\overline{x}) \cdot f'r(\theta)$ in Z'. From the homotopies Φ, Ψ, θ, φ we have $f'r(\theta) \cdot \Phi(x) = \Phi(\overline{x}) \cdot tf(\theta)$, $g's(\varphi) \cdot \Psi(y) = \Psi(\overline{y}) \cdot rg(\varphi)$. Thus $\Psi(\overline{y}) \cdot t(\overline{\xi}) \cdot \Phi^{-1}(\overline{x}) \cdot f'r(\theta) = \Psi(\overline{y}) \cdot t(\overline{\xi}) \cdot tf(\theta) \cdot \Phi^{-1}(x) = \Psi(\overline{y}) \cdot tg(\varphi) \cdot t(\xi) \cdot tf(\theta)^{-1} \cdot tf(\theta) \cdot \Phi^{-1}(x) = g's(\varphi) \cdot \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x)$.

To check that k is an induced map in the sense required, viz., $ru \simeq u'k$, $sv \simeq v'k$ we observe that they are in fact equal, viz., $ru(x, y, \xi) = r(x)$, $u'k(x, y, \xi) = u'(r(x), s(y), \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x)) = r(x)$, $ru((\theta, \varphi)) = r(\theta)$, $u'k((\theta, \varphi)) = u'(r(\theta), s(\varphi)) = r(\theta)$; similarly the other one.

In order to show that k is π_0-surjective consider any (x', y', ξ') in P'; we have to find (x, y, ξ) in P such that $k(x, y, \xi) = (r(x), s(y), \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x)) \simeq (x', y', \xi')$ i.e., we have to find $\varepsilon : r(x) \to x'$, $\delta : s(y) \to y'$ and $\xi : f(x) \to g(y)$ such that $g'(\delta) \cdot \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x) = \xi' \cdot f'(\varepsilon)$. Since r, s are π_0-surjective we get such $\varepsilon : r(x) \to x'$, $\delta : s(y) \to y'$. To have $\xi' = g'(\delta) \cdot \Psi(y) \cdot t(\xi) \cdot \Phi^{-1}(x) \cdot f'(\varepsilon)^{-1}$ it is enough to take $t(\xi) = \Psi(y)^{-1} \cdot$

$g'(\delta) \cdot \xi' \cdot f'(\varepsilon) \cdot \Phi(x)$ and the right hand side is a path $tf(x) \to tg(y)$; this is possible since t is π_1-surjective.

In [3], the Brown Complement Theorem states that if all h-limits exit in a g.e. category $\mathcal{C}$ and if $T : \mathcal{C} \to \mathcal{G}$ is π_0-representable (i.e., there is an object K and a p-natural transformation $\mathrm{Hom}(K, -) \to T(-)$ which is π_0-equivalence for each value of the variable, i.e., $[K, X] \to \pi T(X)$ is a bijection [3]) and π_0-limit preserving then it is pseudo representable (i.e., that the values of the above p-natural transformation, $\mathrm{Hom}(K, X) \to T(X)$ are equivalences of groupoids [3]). We prove that this theorem remains true for weak h-limits replacing h-limits. Also we generalize it in the sense that if all weak h-limits exist in $\mathcal{C}$ and if the p-functor $T : \mathcal{C} \to \mathcal{G}$ is π_0-limit surjective and π_0-representable then it is h-representable in sense now to be defined. We say that a p-functor $T : \mathcal{C} \to \mathcal{G}$ is *h-represented* by the object K in $\mathcal{C}$ if there is a p-natural transformation $\tau : \mathrm{Hom}(K, -) \to T(-)$ which induces for each X, an h-equivalence (i.e., π_0-equivalence+π_1-surjective) $\tau_X : \mathrm{Hom}(K, X) \to T(X)$. We say T is *h-representable* when there exists such a K.

5.4 Theorem. *If all weak h-limits exist in the g.e. category $\mathcal{C}$ and if $T : \mathcal{C} \to \mathcal{G}$ is π_0-limit preserving and π_0-representable then it is pseudo representable.*

5.5 Theorem. (Generalized Brown Complement Theorem) *If all weak h-limits exist in the g.e. category $\mathcal{C}$ and if $T : \mathcal{C} \to \mathcal{G}$ is π_0-limit surjective and π_0-representable then it is h-representable.*

Proof of Theorem 5.5. Since T is π_0-representable there exists an object K in $\mathcal{C}$ and a p-natural transformation $\tau : \mathrm{Hom}(K, -) \to T(-)$ such that for each $X \in \mathcal{C}$, $\tau_X : \mathrm{Hom}(K, X) \to T(X)$ is a π_0-equivalence. By the Yoneda Reduction Lemma [3] it is sufficient to prove that if, in the terminology of [3], the Yoneda transformation $Y : \mathrm{Hom}(K, -) \to T(-)$ defined by $\eta = Y_K(1_K) \in T(K)$ induces a π_0-equivalence $Y_X : \mathrm{Hom}(K, X) \to$

$T(X)$ then for each $f : K \to X$ the induced map $\pi_1 f : \pi_1 K \to \pi_1(T(f)(\eta)) = \pi_1(Y_X(f))$ is a surjection.

We take an indexing set $A + \{0\}$ with base point 0 and the diagram defined by the family of maps $X \to X$ indexed over $A + \{0\}$, each one of which is the identity map. We construct weak h-equalizer cones of this family and its images under T and $\mathrm{Hom}(K, -)$ respectively as $\{(\lambda, (1_X), (\beta_i), \lambda, 1_X) \mid (\beta_i) : (1_X)\lambda \simeq 1_X\lambda, \lambda : C \to X\}$, $\{(l, (1_{T(X)}), (\alpha_i), l, 1_{T(X)} \mid (\alpha_i) : (1_{T(X)})l \simeq 1_{T(X)}l, l : E \to T(X)\}$, $\{(l', (1_{\mathrm{Hom}(K,X)}), (\alpha_i'), l', 1_{\mathrm{Hom}(K,X)}) \mid (\alpha_i') : (1_{\mathrm{Hom}(K,X)})l' \simeq 1_{\mathrm{Hom}(K,X)}l', l' : E' \to \mathrm{Hom}(K, X)\}$. By the definition of weak h-equalizer there exist factorizations $a : T(C) \to E$, $b : \mathrm{Hom}(K, C) \to E'$ such that $T(\lambda) \simeq la$, $\mathrm{Hom}(K, \lambda) \simeq l'b$ and a is π_0-surjective since T is π_0-limit surjective. Also by the definition of weak h-equalizer there exists a map $k : E' \to E$ such that $Y_X l' = lk$. In fact we define k by setting $(f, (\varepsilon_i)) \mapsto (Y_X(f), (Y_X(\varepsilon_i)))$ for $(\varepsilon_i) : (f) \to f$ in $\mathrm{Hom}(K, X)$ and the path $(\xi) : (f, (\varepsilon_i)) \to (g, (\delta_i))$ is sent to $(Y_X(\xi)) : (Y_X(f), (Y_X(\varepsilon_i))) \to (Y_X(\xi), (Y_X(\delta_i)))$. Also we have $T(\lambda)Y_C \simeq Y_X \mathrm{Hom}(K, \lambda)$ and by weak h-equalizer we have $aY_C \simeq kb$. Since a is π_0-surjective and Y_C is a π_0–equivalence, kb and hence k are π_0-surjective and it is the latter which is of significance to us. We put $x = T(f)(\eta) = Y_X(f)$ and take A to be the family of all paths $a : x \to x$. Consider the element $(x, (\gamma_a)) = (x, (a))$ in E. Since k is π_0-surjective there exists an element $y = (f', (\varepsilon_a))$ in E' such that $(x, (a)) \simeq k(f', (\varepsilon_a))$ and since Y_X is a π_0-equivalence it follows that $f \simeq f'$. Hence by a path in E' we may suppose that $y = (f, (\varepsilon_a))$. Thus there is a path $\xi : x \to Y_X(f)$ such that $\xi a = Y_X(\varepsilon_a)\xi$. Hence $Y_X(\varepsilon_a) = \varepsilon \cdot a \cdot \varepsilon^{-1}$ and the desired surjectivity follows.

Proof of Theorem 5.4. In this case a, b and Y_C are π_0-equivalences and hence k is a π_0-equivalence. Thus for $\varepsilon : f \simeq f$ in $\mathrm{Hom}(K, X)$ if $Y_X(\varepsilon)$ is trivial i.e., $Y_X(\varepsilon) = e_{Y_X(f)}$ then, since k is a π_0-equivalence, it is clear that $\varepsilon = e_f$. This establishes the injectivity.

We want to obtain a converse to the Theorem 5.5. We say that $T : \mathcal{C} \to \mathcal{G}$ is π_0-*equalizer* (*multiple*) *surjective* if for every family of maps (f_i), $f_0 : X \to Y$ in $\mathcal{C}$, $i \in A$, A being an index set with base point 0, the induced map $T(E) \to \overline{E}$ is π_0-surjective where E is a multiple h-equalizer of the family of maps f_i, f_0 in $\mathcal{C}$ and $\overline{E}$ is a multiple h-equalizer of the family of maps $T(f_i)$, $T(f_0)$ in $\mathcal{G}$. The following result is the first step.

5.6 Proposotion. *If all weak h-limits exist in $\mathcal{C}$ and if $T : \mathcal{C} \to \mathcal{G}$ is h-representable then* (i) *T is π_0-eqzualizer* (*multiple*) *surjective and* (ii) *T is π_0-product preserving.*

Proof. (i) For the family of maps (f_i), $f_0 : X \to Y$ in $\mathcal{C}$, $i \in A$, A being an index set with base point 0 let E be a multiple weak h-equalizer of f_i, f_0 in $\mathcal{C}$ and (E', e') the multiple h-equalizer of the family of maps $T(f_i)$, $T(f_0)$ in $\mathcal{G}$. We prove that the induced map $t : T(E) \to E'$ is π_0-surjective. Let $(\overline{E}, \overline{e})$ be the multiple h-equalizer of $(\mathrm{Hom}(K, f_i))$ and $\mathrm{Hom}(K, f_0)$; $\tau_E : \mathrm{Hom}(K, E) \to T(E)$; $\tau_X : \mathrm{Hom}(K, X) \to T(E)$, $\tau_Y : \mathrm{Hom}(K, Y) \to T(E)$ are h-equivalences. Let $k : \overline{E} \to E'$ be the induced map with $\tau_X \overline{e} = e'k$. We note that the induced map $s : \mathrm{Hom}(K, E) \to \overline{E}$ is a π_0-equivalence and $ks = t\tau_E$. It is enough to show that k is π_0-surjective and this follows from Proposition 5.1. (ii) Let $\prod X_i$ denote the weak h-product of the family of objects $\{X_i\}$ in $\mathcal{C}$. We have a π_0-equivalence $\mathrm{Hom}(K, \prod X_i) \to \prod \mathrm{Hom}(K, X_i)$ and an h-equivalence $\tau_{\prod X_i} : \mathrm{Hom}(K, \prod X_i) \to T(\prod X_i)$. Also $\prod \tau_{X_i} : \prod \mathrm{Hom}(K, X_i) \to \prod T(X_i)$ is a π_0-equivalence since each $\tau_{X_i} : \mathrm{Hom}(K, X_i) \to T(X_i)$ is an h-equivalence. Now it is clear that there exists a π_0-equivalence $T(\prod X_i) \to \prod T(X_i)$ as required.

5.7 Proposition. *If all weak h-limits exist in $\mathcal{C}$ and if $T : \mathcal{C} \to \mathcal{G}$ is π_0-equalizer* (*multiple*) *surjective and π_0-product preserving then T is π_0-limit surjective.*

Proof. We construct a weak h-limit object of a diagram $F : \mathcal{C} \to \mathcal{G}$ as a weak h-equalizer of the constructed maps $q,\ g : \prod F(i) \to \prod A(f)$ as described in Theorem 3.1. Let L be a weak h-equalizer of $q,\ g$; E a weak h-equalizer of $T(q),\ T(g)$ and E_0 a weak h-equalizer of $s,\ t$: $\prod T(F(i)) \to \prod T(A(f))$. Since T is π_0-product preserving, we have π_0-equivalences $\varphi : T(\prod F(i)) \to \prod T(F(i))$, $\psi : T(\prod A(f)) \to \prod T(A(f))$. Since T is π_0-limit surjective we have a π_0-surjective map $e : T(L) \to E$. We observe from Theorem 5.5 that $\varphi,\ \psi$ are h-equivalences. Hence by Proposition 5.1, k is π_0-surjective.

From Propositions 5.6 and 5.7 we obtain the following.

5.8 Proposition. *If all weak h-limits exist in $\mathcal{C}$ and if $T : \mathcal{C} \to \mathcal{G}$ is h-representable then T is π_0-limit surjective.*

Theorems 5.4 and 5.5 have as applications, the following two theorems which are the main results.

5.9 Theorem. *If the g.e. category $\mathcal{C}$ admits weak h-limits then any weak h-coproduct is a strong h-coproduct.*

Proof. Let L be a weak h-coproduct of a family of objects $\{A_i : i \in \Gamma\}$. Then we have π_0-equivalence $\mathrm{Hom}(L, X) \to \mathrm{Hom}(S, k_X)$ where $S : \Gamma \to \mathcal{C}$ is a diagram defined by $S(i) = A_i,\ i \in \Gamma$ and Γ is discrete. Since $\mathrm{Hom}(L, k_-) \approx \prod_{i\in\Gamma} \mathrm{Hom}(S(i), -)$ and each component $\mathrm{Hom}(S(i), -)$ is π_0-limit preserving (i.e., Hom functors are π_0-limit preserving ([3], p.404)) it follows that the p-functor $\mathrm{Hom}(L, k_-) : \mathcal{C} \to \mathcal{G}$ is π_0-limit preserving. By Theorem 5.4, $\mathrm{Hom}(L, X) \to \mathrm{Hom}(S, k_X)$ is an equivalence and thus any weak h-coproduct is a strong h-coproduct.

Dually, we see that in the presence of weak h-colimits every weak h-product is a strong h-product.

5.10 Theorem. *Assume that all h-limits and weak h-colimits exist in $\mathcal{C}$. Then each weak h-colimit is an h-colimit.*

Proof. For a diagram $T : \Gamma \to \mathcal{C}$ we have π_0-equivalence $\mathrm{Hom}(L, X) \to \mathrm{Hom}(T, k_X)$ where L is a weak h-colimit of T. We show that the functor $\mathrm{Hom}(T, k_-) : \mathcal{C} \to \mathcal{G}$ is π_0-limit surjective. Hence by Theorem 5.5, it follows that $\mathrm{Hom}(L, X) \to \mathrm{Hom}(T, k_X)$ is an h-equivalence and thus the weak h-colimit is an h-colimit.

To prove the required π_0-limit surjectivity we proceed as follows: Corresponding to $T : \Gamma \to \mathcal{C}$ we form $\mathrm{Hom}(T, X)$ which is the image of the diagram T through the functor $\mathrm{Hom}(-, X)$. Let $L\mathrm{Hom}(T, X)$ be the 2-limit groupoid of this functor. So the induced $\mathrm{Hom}(T, k_x) \to L\mathrm{Hom}(T, X)$ is an isomorphism of groupoids. We observe that strict p-maps of diagrams of groupoids and their homotopies induce maps and homotopies of the 2-limit groupoids canonically so that 2-limit formation gives a strict functor. Hence if we put $L\mathrm{Hom}(T, X) = M(X)$ each f, $g : X \to Y$ and $\alpha : f \simeq g$ induces $M(f)$, $M(g) : M(X) \to M(Y)$, $M(\alpha) : M(f) \simeq M(g)$ and $M(-)$ becomes a strict functor and this strict functor corresponds through the above isomorphism to the strict functor structure on $\mathrm{Hom}(T, k_-)$, alluded to above. If we apply M to a diagram $S : \Delta \to \mathcal{C}$ we get $L\mathrm{Hom}(T, S) : \Delta \to \mathcal{G}$ (to remove ambiguity we write this as $L_\Gamma\mathrm{Hom}(T, S)$). This diagram itself has a 2-limit groupoid which we write as $L_\Delta L_\Gamma \mathrm{Hom}(T, L)$. We note that we could have also alternatively formed first the diagram $L_\Delta\mathrm{Hom}(T, S) : \Gamma \to \mathcal{G}$ and then its 2-limit $L_\Gamma L_\Delta \mathrm{Hom}(T, S)$. To see what is required let $k_L \to S$ be an h-limit cone. This induces $k_N \to L_\Gamma\mathrm{Hom}(T, S)$ where $N = L_\Gamma\mathrm{Hom}(T, L)$ and then an induced map $\chi : N \to L_\Gamma\mathrm{Hom}(T, S)$; we must show that χ is π_0-surjective. We note that $\mathrm{Hom}(T, S)$ can itself be ascribed an obvious meaning through the various induced maps as a diagram $(T, S) : \Gamma^{\mathrm{opp}} \times \Delta \to \mathcal{G}$. We note that for any diagram $U : \Gamma^{\mathrm{opp}} \times \Delta \to \mathcal{G}$ we can form in addition to its 2-limit directly first the iterated partial 2-limits that are diagrams $L_\Gamma U : \Delta \to \mathcal{G}$, $L_\Delta U : \Gamma \to \mathcal{G}$ and then the iterated 2-limit groupoids $L_\Delta L_\Gamma U$, $L_\Gamma L_\Delta U$ and this iterated procedure when applied to (T,S) gives these $L_\Delta L_\Gamma$, $L_\Gamma L_\Delta$ on the symbol $\mathrm{Hom}(T,S)$.

In the following Lemma 5.11 we shall show that $L_\Delta L_\Gamma U$, $L_\Gamma L_\Delta U$ are isomorphic. Through this isomorphism we shall also show that χ appears as the induced map, under L_Γ of $\mathrm{Hom}(T,L) \to L_\Delta \mathrm{Hom}(T,S)$ which is an induced map of limits under $\mathrm{Hom}(T,-)$. The result will then follow by Corollary 5.2 (and this also explains why it is not sufficient to suppose merely weak h-limits).

To introduce Lemma 5.11 we first define for graphs Γ, Δ the graph $\Gamma^{\mathrm{opp}} \times \Delta$; this comprises points (i,j) with $i \in \Gamma_0$, $j \in \Delta_0$ and paths that are of two kinds $(i,b) : (i,k) \to (i,l)$ for $i \in \Gamma_0$, $b : k \to l$ and $(a,k) : (i,k) \to (j,k)$ for $a : i \to j$, $k \in \Delta_0$. Thus for $U : \Gamma^{\mathrm{opp}} \times \Delta \to \mathcal{G}$ and for each $i \to j$, $k \to l$ we suppose that $U(-,l)U(i,-) = U(j,-)U(-,k)$. We form for each $i \in \Delta_0$ the 2-limit $L_\Delta U(i,-)$ of $U(i,-) : \Delta \to \mathcal{G}$ comprising points $((x_{i,k}),(\alpha_{i,g}))$, $k \in \Delta_0$, $g \in \Delta$, $\alpha_{i,g} : U(i,g)(x_{i,k}) \simeq x_{i,l}$ for $g : k \to l$ and paths $(\gamma_{i,k}) : ((x_{i,k}),(\alpha_{i,g})) \simeq ((x'_{i,k}),(\alpha'_{i,g}))$, $k \in \Delta_0$, $g \in \Delta$, $\gamma_{i,k} : x_{i,k} \simeq x'_{i,k}$ for $g : k \to l$, $\alpha'_{i,g} \cdot U(i,g)(\gamma_{i,k}) = \gamma_{i,l} \cdot \alpha_{i,g}$.

For S, $T : \Omega \to \mathcal{G}$ and $\theta : S \to T$ the canonical limit map $L_\theta : L_S \to L_T$ is defined as follows: for $f : i \to j$, $((x_i),(\alpha_f)) \mapsto ((\theta_i(x_i)),(\theta_j(\alpha_f) \cdot \theta_f(x_i)))$, $(\gamma_i) \mapsto (\theta_i(\gamma_i))$. Also for $f : i \to j$ and the diagram $U : \Gamma^{\mathrm{opp}} \times \Delta \to \mathcal{G}$ the maps $U(f,k) : U(i,k) \to U(j,k)$ define a strict map $U(f,-) : U(i,-) \to U(j,-)$ and we obtain the induced $L_\Delta U(f,-) : L_\Delta U(i,-) \to L_\Delta U(j,-)$ and these maps go together to give a map $L_\Delta U : \Gamma^{\mathrm{opp}} \to \mathcal{G}$. The 2-limit $L_\Gamma L_\Delta U$ of this comprises for its points: $((x_{i,k}),(\alpha_{i,g}),(\alpha_{f,k}))$, $\alpha_{i,g} : U(i,g)(x_{i,k}) \simeq x_{i,l}$, $g : k \to l$ and for $f : i \to j$, $\alpha_{f,k} : U(i,k)(x_{i,k}) \simeq x_{j,l}$ describes a path $((U(f,k)(x_{i,k})), ((U(f,l)(\alpha_{i,g}))_{g:k\to l})) \simeq ((x_{j,k}),(\alpha_{j,g}))$ provided the corresponding limit path condition is satisfied and this is easily seen to be for points $\alpha_{f,l} \cdot U(f,l)(\alpha_{i,g})) = \alpha_{j,g} \cdot U(j,g)(\alpha_{f,k})$, and for paths $\gamma_{i,k} : ((x_{i,k}),(\alpha_{j,g}),(\alpha_{f,k})) \simeq ((x'_{i,k}),(\alpha'_{i,g}),(\alpha'_{f,k}))$, for each i, $(\gamma_{i,k}) : ((x_{i,k}),(\alpha_{j,g})) \simeq ((x'_{i,k}),(\alpha'_{i,g}))$ with $\alpha'_{i,g} \cdot U(i,g)(\gamma_{i,k}) = \gamma_{i,l} \cdot \alpha_{j,g}$ together with a path collecting (with respect to Γ^{opp}) condition of its own, which reduces to: $\alpha'_{f,k} \cdot U(f,k)(\gamma_{f,k}) = \gamma_{j,k} \cdot \alpha_{f,k}$.

5.11 Lemma. $L_\Gamma L_\Delta U \simeq L_\Delta L_\Gamma U$.

Proof. This follows by symmetry.

Proof of Theorem 5.10 continued: For the final point, we examine the structure of the mapping χ. From the limit cone $((h_i),(h_f)) : k_L \to S(h_k : L \to S(k), h_g : S(g)h_k \simeq h_l$, for all $k \in \Gamma_0$ and $g : k \to l)$ the mapping χ collects the assignments $(d_i : T(i) \to L,\ d_f : d_jT(f) \simeq d_i) \to (h_kd_i : T(i) \to S(k), h_k * d_f)$, $(\delta_i : d_i \simeq d'_i) \to (h_k * d_i)$. The first part (i.e., point part) puts itself into (i.e., in the language of the Lemma 5.11) a point of $L_\Delta L_\Gamma(T,S)$ by $(A) : ((d_i),(d_f)) \to ((h_kd_i),(h_k * d_f),(h_g * d_i))$ (through the correspondence of the Lemma 5.11 which is ultimately symbolized as an identity) which is same as the result of applying L_Γ to $\mathrm{Hom}(T,L) \to L_\Delta(T,S)$ because this map itself is merely (A) with the indices from Γ individualized. By contrast the second part (i.e., path part) is already presented in $L_\Delta L_\Gamma(T,S)$.

6. Smallness conditions. We call a g.e. category $\mathcal{C}$, *h-complete* (*weakly h-complete*) if it admits all h-limits (all weak h-limits) i.e., for any diagram $S : \Gamma \to \mathcal{C}$, Γ an arbitrary graph, the h-limit (weak h-limit) of S exists in $\mathcal{C}$. In this section our aim is to find appropriate "*smallness conditions*" (as they are called) for an h-complete g.e. category to admit an h-colimit or a weak h-colimit of a given diagram $S : \Gamma \to \mathcal{C}$. We shall fit this into a general scheme. Let $\mathcal{C}$ be h-complete. First we consider a π_0-limit preserving pseudo functor $\tilde{T} : \mathcal{C} \to \mathcal{G}$. Let $\pi\tilde{T} = T : \pi\mathcal{C} \to \mathcal{S}$ (category of sets and functions). Our aim is to discuss smallness conditions on certain classes for $\tilde{T}$ to be π_0-representable, i.e., T to be representable. In fact we discuss conditions for the existence of h-colimits assuming always the existence of h-limits. By the previous work of this paper, this problem reduces to the discussion of the π_0-representability of the functor $\mathrm{Hom}(S, k_-) = \tilde{T}$, $S : \Gamma \to \mathcal{C}$ where the condition of π_0-limit preservation is to be relaxed.

The smallness conditions appearing as (S_1) and (S_2) below are already familiar enough in ordinary category theory as representability conditions for T. In such contexts in ordinary category theory we suppose the category to be closed to limits (especially to products and equalizers, which are the limits the analysis reduces to). But here we suppose, instead, closure to h-limits in $\mathcal{C}$. The puzzles, alluded to above, stem from this shift: limits to h-limits. Namely, we work on how h-limits, which fundamentally involve the structure of $\mathcal{C}$ itself (in fact the homotopies, which disappear in the formation of $\pi\mathcal{C}$), can leave anything more than indistinct traces of themselves on $\pi\mathcal{C}$; in fact these traces reduce to what we have pointed out already that h-products in $\mathcal{C}$ become products in $\pi\mathcal{C}$ and that h-limits in $\mathcal{C}$ become quasi-limits in $\pi\mathcal{C}$. However as we shall see, below, there is one more effect of h-limits on $\pi\mathcal{C}$ which is possibly more significant. But for the moment we shall dwell a little on quasi-limits and quasi-equalizers.

Firstly we note that an h-equalizer is a quasi-equalizer. Secondly if $\tilde{T}$ is π_0-limit preserving, T applied to an h-equalizer diagram $E \to Y \rightrightarrows Z$ gives a quasi-equalizer diagram $T(E) \to T(Y) \rightrightarrows T(Z)$. Moreover if a functor (still denoted T, for brevity) sends one quasi-equalizer of a pair of maps $Y \rightrightarrows Z$ to a quasi-equalizer then it does so to all quasi-equalizers of $Y \rightrightarrows Z$. Indeed if $E \to Y$, $E' \to Y$ are quasi-equalizers of the pair, we get a factorization $E \to E' \to Y$ of the first through the second (to within homotopy in $\mathcal{C}$). Hence if the image of the first $T(E) \to T(Y)$ is an equalizer of the image pair so is the image of the second $T(E') \to T(Y)$ because when $M \to T(Y)$ equalizes the image pair then it factorizes as $M \to T(E) \to T(E') \to T(Y)$ to within homotopy. Thus if we forget how T arises through h-limits in $\mathcal{C}$ there nevertheless remains the crucial fact that it sends quasi-equalizers to quasi-equalizers. The following result is worthy of notice in any discussion involving quasi-equalizers (and corresponding results are easily obtained for quasi-limits).

6.1 Lemma. *If* $E \xrightarrow{u} Y \rightrightarrows Z$ *is a quasi-equalizer diagram in the homotopy class category* $\pi\mathcal{C}$ *then any other quasi-equalizer in* $\pi\mathcal{C}$ *is of the form* $E' \xrightarrow{v} E \xrightarrow{v} Y$ *and* uv *is itself a quasi-equalizer if and only if* u *can be factored as* $E \xrightarrow{w} E' \xrightarrow{v} E \xrightarrow{u} Y$.

Proof. Clearly if $E' \xrightarrow{t} Y \rightrightarrows Z$ is a quasi-equalizer then $t = uv$ for some $v : E' \to E$. If uv is a quasi-equalizer then $uv = uvw$ for some $w : E \to E'$. Conversely if the factorization of u exists and s equalizes the pair of maps $Y \rightrightarrows Z$ then $s = uk$ for some $k : U \to E$ because u is a quasi-equalizer. Thus $s = uvwk = (uv)(wk)$. Hence uv is a quasi-equalizer.

A more significant effect of h-limits on $\pi\mathcal{C}$ appears in [3] in a rather hidden form as a lemma to the effect that if $r : X \to X$ is a homotopy idempotent (i.e., $rr \simeq r$) then there exist maps $u : Y \to X$ and $v : X \to Y$ such that $r = uv$, $vu = 1_Y$. In this context in ordinary category theory (i.e., in $\pi\mathcal{C}$) $u : Y \to X$ is called a *retract* for or associated to r and $v : X \to Y$ the corresponding *retraction.* In ordinary category theory, in the presence of equalizers the existence of a retract for an idempotent r is abundantly clear as it appears as the equalizer of r and 1_X. In [3] the retract is provided through the h-limit cone of the sequential diagram $\cdots \to X \xrightarrow{r} X \xrightarrow{r} X$. This h-limit cone is defined by a family of maps $Y \to X$ labelled over the integers; these maps are evidently homotopic to each other and may be replaced (via., the homotopy of their limit cones) by a single map $u : Y \to X$ with $u \simeq ru$. However all we need as given in [3] is that: *Idempotents in* $\pi\mathcal{C}$ *admits retracts.*

6.2 Smallness conditions. The conditions on $\tilde{T}$ (or T) for π_0-representability (or on T for representability) referred to above are:

(S_0). $\tilde{T}$ *is* π_0*-limit preserving.*

From the above discussion this can be replaced by its effect on T, namely, that T preserves products and sends quasi-equalizers to quasi-equalizers. To be as general as possible we temporarily replace $\pi\mathcal{C}$ by an ordinary category $\mathcal{D}$ and T by $U : \mathcal{D} \to \mathcal{S}$ and then in place of (S_0) we suppose that:

(S_0'). *$\mathcal{D}$ admits quasi-products and quasi-equalizers and its idempotents admit retracts; furthermore, U sends quasi-products to quasi-products and sends quasi-equalizers to quasi-equalizers.*

One advantage of this more general form is that it covers the case $T = \pi\tilde{T} = \mathrm{Hom}(S, k_{-})$, for a diagram $S : \Gamma \to \mathcal{D} = \pi\mathcal{C}$ where $\mathcal{D}$ admits actual products, which are preserved by T. Also the condition of π_0-equalizer surjectivity of $\tilde{T}$ obviously implies that T sends h-equalizers to quasi-equalizers. Expressed in terms of the original T, the next condition appears as:

(S_1). *There is a set-indexed family $y_i \in T(Y_i)$, $i \in I$, such that for any X and for any $x \in T(X)$, there is an $i \in I$ and a map $f : Y \to X$ with $T(f)(y_i) = x$.*

We note that if we replace set-indexed family here by class then we can say that the class generates the functor T. We note that the class which is the union of all Y in $\mathcal{C}$ is clearly a generating class. Hence we may interpret (S_1) by saying that among the various generating classes there is one that is a set.

By (S_0') we consider a quasi-product Y of $\{Y_i : i \in I\}$ with quasi-projections p_i and $y \in T(Y)$ with $T(p_i)(y) = y_i$, for all $i \in I$. By factorizing through the various projections for each $x \in T(X)$, we have a map $f : Y \to X$ such that $T(f)(y) = x$. This means that the generating family is replaced by a single element — a *generator* of T, as we shall call it. If we write all this relative to $U : \mathcal{D} \to \mathcal{S}$ this may be expressed (equivalent to (S_1)) by

(S_1'). *The Yoneda transformation τ. $\mathrm{Hom}_{\mathcal{D}}(Y, X) \to U(X)$ generated by a certain $y \in U(Y)$ is a surjection.*

Hence we envisage $U(X)$ as a quotient of $\mathrm{Hom}_{\mathcal{D}}(Y,X)$ through the relation $f \equiv f'$: $Y \to X$ meaning $U(f)(y) = U(f')(y)$. The role of further condition - (S_2) below, is to replace Y so that the surjection becomes a bijection and by this way we may say that the relation $f \equiv f'$ is *representable.* By condition (S'_1) there exists of a single generator $S \to k_Y$; in general this cone precisely refers to condition (S'_1) as the *quasi-representability* of U. Assuming (S'_1), we now require that the class of equivalent pairs as described above is generated through composition by a class of such pairs.

(S_2). *There is a family f_i, $g_i : Y \rightrightarrows Z_i$ with $f_i \equiv g_i$ and such that if $h \equiv k$ then for some $i \in I$, there is a factorization $h = mf_i$, $k = mg_i$.*

As we reduced (S_1) to (S'_1), we may take Z a quasi-product of $\{Z_i : i \in I\}$ and replace the family $\{f_i, g_i : i \in I\}$ by a single *generating equalizing pair* $Y \rightrightarrows Z$ (say). Thus we have a diagram $\mathrm{Hom}_{\mathcal{D}}(Z,X) \rightrightarrows \mathrm{Hom}_{\mathcal{D}}(Y,X) \to U(X)$ in which $U(X)$ appears as the coequalizer of the maps f^*, g^* on the left. We note that this coequalizer is obtained from the second set $\mathrm{Hom}_{\mathcal{D}}(Y,X)$ by an equivalence relation which is merely $x \equiv y \Leftrightarrow x = f^*(z)$, $y = g^*(z)$ and not as we might expect in general, its transitive, symmetric, reflexive closure. We prove the next result essentially following an argument of [3].

6.3 Theorem. *Under the above smallness conditions U is representable.*

Proof. Since U sends quasi-equalizers to quasi-equalizers and $U(f)(y)$ $= U(g)(y)$ it follows that there is $\overline{y} \in U(E)$ such that $U(q)(\overline{y}) = y$. Thus there is $s : Y \to E$ with $U(s)(y) = \overline{y}$ (because y generates U). The maps $s^* : \mathrm{Hom}_{\mathcal{D}}(E,X) \to \mathrm{Hom}_{\mathcal{D}}(Y,X)$, $\tau : \mathrm{Hom}_{\mathcal{D}}(Y,X) \to U(X)$, $q^* :$ $\mathrm{Hom}_{\mathcal{D}}(Y,X) \to \mathrm{Hom}_{\mathcal{D}}(E,X)$, $\tau' : \mathrm{Hom}_{\mathcal{D}}(E,X) \to U(X)$, yield that $\tau' =$ τs^*, $\tau' q^* = \tau$. We observe that $q^* s^*$ is idempotent. In fact if $\tau(h) = \tau(k)$ then $U(h)(y) = U(k)(y)$ (because y generates U) and so $hq = kq$ since $U(hq)(\overline{y}) = U(kq)(\overline{y})$; this means that if τ identifies two elements, so does q^*. Hence τ' is injective on the image of q^* and since τ is surjective τ' must be bijective on this image. Thus $\tau' q^*$ and $q^* s^*$ are bijective on the image. Hence $q^* s^*(1_E)$ i.e., $sq = r$ (say) is an idempotent. Thus there is a retract $u : K \to E$ associated to r and if $v : E \to K$ is the corresponding retraction then the composition of τ' with v^* gives a bijection $\mathrm{Hom}_{\mathcal{D}}(K, X) \to U(X)$.

6.2 Remarks on the above proof. From the above bijection it is clear that $q' = qu : K \to Y$ is an equalizer of f, g. Also if we replace the quasi-equalizer $q : E \to Y$ by $q' : K \to Y$ and with s' corresponding to s we see that $q's'$ is the identity and hence q' is itself a retract of Y.

The above discussion for a g.e. category $\mathcal{C}$ admitting weak h-limits gives, equally, conditions for the existence of a weak h-colimits in $\mathcal{C}$ itself and an actual colimit in $\pi\mathcal{C}$. Our discussion above relates (S_1') to the existence of a quasi-colimit: We consider a pair of maps u, $v : A \rightrightarrows B$ with a quasi-coequalizing diagram $A \rightrightarrows B \to Y$ together with a homotopy in the h-case. By condition (S_2) we have $A \rightrightarrows B \to Y \rightrightarrows Z$ $(f, g : Y \rightrightarrows Z, s : B \to Y)$. In the h-case we have $\alpha : su \simeq gs$ and $\varphi : fs \simeq gs$ with $(\varphi * v) \cdot (f * \alpha) \cdot (\varphi^{-1} * u) = g * \alpha$ such that for any h, k with $\psi : hs \simeq ks$ and $(\psi * v) \cdot (h * \alpha) \cdot (\psi^{-1} * u) = k * \alpha$ there is $t : Z \to X$ with $h = tf$, $k = tg$. In the ordinary case the conditions on f, g are merely: $fs = gs$ such that if $hs = ks$ then $h = tf$, $k = tg$ for appropriate t.

Thus we obtain the existence of any weak h-colimits and these are specified from a quasi-colimit $S \to k_Y$ via the existence of an appropriate pair of maps $Y \rightrightarrows Z$ with the coequalizing properties as described above. By the earlier results (Section 5) we conclude that all such weak h-colimits are actual h-colimits when $\mathcal{C}$ is h-complete rather than merely weakly h-complete.

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Department of Mathematics, National Institute of Technology, Rourkela - 769 008, India.

E-mail: abehera@nitrkl.ac.in